Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2011, Article ID 808276, 14 pages
doi:10.1155/2011/808276

Research Article

A Direct Method for the Analyticity of the Pressure for Certain Classical Unbounded Models

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Received 24 November 2010; Accepted 20 January 2011

Academic Editor: Giorgio Kaniadakis

The goal of this paper is to provide estimates leading to a direct proof of the analyticity of the pressure for certain classical unbounded models. We use our new formula (Lo, 2007) to establish the analyticity of the pressure in the thermodynamic limit for a wide class of classical unbounded models in statistical mechanics.

1. Introduction

This paper is a continuation of [1] on the analyticity of the pressure. It attempts to study a direct method for the analyticity of the pressure for certain classical unbounded spin systems. The paper presents a simple hypothesis, on a C^n-estimate of the moments of the source term to show that it does yield analyticity in the infinite volume limit.

The study of the analyticity of the pressure is very important in Statistical Mechanics. In fact the analytic behavior of the pressure is the classical thermodynamic indicator for the absence or existence of phase transition [2–19].

Because the nth-derivatives of the pressure are commonly represented in terms of the truncated functions, most of the techniques available so far for proving analyticity of the pressure take advantage of a sufficiently rapid decay of correlations and cluster expansion methods or Brascamp-Lieb inequality [1, 5, 20–35].

In this paper, we propose a new method for proving the analyticity of the pressure for a wide class of classical unbounded models. The method is based on a powerful representation of the nth-derivatives of the pressure by means of the Witten Laplacians [36] given by

$$\mathbf{W}_{\Phi}^{(0)} = \left(-\boldsymbol{\Delta} + \frac{|\nabla\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2}\right),$$
$$\mathbf{W}_{\Phi}^{(1)} = -\boldsymbol{\Delta} + \frac{|\nabla\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2} + \mathbf{Hess}\,\Phi. \tag{1.1}$$

These operators are in some sense deformations of the standard Laplace Beltrami operator. They are, respectively, equivalent to

$$A_{\Phi}^{(0)} := -\boldsymbol{\Delta} + \nabla\Phi\cdot\nabla,$$
$$A_{\Phi}^{(1)} := -\boldsymbol{\Delta} + \nabla\Phi\cdot\nabla + \mathbf{Hess}\,\Phi. \tag{1.2}$$

Indeed,

$$W_{\Phi}^{(\cdot)} = e^{-\Phi/2} \circ A_{\Phi}^{(\cdot)} \circ e^{\Phi/2}, \tag{1.3}$$

and the map

$$U_{\Phi} : L^2\left(\mathbb{R}^{\Lambda}\right) \longrightarrow L^2\left(\mathbb{R}^{\Lambda}, e^{-\Phi}dx\right)$$
$$u \longmapsto e^{\Phi/2}u \tag{1.4}$$

is unitary. More precisely, we will use the formula

$$\mathbf{cov}(g,h) = Z^{-1}\int\left(A_{\Phi}^{(1)^{-1}}\nabla g\cdot\nabla h\right)e^{-\Phi(x)}dx, \tag{1.5}$$

where

$$Z = \int e^{-\Phi(x)}dx. \tag{1.6}$$

This formula, due to Helffer and Sjöstrand [29, 37], is a stronger and more flexible version of the Brascamp-Lieb inequality [20]. It allowed us in [1] to obtain an exact formula for the nth-derivatives of the pressure. In this paper, we will use this exact formula to show that a simpler assumption on the source term similar to the weak decay used in [3, 11] will guarantee the analyticity of the pressure in the infinite volume limit for a wide class of classical unbounded models.

We will consider classical unbounded systems, where each component is located at a site i of a crystal lattice $\Lambda \subset \mathbb{Z}^d$ and is described by a continuous real parameter $x_i \in \mathbb{R}$. A particular configuration of the total system will be characterized by an element $x = (x_i)_{i\in\Lambda}$ of the product space $\mathbb{R}^{\Lambda}$.

The $\Phi = \Phi^{\Lambda}$ will denote the Hamiltonian which assigns to each configuration $x \in \mathbb{R}^{\Lambda}$ a potential energy $\Phi(x)$. The probability measure that describes the equilibrium of the system is then given by the Gibbs measure

$$d\mu^{\Lambda}(x) = Z_{\Lambda}^{-1} e^{-\Phi(x)} dx. \tag{1.7}$$

The $Z_{\Lambda} > 0$ is a normalization constant.

We are eventually interested in the behavior of the system in the thermodynamic limit, that is, $\lim_{\Lambda \to \mathbb{Z}^d}$.

Assume that Λ is finite, and consider a Hamiltonian Φ of the phase space $\mathbb{R}^{\Lambda}$ satisfying the assumptions of the following theorem.

Theorem 1.1 (see [30]). *Let Λ be a finite domain in $\mathbb{Z}^d$. If Φ satisfies the following:*

(1) $\lim_{|x| \to \infty} |\nabla \Phi(x)| = \infty$,

(2) *for some M, any $\partial^{\alpha}\Phi$ with $|\alpha| = M$ is bounded on $\mathbb{R}^{\Lambda}$,*

(3) *for $|\alpha| \geq 1$, $|\partial^{\alpha}\Phi(x)| \leq C_{\alpha}(1 + |\nabla\Phi(x)|^2)^{1/2}$ for some $C_{\alpha} > 0$,*

(4) **Hess** $\Phi \geq \delta$ *for some $0 < \delta \leq 1$,*

then for any C^{∞}-function g satisfying

$$\left|\partial^{\alpha} g\right| \leq C_{\alpha}(1 + Z_{\Phi})^{q\alpha}, \tag{1.8}$$

where

$$Z_{\Phi} = \frac{|\nabla\Phi|}{2}, \tag{1.9}$$

$\alpha \in \mathbb{N}^{|\Lambda|}$ with some C_{α} and some $q_{\alpha} > 0$, there exists a unique C^{∞}-function u solution of

$$\begin{gathered} A_{\Phi}^{(0)} v = g - \langle g \rangle_{L^2(\mu^{\Lambda})}, \\ \langle v \rangle_{L^2(\mu^{\Lambda})} = 0. \end{gathered} \tag{1.10}$$

Remark 1.2. This theorem was established by Johnsen [30]. A detailed proof of this theorem in the convex case that includes the regularity theory may also be found in [38]. The function spaces to be considered are the Sobolev spaces $B_{\Phi}^{k}(\mathbb{R}^{\Lambda})$ defined by

$$B_{\Phi}^{k}\left(\mathbb{R}^{\Lambda}\right) = \left\{u \in L^2\left(\mathbb{R}^{\Lambda}\right) : Z_{\Phi}^{\ell}\partial^{\alpha}u \in L^2\left(\mathbb{R}^{\Lambda}\right), \forall \ell + |\alpha| \leq k\right\}, \tag{1.11}$$

where

$$Z_{\Phi} = \frac{|\nabla\Phi|}{2}. \tag{1.12}$$

These are subspaces of the well-known Sobolev spaces $W^{k,2}(\mathbb{R}^{\Lambda})$, $k \in \mathbb{N}$. By regularity arguments, one may prove that the solution of (1.10) belongs to each $e^{\Phi/2}B_{\Phi}^{k}(\mathbb{R}^{\Lambda})$ for all $k \in \mathbb{N}$.

2. The Analyticity of the Pressure

2.1. Preliminaries

We first recall the context over which the formula for the nth-derivative of the pressure was derived in [1].

Let Λ be a finite domain in $\mathbb{Z}^d$ $(d \geq 1)$, and consider as above the Hamiltonian Φ of the phase space $\mathbb{R}^\Lambda$ satisfying the assumptions of Theorem 1.1.

Let g be a smooth function on $\mathbb{R}^{\mathbb{Z}^d}$ satisfying

$$\begin{gathered} \left|\partial^\alpha \nabla g\right| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^\Lambda, \\ \textbf{Hess}\, g \leq C \quad \text{for some positive constant } C. \end{gathered} \tag{2.1}$$

Let

$$\Phi_\Lambda^t(x) = \Phi(x) - t g(x), \tag{2.2}$$

where $x = (x_i)_{i \in \Lambda}$, and $t \in [0, \infty)$ is a thermodynamic parameter.

The finite volume pressure is defined by

$$P_\Lambda(t) = \frac{1}{|\Lambda|} \log\left[\int_{\mathbb{R}^\Lambda} dx e^{-\Phi_\Lambda^t(x)}\right]. \tag{2.3}$$

Denote that

$$\begin{gathered} Z = \int_{\mathbb{R}^\Lambda} dx e^{-\Phi_\Lambda(x)}, \\ Z_t = \int_{\mathbb{R}^\Lambda} dx e^{-\Phi_\Lambda^t(x)}, \\ \langle \cdot \rangle_{t,\Lambda} = \frac{\int \cdot dx e^{-\Phi_\Lambda^t(x)}}{Z_t}. \end{gathered} \tag{2.4}$$

Because of the assumptions made on g, one may find $T > 0$ such that, for all $t \in [0, T)$, $\Phi_\Lambda^t(x)$ satisfies all the assumptions of Theorem 1.1. Thus, each $t \in [0, T)$ is associated with a unique C^∞-solution $f(t)$ of the equation

$$\begin{gathered} A_{\Phi_\Lambda^t}^{(0)} f(t) = g - \langle g \rangle_{L^2(\mu)}, \\ \langle f(t) \rangle_{L^2(\mu)} = 0. \end{gathered} \tag{2.5}$$

Hence,

$$A_{\Phi_\Lambda^t}^{(1)} \mathbf{v}(t) = \nabla g, \tag{2.6}$$

where $\mathbf{v}(t) = \nabla f(t)$. Notice that the map

$$t \longmapsto \mathbf{v}(t) \tag{2.7}$$

is well defined [1] and that

$$\{\mathbf{v}(t) : t \in [0, T)\} \tag{2.8}$$

is a family of smooth solutions on $\mathbb{R}^{\Lambda}$ corresponding to the family of potential

$$\left\{\Phi_{\Lambda}^{t} : t \in [0, T)\right\}. \tag{2.9}$$

We proved in [1] that $\mathbf{v}$ is a smooth function of t by means of regularity arguments. The following proposition proved in [1] gives an exact formula for the nth-derivatives of the pressure.

Proposition 2.1 (see [1]). *If*

$$P_{\Lambda}(t) = \frac{1}{|\Lambda|} \ln\left[\int_{\mathbb{R}^{\Lambda}} dx e^{-\Phi^{t}(x)}\right], \tag{2.10}$$

where

$$\begin{gathered}\Phi^{t}(x) = \Phi_{\Lambda}(x) - t g(x), \\ \left|\partial^{\alpha} \nabla g\right| \leq C_{\alpha}, \quad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \\ \mathbf{Hess}\, g \leq C \quad \text{for some positive constant } C,\end{gathered} \tag{2.11}$$

and $\Phi_{\Lambda}(x)$ satisfies the assumptions of Theorem 1.1, then there exist $T > 0$ such that, for all $t \in [0, T)$, the nth-derivative of the finite volume pressure $P_{\Lambda}(t)$ is given by the formula

$$P_{\Lambda}^{(n)}(t) = \frac{(n-1)!\left\langle A_{g}^{n-1} g\right\rangle_{t,\Lambda}}{|\Lambda|}, \quad n \geq 1, \tag{2.12}$$

where

$$A_{g} h := A_{\Phi_{\Lambda}^{t}}^{(1)^{-1}} \nabla h \cdot \nabla g. \tag{2.13}$$

This formula gives a direction towards proving the analyticity of the pressure in the thermodynamic limit. In fact one only needs to provide a suitable estimate for $\langle A_{g}^{n-1} g\rangle_{\Lambda,t}$.

Remark 2.2. Though formula (2.12) was derived in [1] for models of Kac type, it is clear from the proof that it remains valid for Hamiltonians Φ for which the Helffer-Sjöstrand formula

$$\mathbf{cov}(f,g) = \left\langle A_{\Phi}^{(1)^{-1}} (\nabla f) \cdot \nabla g \right\rangle_{\Lambda} \tag{2.14}$$

holds. In [30], Johnsen proved that this formula remains valid for a wide class of none convex Hamiltonians.

3. An Estimate for the Coefficients

In this section we propose to provide an estimate that establishes the analyticity of the pressure in the infinite volume limit.

Recall that if f is an infinitely differentiable function defined on an open set D, then f is real analytic if for every compact set $K \subset D$ there exists a constant C such that for every $x \in K$ and every nonnegative integer n the following estimate holds:

$$\left| \frac{\partial^n}{\partial x^n} f(x) \right| \leq C^{n+1} n!. \tag{3.1}$$

We propose to establish the above estimate for the nth-derivatives of the pressure. First we have the following convolution formula.

Proposition 3.1. *Under the assumptions and notations of Proposition 2.1, one has*

$$\sum_{k=0}^{n-1} \frac{\langle g^k \rangle_{\Lambda,t} \left\langle A_g^{n-k-1} g \right\rangle_{\Lambda,t}}{k!} = \frac{1}{(n-1)!} \langle g^n \rangle_{\Lambda,t}. \tag{3.2}$$

Proof. First observe that

$$\begin{aligned}
\langle g^p A_g h \rangle_{\Lambda,t} &= \left\langle g^p A_{\Phi^t}^{(1)^{-1}} \nabla h \cdot \nabla g \right\rangle_{\Lambda,t} \\
&= \frac{1}{p+1} \left\langle A_{\Phi^t}^{(1)^{-1}} \nabla h \cdot \nabla g^{p+1} \right\rangle_{\Lambda,t} \\
&= \frac{1}{p+1} \mathbf{cov}\left(g^{p+1}, h\right) \\
&= \frac{1}{p+1} \left[\left\langle g^{p+1} h \right\rangle_{\Lambda,t} - \left\langle g^{p+1} \right\rangle_{\Lambda,t} \langle h \rangle_{\Lambda,t} \right], \quad p = 0, 1, \ldots.
\end{aligned} \tag{3.3}$$

Setting

$$k = p+1, \qquad h = A_g^{n-k-1} g \tag{3.4}$$

yields

$$\left\langle g^k \right\rangle_{\Lambda,t} \left\langle A_g^{n-k-1} g \right\rangle_{\Lambda,t} = \left\langle g^k A_g^{n-k-1} g \right\rangle_{\Lambda,t} - k \left\langle g^{k-1} A_g^{n-k} g \right\rangle_{\Lambda,t}. \tag{3.5}$$

Now dividing by $k!$, summing over k, and noticing that on the right-hand side one obtains a telescoping sum yield

$$\sum_{k=0}^{n-1} \frac{\langle g^k \rangle_{\Lambda,t} \left\langle A_g^{n-k-1} g \right\rangle_{\Lambda,t}}{k!} = \frac{1}{(n-1)!} \langle g^n \rangle_{\Lambda,t}. \tag{3.6}$$

□

Next, we need the following lemma.

Lemma 3.2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of real numbers such that*

$$a_0 = 1, \qquad |b_0| \leq C, \quad |a_n| \leq C^n, \qquad \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \leq (n+1) C^{n+1} \tag{3.7}$$

for some positive constant C*. Then*

$$|b_n| \leq \left[n + 1 + \sum_{k=0}^{n-1} 2^k (n-k) \right] C^{n+1}, \quad \forall n \geq 1. \tag{3.8}$$

Proof. Let $\{\alpha_n\}$ be the sequence defined recursively by

$$\begin{gathered} \alpha_0 = 1, \\ \alpha_n = (n+1) + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_1 + \alpha_0 \text{if } n \geq 1. \end{gathered} \tag{3.9}$$

It is clear that

$$\alpha_n = n + 1 + \sum_{k=0}^{n-1} 2^k (n-k), \quad \forall n \geq 0. \tag{3.10}$$

We need to prove that

$$|b_n| \leq \alpha_n C^{n+1}, \quad \forall n. \tag{3.11}$$

We have

$$|b_0| \leq C = \alpha_0 C^{0+1}. \tag{3.12}$$

By induction assume that the result is true if n is replaced by $\tilde{n} \leq n$. We have

$$\begin{aligned}
\left|\sum_{k=0}^{n+1} a_k b_{n+1-k}\right| \leq (n+2)C^{n+2} &\Longleftrightarrow \\
\left|b_{n+1} + \sum_{k=1}^{n+1} a_k b_{n+1-k}\right| \leq (n+2)C^{n+2} &\Longrightarrow \\
\left||b_{n+1}| - \left|\sum_{k=1}^{n+1} a_k b_{n+1-k}\right|\right| \leq (n+2)C^{n+2} &\Longrightarrow \\
|b_{n+1}| \leq (n+2)C^{n+2} + \left|\sum_{k=1}^{n+1} a_k b_{n+1-k}\right| &\Longrightarrow \\
|b_{n+1}| \leq (n+2)C^{n+2} + \left|\sum_{k=1}^{n+1} C^k \alpha_{n+1-k} C^{n+2-k}\right| &\Longleftrightarrow \\
|b_{n+1}| \leq \left[(n+2) + \sum_{k=1}^{n+1} \alpha_{n+1-k}\right] C^{n+2}&, \\
|b_n| \leq \alpha_{n+1} C^{n+2}&.
\end{aligned} \tag{3.13}$$

□

Proposition 3.3. *In addition to the assumptions of Proposition 2.1 assume that for $t_0 \in [0, T)$ the thermodynamic limit of $P_\Lambda(t_0)$ exists, and*

$$\left|\langle g^n \rangle_{\Lambda, t_0}\right| \leq n! \left(\frac{C}{2}\right)^n u\left[\max_{x_1, x_2 \in \Lambda} d(x_1, x_2)\right], \quad \forall n \geq 1, \tag{3.14}$$

where C is a positive constant and u is a positive real variable function satisfying

$$u(s) \leq 1 \quad \text{if } s \geq M \text{ for some } M > 0. \tag{3.15}$$

Then the infinite volume pressure $\lim_{\Lambda \to \mathbb{Z}^d} P_\Lambda(t)$ is analytic at t_0.

Proof. First choose Λ large enough so that $u[\max_{x_1, x_2 \in \Lambda} d(x_1, x_2)] \leq 1$. We then have

$$\left|\langle g^n \rangle_{\Lambda, t_0}\right| \leq n! \left(\frac{C}{2}\right)^n. \tag{3.16}$$

Let

$$a_n = \frac{\langle g^n \rangle_{\Lambda, t_0}}{n!}, \qquad b_n = \frac{1}{|\Lambda|} \left\langle A_g^n g \right\rangle_{\Lambda, t_0}, \qquad C_1 = \frac{C}{2}. \tag{3.17}$$

We have $a_0 = 1, |b_0| = (1/|\Lambda|)|\langle g\rangle_{\Lambda,t_0}| \leq C_1, |a_n| \leq C_1^n$, and by Proposition 3.1

$$\sum_{k=0}^{n} a_k b_{n-k} = \frac{1}{|\Lambda|}(n+1)a_{n+1}, \tag{3.18}$$

from which we have

$$\left|\sum_{k=0}^{n} a_k b_{n-k}\right| \leq (n+1)C_1^{n+1}. \tag{3.19}$$

Now applying Lemma 3.2 we get

$$\left|\left\langle A_g^n g\right\rangle_{\Lambda,t_0}\right| \leq |\Lambda|\alpha_n C_1^{n+1}, \quad \forall n \geq 1, \tag{3.20}$$

where

$$\alpha_n = n+1+\sum_{k=0}^{n} 2^k(n-k). \tag{3.21}$$

Now using the fact that, for $n \geq 2$

$$\begin{aligned}
\alpha_{n-1} &= n+\sum_{k=0}^{n-1} 2^k(n-1-k) \\
&= n\left[1+\sum_{k=0}^{n-1} 2^k\frac{(n-1-k)}{n}\right] \\
&\leq n\left(1+\sum_{k=0}^{n-1} 2^k\right) \\
&= n\left(1+\frac{1-2^n}{1-2}\right) \\
&= n2^n,
\end{aligned} \tag{3.22}$$

we have

$$\begin{aligned}
\left|\left\langle A_g^{n-1} g\right\rangle_{\Lambda,t_0}\right| &\leq |\Lambda| n2^n C_1^n, \quad \forall n \geq 1, \\
\left|P_\Lambda^{(n)}(t_0)\right| &\leq (n-1)!n(2C_1)^n \\
&= n!C^n, \quad \forall n \geq 1.
\end{aligned} \tag{3.23}$$

This shows that the Taylor series of the infinite volume pressure at t_0 has a nonvanishing radius of convergence.

Next, we propose to prove that the pressure is equal to its Taylor series in a neighborhood of t_0.

By (3.16), the power series

$$\sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{\Lambda,t_0}}{n!} (t-t_0)^n \tag{3.24}$$

has nonvanishing radius of convergence R_Λ. Put

$$G(t) = \sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{\Lambda,t_0}}{n!} (t-t_0)^n, \qquad A(t) = \sum_{n=0}^{\infty} \left\langle A_g^n g \right\rangle_{\Lambda,t_0} (t-t_0)^n. \tag{3.25}$$

Inside the interval of convergence of $G(t)$, the convolution formula of Proposition 3.1 gives

$$G(t)A(t) = G'(t). \tag{3.26}$$

This implies that

$$A(t) = \frac{G'(t)}{G(t)} \quad \text{if } t \neq t_0 \Longrightarrow \int A(t)dt = \ln|G(t)| + \text{const}, \quad \text{if } t \neq t_0. \tag{3.27}$$

Equivalently

$$\sum_{n=0}^{\infty} \frac{\left\langle A_g^n g \right\rangle_{\Lambda,t_0}}{n+1} (t-t_0)^{n+1} = \ln\left| \sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{\Lambda,t_0}}{n!} (t-t_0)^n \right| \quad \text{if } t \neq t_0 \tag{3.28}$$

or

$$\sum_{n=1}^{\infty} \frac{\left\langle A_g^{n-1} g \right\rangle_{\Lambda,t_0}}{n} (t-t_0)^n = \ln\left| \sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{\Lambda,t_0}}{n!} (t-t_0)^n \right| \quad \text{if } t \neq t_0. \tag{3.29}$$

Thus

$$\sum_{n=1}^{\infty} \frac{\left\langle A_g^{n-1} g \right\rangle_{\Lambda,t_0}}{|\Lambda| n} (t-t_0)^n = \frac{1}{|\Lambda|} \ln\left| \sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{\Lambda,t_0}}{n!} (t-t_0)^n \right|, \quad \text{if } t \neq t_0. \tag{3.30}$$

Now using Proposition 2.1, we obtain

$$\sum_{n=1}^{\infty} \frac{P_\Lambda^{(n)}(t_0)}{n!} (t-t_0)^n = \frac{1}{|\Lambda|} \ln\left| \sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{\Lambda,t_0}}{n!} (t-t_0)^n \right|, \quad \text{if } t \neq t_0. \tag{3.31}$$

Now adding $P_\Lambda(t_0)$ on both sides of this above equality, we get

$$\sum_{n=0}^{\infty}\frac{P_\Lambda^{(n)}(t_0)}{n!}(t-t_0)^n = \frac{1}{|\Lambda|}\ln\left|\sum_{n=0}^{\infty}\frac{Z_t\langle g^n\rangle_{\Lambda,t_0}}{n!}(t-t_0)^n\right|, \quad \text{if } t\neq t_0$$
$$= \frac{1}{|\Lambda|}\ln\left|\sum_{n=0}^{\infty}\int_{\mathbb{R}^\Lambda}\frac{(t-t_0)^n}{n!}g^n e^{-\Phi_\Lambda(x)+t_0 g(x)}dx\right|, \quad \text{if } t\neq t_0. \tag{3.32}$$

Because g has bounded derivatives, we have

$$\int_{\mathbb{R}^\Lambda}\sum_{n=0}^{\infty}\left|\frac{(t-t_0)^n}{n!}g^n e^{-\Phi_\Lambda(x)+t_0 g(x)}\right|dx = \int_{\mathbb{R}^\Lambda} e^{|(t-t_0)g|-\Phi_\Lambda(x)+t_0 g(x)}dx < \infty. \tag{3.33}$$

Thus by permuting sum and integral we obtain

$$\sum_{n=0}^{\infty}\frac{P_\Lambda^{(n)}(t_0)}{n!}(t-t_0)^n = \frac{1}{|\Lambda|}\ln\left|\int_{\mathbb{R}^\Lambda} e^{-\Phi_\Lambda(x)+t g(x)}dx\right|, \quad \text{if } t\neq t_0$$
$$= P_\Lambda(t), \quad \text{if } t\neq t_0. \tag{3.34}$$

□

4. Comparison with Known Results

In [11], Lebowitz derived some regularity properties of the infinite volume pressure by assuming that the truncated functions have a weak decay of the type

$$\left|\langle x_{i_1,\dots,}x_{i_n}\rangle_\Lambda^T\right| \leq C^{n-1}(n-1)!\,u\left(\max_{x_1,x_2} d(x_1,x_2)\right), \tag{4.1}$$

where u is a rapidly decreasing function independent of Λ. However, he only obtained infinite differentiability rather than analyticity. The obstacle from getting analyticity is that, when u is rapidly or exponentially decaying, the bounds obtained increase too fast with n. In [3], Duneau et al. considered stronger decay assumptions of the truncated functions and showed that they do yield analyticity.

We showed in this paper that if the decay assumption is made on the nth moments of $g = \sum_{i\in\Lambda} x_i$ for instance, then an even weaker assumption would imply analyticity.

Let us also mention that, even though our results concern unbounded models whose Hamiltonians satisfy the assumptions of Theorem 1.1, it could be useful for the study of certain bounded models. Indeed, it has been shown in [32] that the investigation of the critical behavior of the two-dimensional Kac models may be reduced in the mean-field approximation to the study of unbounded models of the type discussed above.

It is also clear that if the thermodynamic limit

$$\lim_{\Lambda \to \mathbb{Z}^d} \int_{\mathbb{R}^\Lambda} dx e^{-\Phi^{t_0}(x)} \tag{4.2}$$

exists, then the assumption

$$\left| \langle g^n \rangle_{\Lambda, t_0} \right| \leq n! \left(\frac{C}{2} \right)^n \tag{4.3}$$

is equivalent to saying that the partition function

$$Z_{t,\Lambda} = \int_{\mathbb{R}^\Lambda} dx e^{-\Phi^t(x)} \tag{4.4}$$

is analytic at t_0. Thus, Proposition 3.3 provides a simple and direct proof of the analyticity of the pressure from the analyticity of the partition function. Recall that even in the grand canonical ensemble, where the partition function is directly given as a power series, the classical proofs of the analyticity of the pressure that are available in the literature involve in general cluster expansions, sometimes with complicated renormalization arguments.

Acknowledgment

The support of King Fahd University is deeply acknowledged. The author also would like to express his appreciation to Professor Haru Pinson.

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