Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2011, Article ID 957592, 22 pages
doi:10.1155/2011/957592

Research Article

Partial Inner Product Spaces: Some Categorical Aspects

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Received 16 March 2011; Revised 20 July 2011; Accepted 10 August 2011

Academic Editor: Partha Guha

We make explicit in terms of categories a number of statements from the theory of partial inner product spaces (PIP-spaces) and operators on them. In particular, we construct sheaves and cosheaves of operators on certain PIP-spaces of practical interest.

1. Motivation

Partial inner product spaces (PIP-spaces) were introduced some time ago by Grossmann and one of us (JPA) as a structure unifying many constructions introduced in functional analysis, such as distributions or generalized functions, scales of Hilbert or Banach spaces, interpolation couples, and so forth [1–4]. Since these structures have regained a new interest in many aspects of mathematical physics and in modern signal processing, a comprehensive monograph was recently published by two of us [5], as well as a review paper [6].

Roughly speaking, a PIP-space is a vector space equipped with a partial inner product, that is, an inner product which is not defined everywhere, but only for specific pairs of vectors. Given such an object, operators can be defined on it, which generalize the familiar notions of operators on a Hilbert space, while admitting extremely singular ones.

Now, in the previous work, many statements have a categorical "flavor", but the corresponding technical language was not used; only some hints in that direction were given in [7]. Here we fill the gap and proceed systematically. We introduce the category **PIP** of (indexed) PIP-spaces, with homomorphisms as arrows (they are defined precisely to play that role), as well as several other categories of PIP-spaces.

In a second part, we consider a single PIP-space V_I as a category by itself, called $\mathbf{V_I}$, with natural embeddings as arrows. For this category $\mathbf{V_I}$, we show, in Sections 4 and 5, respectively, that one can construct sheaves and cosheaves of operators. There are some restrictions on the PIP-space V_I, but the cases covered by our results are the most useful ones for applications. Then, in Section 6, we describe the cohomology of these (co)sheaves and prove that, in many cases, the sheaves of operators are acyclic; that is, all cohomology groups of higher order are trivial.

Although sheaves are quite common in many areas of mathematics, the same cannot be said of cosheaves, the dual concept of sheaves, for which very few concrete examples are known. Actually, cosheaves were recently introduced in the context of nonclassical logic (see Section 7) and seem to be related to certain aspects of quantum gravity. Hence the interest of having at one's disposal new, concrete examples of cosheaves, namely, cosheaves of operators on certain types of PIP-spaces.

2. Preliminaries

2.1. Partial Inner Product Spaces

We begin by fixing the terminology and notations, following our monograph [5], to which we refer for a full information. For the convenience of the reader, we have collected in Appendix A the main features of partial inner product spaces and operators on them.

Throughout the paper, we will consider an indexed PIP-space $V_I = \{V_r,\ r \in I\}$, corresponding to the linear compatibility #. The assaying subspaces are denoted by $V_p, V_r, \ldots,$ $p, r \in I$. The set I indexes a generating involutive sublattice of the complete lattice $\mathcal{F}(V,\#)$ of all assaying subspaces defined by #; that is,

$$f\#g \Longleftrightarrow \exists r \in I \quad \text{such that } f \in V_r, g \in V_{\overline{r}}. \tag{2.1}$$

The lattice properties are the following:

(i) involution: $V_r \leftrightarrow V_{\overline{r}} := (V_r)^{\#}$,

(ii) infimum: $V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q, (p, q, r \in I)$,

(iii) supremum: $V_{p\vee q} := V_p \vee V_q = (V_p + V_q)^{\#\#}$.

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$, and the greatest element is $V = \bigcup_r V_r$, but often they do not belong to V_I.

Each assaying subspace V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$, and $V^{\#}$ is dense in every V_r, since the indexed PIP-space V_I is assumed to be nondegenerate. In the sequel, we consider projective and additive indexed PIP-spaces (see Appendix A) and, in particular, lattices of the Banach or Hilbert spaces (LBS/LHS).

Given two indexed PIP-spaces V_I, Y_K, an operator $A : V_I \to Y_K$ may be identified with the coherent collection of its representatives $A \simeq \{A_{ur}\}$, where each $A_{ur} : V_r \to Y_u$ is a continuous operator from V_r into Y_u. We will also need the set $\mathsf{d}(A) = \{r \in I :$ there is a $u \in K$ such that A_{ur} exists$\}$. Every operator A has an adjoint $A^{\times}$, and a partial multiplication between operators is defined.

A crucial role is played by homomorphisms, in particular, mono-, epi-, and isomorphisms. The set of all operators from V_I into Y_K is denoted by $\mathrm{Op}(V_I, Y_K)$ and the set of all homomorphisms by $\mathrm{Hom}(V_I, Y_K)$.

For more details and references to related work, see Appendix A or our monograph [5].

2.2. Categories

According to the standard terminology [8], a (small) category $\mathbf{C}$ is a collection of objects $X, Y, Z, \ldots$ and arrows or morphisms $\alpha, \beta, \ldots$, where we note $\alpha \in \hom(X, Y)$ or $X \xrightarrow{\alpha} Y$, satisfying the following axioms.

(i) *Identity:* for any object X, there is a unique arrow $1_X : X \to X$.

(ii) *Composition:* whenever $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$, there is a unique arrow $\beta \circ \alpha$ such that $X \xrightarrow{\beta\circ\alpha} Z$.

(iii) *Associativity:* whenever $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} T$, one has $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.

(iv) *Unit law:* whenever $X \xrightarrow{\alpha} Y$, one has $\alpha \circ 1_X = \alpha$ and $1_Y \circ \alpha = \alpha$.

In a category, an object S is *initial* if, for each object X, there is exactly one arrow $S \to X$. An object T is *final* or *terminal* if, for each object X, there is exactly one arrow $X \to T$. Two terminal objects are necessarily isomorphic (isomorphisms in categories are defined exactly as for indexed PIP-spaces, see Appendix A).

Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *covariant functor* $\mathsf{F} : \mathbf{C} \to \mathbf{D}$ is a morphism between the two categories. To each object X of $\mathbf{C}$, it associates an object $\mathsf{F}(X)$ of $\mathbf{D}$, and, to each arrow $\alpha : X \to Y$ in $\mathbf{C}$, it associates an arrow $\mathsf{F}(\alpha) : \mathsf{F}(X) \to \mathsf{F}(Y)$ of $\mathbf{D}$ in such a way that

$$\mathsf{F}(1_X) = 1_{\mathsf{F}(X)}, \qquad \mathsf{F}(\beta \circ \alpha) = \mathsf{F}(\beta) \circ \mathsf{F}(\alpha), \tag{2.2}$$

whenever the arrow $\beta \circ \alpha$ is defined in $\mathbf{C}$.

Given a category $\mathbf{C}$, the *opposite* category $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$ and all arrows reversed: to each arrow $\alpha : X \to Y$, there is an arrow $\alpha^{\mathrm{op}} : Y \to X$, so that $\alpha^{\mathrm{op}} \circ \beta^{\mathrm{op}} = (\beta \circ \alpha)^{\mathrm{op}}$.

A *contravariant functor* $\overline{\mathsf{F}} : \mathbf{C} \to \mathbf{D}$ may be defined as a functor $\mathsf{F} : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, or directly on $\mathbf{C}$, by writing $\overline{\mathsf{F}}(\alpha) = \mathsf{F}(\alpha^{\mathrm{op}})$. Thus, we have

$$\overline{\mathsf{F}}(1_X) = 1_{\overline{\mathsf{F}}(X)}, \qquad \overline{\mathsf{F}}(\beta \circ \alpha) = \overline{\mathsf{F}}(\alpha) \circ \overline{\mathsf{F}}(\beta). \tag{2.3}$$

Some standard examples of categories are

(i) **Set**, the category of sets with functions as arrows.

(ii) **Top**, the category of topological spaces with continuous functions as arrows.

(iii) **Grp**, the category of groups with group homomorphisms as arrows.

For more details, we refer to standard texts, such as Mac Lane [8].

3. Categories of PIP-Spaces

3.1. A Single PIP-Space as Category

We begin by a trivial example.

A single-indexed PIP-space $V_I = \{V_r,\ r \in I\}$ may be considered as a category $\mathbf{V_I}$, where

(i) the objects are the assaying subspaces $\{V_r,\ r \in I,\ V^\#,\ V\}$;

(ii) the arrows are the natural embeddings $\{E_{rs} : V_r \to V_s,\ r \leqslant s\}$, that is, the representatives of the identity operator on V_I.

The axioms of categories are readily checked as follows.

(i) For every V_r, there exists an identity, $E_{rr} : V_r \to V_r$, the identity map.

(ii) For every V_r, V_s with $r \leqslant s$, one has $E_{ss} \circ E_{sr} = E_{sr}$ and $E_{sr} \circ E_{rr} = E_{sr}$.

(iii) For every V_r, V_s, V_t with $r \leqslant s \leqslant t$, one has $E_{ts} \circ E_{sr} = E_{tr}$.

(iv) For every V_r, V_s, V_t, V_u with $r \leqslant s \leqslant t \leqslant u$, one has $(E_{ut} \circ E_{ts}) \circ E_{sr} = E_{ut} \circ (E_{ts} \circ E_{sr})$.

In the category $\mathbf{V_I}$, we have:

(i) $V_{\overline{\infty}} := V^\# = \bigcap_{r\in I}$ is an initial object: for every $V_r \in V_I$, there is a unique arrow $E_{r\overline{\infty}} : V^\# \to V_r$.

(ii) $V_\infty := V = \sum_{r\in I}$ is a terminal object: for every $V_r \in V_I$, there is a unique arrow $E_{\infty r} : V_r \to V$.

(iii) The compatibility $\# : V_r \mapsto (V_r)^\# = V_{\overline{r}}$ defines a contravariant functor $\mathbf{V_I} \to \mathbf{V_I}$.

Although this category seems rather trivial, it will allow us to define sheaves and cosheaves of operators, a highly nontrivial (and desirable) result.

3.2. A Category Generated by a Single Operator

In the indexed PIP-space $V_I = \{V_r,\ r \in I\}$, take a single totally regular operator, that is, an operator A that leaves every V_r invariant. Hence so does each power $A^n,\ n \in \mathbb{N}$. Then this operator induces a category $\mathbf{A(V_I)}$, as follows.

(i) The objects are the assaying subspaces $\{V_r,\ r \in I\}$.

(ii) The arrows are the operators $A^n_{pq} : V_q \to V_p,\ q \leqslant p,\ n \in \mathbb{N}$.

The axioms of categories are readily checked as follows.

(i) For every V_r, there exists an identity, $A_{rr} : V_r \to V_r$, since A is totally regular.

(ii) For every V_r, V_s with $r \leqslant s$, one has $A_{ss} \circ A^n_{sr} = A^{n+1}_{sr}$ and $A^n_{sr} \circ A_{rr} = A^{n+1}_{sr}$, for all $n \in \mathbb{N}$.

(iii) For every V_r, V_s, V_t with $r \leqslant s \leqslant t$, one has $A^m_{ts} \circ A^n_{sr} = A^{m+n}_{tr}$, for all $m, n \in \mathbb{N}$.

(iv) For every V_r, V_s, V_t, V_u with $r \leqslant s \leqslant t \leqslant u$, one has $(A^m_{ut} \circ A^n_{ts}) \circ A^k_{sr} = A^m_{ut} \circ (A^n_{ts} \circ A^k_{sr})$, for all $m, n, k \in \mathbb{N}$.

As for $\mathbf{V_I}$, the space $V_{\overline{\infty}} := V^\#$ is an initial object in $\mathbf{A(V_I)}$ and $V_\infty := V$ is a terminal object.

The adjunction $A \mapsto A^\#$ defines a contravariant functor from $\mathbf{A(V_I)}$ into $\mathbf{A(V_I)}^\#$, where the latter is the category induced by $A^\#$. The proof is immediate.

3.3. The Category PIP *of Indexed* PIP*-Spaces*

The collection of all indexed PIP-spaces constitutes a category, which we call **PIP**, where

(i) objects are indexed PIP-spaces $\{V_I\}$;

(ii) arrows are homomorphisms $A : V_I \to Y_K$, where an operator $A \in \mathrm{Op}(V_I, Y_K)$ is called a *homomorphism* if

  (a) for every $r \in I$ there exists $u \in K$ such that both A_{ur} and $A_{\overline{ur}}$ exist;

  (b) for every $u \in K$ there exists $r \in I$ such that both A_{ur} and $A_{\overline{ur}}$ exist.

For making the notation less cumbersome (and more automatic), we will, henceforth, denote by A_{KI} an element $A \in \mathrm{Hom}(V_I, Y_K)$. Then the axioms of a category are obviously satisfied as follows.

(i) For every V_I, there exists an identity, $1_I \in \mathrm{Hom}(V_I, V_I)$, the identity operator on V_I.

(ii) For every V_I, Y_K, one has $1_K \circ A_{KI} = A_{KI}$ and $A_{KI} \circ 1_I = A_{KI}$.

(iii) For every V_I, Y_K, W_L, one has $B_{LK} \circ A_{KI} = C_{LI} \in \mathrm{Hom}(V_I, W_L)$.

(iv) For every V_I, Y_K, W_L, Z_M, one has $(C_{ML} \circ B_{LK}) \circ A_{KI} = C_{ML} \circ (B_{LK} \circ A_{KI}) \in \mathrm{Hom}(V_I, Z_M)$.

The category **PIP** has no initial object and no terminal object; hence, it is not a topos.

One can define in the same way smaller categories **LBS** and **LHS**, whose objects are, respectively, lattices of the Banach spaces (LBS) and lattices of the Hilbert spaces (LHS), the arrows being still the corresponding homomorphisms.

3.3.1. Subobjects

We recall that a homomorphism $M_{KI} \in \mathrm{Hom}(V_I, Y_K)$ is a monomorphism if $M_{KL}A_{LI} = M_{KL}B_{LI}$ implies $A_{LI} = B_{LI}$, for any pair A_{LI}, B_{LI} and any indexed PIP-space W_L (a typical example is given in Appendix A). Two monomorphisms M_{LI}, N_{LK} with the same codomain W_L are *equivalent* if there exists an isomorphism U_{KI} such that $N_{LK}U_{KI} = M_{LI}$. Then a *subobject* of V_I is an equivalence class of monomorphisms into V_I. A PIP-subspace W of an indexed PIP-space V is defined as an orthocomplemented subspace of V, and this holds if and only if W is the range of an orthogonal projection, $W_I = PV_I$. Now the embedding $M : W_I = PV_I \longmapsto V_I$ is a monomorphism; thus, orthocomplemented subspaces are subobjects of **PIP**.

However, the converse is not true, at least for a general indexed PIP-space. Take the case where V is a noncomplete prehilbert space (i.e., $V = V^{\#}$). Then every subspace is a subobject, but need not to be the range of a projection. To give a concrete example [5, Section 3.4.5], take $V = \mathcal{S}(\mathbb{R})$, the Schwartz space of test functions. Let $W = \mathcal{S}_+ := \{\varphi \in \mathcal{S} : \varphi(x) = 0 \text{ for } x \leqslant 0\}$. Then $W^\perp = \mathcal{S}_- :=\{\psi \in \mathcal{S} : \psi(x) = 0 \text{ for } x \geqslant 0\}$; hence, $W^{\perp\perp} = W$. However, W is not orthocomplemented, since every $\chi \in W + W^\perp$ satisfies $\chi(0) = 0$, so that $W + W^\perp \neq \mathcal{S}$. Yet W is the range of a monomorphism (the injection), hence, a subobject. However, this example addresses an indexed PIP-space which is *not* a LBS/LHS.

Take now a LBS/LHS $V_I = \{V_r,\ r \in I\}$ and a vector subspace W. In order that W becomes a LBS/LHS W_I in its own right, we must require that, for every $r \in I$, $W_r = W \cap V_r$ and $W_{\overline{r}} = W \cap V_{\overline{r}}$ are a dual pair with respect to their respective Mackey topologies and that the intrinsic Mackey topology $\tau(W_r, W_{\bar{r}})$ coincides with the norm topology induced by V_r. In other words, W must be topologically regular, which is equivalent that it be orthocomplemented [5, Section 3.4.2]. Now the injection $M_I : W_I \to V_I$ is clearly a monomorphism, and W_I is a subobject of V_I. Thus, we have shown that, in a LBS/LHS, the subobjects are precisely the orthocomplemented subspaces.

Coming back to the previous example of a noncomplete prehilbert space, we see that an arbitrary subspace W need *not* be orthocomplemented, because it may fail to be topologically regular. Indeed the intrinsic topology $\tau(W, W)$ does not coincide with the norm topology, unless W is orthocomplemented (see the discussion in [5, Section 3.4.5]). In the Schwartz example above, one has $W^{\perp\perp} = W$, which means that W is $\tau(W, W)$ closed, hence, norm closed, but it is not orthocomplemented.

Remark 3.1. Homomorphisms are defined between arbitrary PIP-spaces. However, when it comes to *indexed* PIP-spaces, the discussion above shows that the notion of homomorphism is more natural between two indexed PIP-spaces *of the same type*, for instance, two LBSs or two LHSs. This is true, in particular, when trying to identify subobjects. This suggests to define categories **LBS** and **LHS**, either directly as above or as subcategories within **PIP**, and then define properly subobjects in that context.

3.3.2. Superobjects

Dually, one may define superobjects in terms of epimorphisms. We recall that a homomorphism $N_{KL} \in \mathrm{Hom}(V_I, W_K)$ is an epimorphism if $A_{IK}N_{KL} = B_{IK}N_{KL}$ implies $A_{IK} = B_{IK}$, for any pair A_{IK}, B_{IK} and any indexed PIP-space Y_L. Then a *superobject* is an equivalence class of epimorphisms, where again equivalence means modulo isomorphisms.

Whereas monomorphisms are natural in the context of sheaves; epimorphisms are natural in the dual structure, that is, cosheaves.

4. Sheaves of Operators on PIP-Spaces

4.1. Presheaves and Sheaves

Let X be a topological space, and let $\mathbf{C}$ be a (concrete) category. Usually $\mathbf{C}$ is the category of sets, the category of groups, the category of abelian groups, or the category of commutative rings. In the standard fashion [8, 9], we proceed in two steps.

Definition 4.1. A *presheaf* F on X with values in $\mathbf{C}$ is a map F defined on the family of open subsets of X such that

(PS_1) for each open set U of X, there is an object $\mathsf{F}(U)$ in $\mathbf{C}$;

(PS_2) for each inclusion of open sets $T \subseteq U$, there is given a *restriction morphism* ρ_V^T : $\mathsf{F}(U) \to \mathsf{F}(T)$ in the category $\mathbf{C}$, such that ρ_U^U is the identity for every open set U and $\rho_S^T \circ \rho_T^U = \rho_S^U$ whenever $S \subseteq T \subseteq U$.

Definition 4.2. Let F be a presheaf on X, and let U be an open set of X. Every element $s \in \mathsf{F}(U)$ is called a *section* of F over U. A section over X is called a *global section*.

Example 4.3. Let X be a topological space; let $\mathbf{C}$ be the category of vector spaces. Let F associate to each open set U the vector space of continuous functions on U with values in $\mathbb{C}$. If $T \subseteq U$,

ρ_T^U associates to each continuous function on U its restriction to T. This is a presheaf. Any continuous function on U is a section of F on U.

Definition 4.4. Let F be a presheaf on the topological space X. One says that F is a *sheaf* if, for every open set $U \subset X$ and for every open covering $\{U_i\}_{i\in I}$ of U, the following conditions are fulfilled:

(S_1) given $s, s' \in \mathsf{F}(U)$ such that $\rho_{U_i}^U(s) = \rho_{U_i}^U(s')$, for every $i \in I$, then $s = s'$ (local identity);

(S_2) given $s_i \in \mathsf{F}(U_i)$ such that $\rho_{U_i\cap U_j}^{U_i}(s_i) = \rho_{U_i\cap U_j}^{U_j}(s_j)$, for every $i, j \in I$, then there exists a section $s \in \mathsf{F}(U)$ such that $\rho_{U_i}^U(s) = s_i$, for every $i \in I$ (gluing).

The section s whose existence is guaranteed by axiom (S_2) is called the *gluing*, *concatenation*, or *collation* of the sections s_i. By axiom (S_1), it is unique. The sheaf F may be seen as a contravariant functor from the category of open sets of X into $\mathbf{C} : = (\{\mathsf{F}(U_i)\}, \{\rho_{U_j}^{U_i}\})$.

4.2. A Sheaf of Operators on an Indexed PIP*-Space*

Let $V_I = \{V_r,\ r \in I\}$ be an indexed PIP-space, and let $\mathbf{V}_I$ be the corresponding category defined in Section 3.1. If we put on I the discrete topology, then I defines an open covering of V. Each V_r carries its Mackey's topology $\tau(V_r, V_{\overline{r}})$.

We define a sheaf on V_I by the contravariant functor $\mathsf{F} : \mathbf{V}_I \to \mathbf{Set}$ given by

$$\mathsf{F} : V_r \longmapsto \mathrm{Op}_r := \{A \upharpoonright V_r : A \in \mathrm{Op}(V_I),\ r \in \mathsf{d}(A)\}. \tag{4.1}$$

This means that an element of Op_r is a representative A_{wr} from V_r into some V_w. In the sequel, we will use the notation A_{*r} whenever the dependence on the first index may be neglected without creating ambiguities. By analogy with functions, the elements of Op_r may be called germs of operators. Op_r is the restriction to V_r of the set L_r of left multipliers [5, Section 6.2.3]:

$$L_r := \{A \in \mathrm{Op}(V_I) : \exists w \text{ such that } A_{wr} \text{ exists}\} = \{A \in \mathrm{Op}(V_I) : r \in \mathsf{d}(A)\}. \tag{4.2}$$

Then we have:

(i) When $V_q \subseteq V_p$, define $\rho_q^p : \mathrm{Op}_p \to \mathrm{Op}_q$ by $\rho_q^p(A_{*p}) := A_{*q}$, for $A_{*p} \in \mathrm{Op}_p$. Clearly, $\rho_p^p = id_{\mathrm{Op}_p}$ and $\rho_r^q \circ \rho_q^p = \rho_r^p$ if $V_r \subset V_q \subset V_p$. Hence, F is a presheaf.

(ii) (S_1) is clearly satisfied. As for (S_2), if $A_{*r} \in \mathrm{Op}_r$ and $A'_{*s} \in \mathrm{Op}_s$ are such that $\rho_{r\wedge s}^r(A_{*r}) = \rho_{r\wedge s}^s(A'_{*s})$; that is, $A_{*r\wedge s} = A'_{*r\wedge s}$, then these two operators are the $(r \wedge s,\ *)$ representative of a unique operator $A \in \mathrm{Op}(V_I)$. It remains to prove that A extends to $V_{r\vee s}$ and that its representative $A_{*r\vee s}$ extends both A_{*r} and A'_{*s}.

Proposition 4.5. *Let the indexed* PIP*-space* V_I *be additive; that is,* $V_{r\vee s} = V_r + V_s$*, for all* $r, s \in I$*. Then the map* F *given in* (4.1) *is a sheaf of operators on* V_I*.*

Proof. By linearity, A_{*r} and A_{*s} can be extended to an operator $A^{(r\vee s)}$ on $V_r + V_s$ as follows:

$$A^{(r\vee s)}(f_r + f_s) := A_{*r} f_r + A_{*s} f_s. \tag{4.3}$$

This operator is well defined. Let indeed $f = f_r + f_s = f'_r + f'_s$ be two decompositions of $f \in V_r + V_s$, so that $f_r - f'_r = f'_s - f_s \in V_r \cap V_s$. Then $A^{(r\vee s)}f = A_{*r}f_r + A_{*s}f_s = A_{*r}f'_r + A_{*s}f'_s$. Hence, $A_{*r}(f_r - f'_r) = A_{*s}(f'_s - f_s)$. Taking the restriction to $V_r \cap V_s = V_{r\wedge s}$, this relation becomes $A_{*r\wedge s}(f - f')_{r\wedge s} = A_{*r\wedge s}(f' - f)_{r\wedge s} = 0$, so that the vector $A^{(r\vee s)}f$ is uniquely defined.

Next, by additivity, $V_r + V_s$, with its inductive topology, coincides with $V_{r\vee s}$, and, thus, $A_{*r\vee s}$ is the $(r \vee s,\ *)$-representative of the operator $A \in \mathrm{Op}(V_I)$. Therefore, F is a sheaf. □

We recall that the most interesting classes of indexed PIP-spaces are additive, namely, the projective ones and, in particular, LBSs and LHSs. Thus the proposition just proven has a widely applicable range.

5. Cosheaves of Operators on PIP-Spaces

5.1. Pre-Cosheaves and Cosheaves

Pre-cosheaves and cosheaves are the dual notions of presheaves and sheaves, respectively. Let again X be a topological space, with closed sets W_i so that $X = \bigcup_{i\in I} W_i$, and let $\mathbf{C}$ be a (concrete) category.

Definition 5.1. A *pre-cosheaf* G on X with values in $\mathbf{C}$ is a map G defined on the family of closed subsets of X such that

(PC$_1$) for each closed set W of X, there is an object $\mathsf{G}(W)$ in $\mathbf{C}$;

(PC$_2$) for each inclusion of closed sets $Z \supseteq W$, there is given a *extension morphism* δ^Z_W : $\mathsf{G}(W) \to \mathsf{G}(Z)$ in the category $\mathbf{C}$, such that δ^W_W is the identity for every closed set W and $\delta^T_Z \circ \delta^Z_W = \delta^T_W$ whenever $T \supseteq Z \supseteq W$.

Definition 5.2. Let G be a pre-cosheaf on the topological space X. We say that G is a *cosheaf* if, for every nonempty closed set $W = \bigcap_{j\in J} W_j,\ J \subseteq I$ and for every family of (local) sections $\{t_j \in \mathsf{G}(W_j)\}_{j\in J}$, the following conditions are fulfilled:

(CS$_1$) given $t, t' \in \mathsf{G}(W)$ such that $\delta^{W_j}_W(t) = \delta^{W_j}_W(t')$, for every $j \in I$ such that $W \subseteq W_j$, then $t = t'$;

(CS$_2$) if $\delta^{W_j\cup W_k}_{W_j}(t_j) = \delta^{W_j\cup W_k}_{W_k}(t_k)$, for every $j, k \in J$, then there exists a unique section $t \in \mathsf{G}(W)$ such that $\delta^{W_j}_W(s) = t_j$, for every $j \in J$.

The cosheaf G may be seen as a covariant functor from the category of closed sets of X into $\mathbf{C} := (\{\mathsf{G}(W_i)\}, \{\delta^{W_i}_{W_j}\})$.

Now, there are situations where extensions do not exist for all inclusions $Z \supseteq W$ but only for certain pairs. This will be the case for operators on a PIP-space, as will be seen below. Thus, we may generalize the notions of (pre-) cosheaf as follows (there could be several variants).

Definition 5.3. Let X be a topological space, let $\mathcal{C}l(X)$ be the family of closed subsets of X, and let $\preccurlyeq$ a be coarsening of the set inclusion in $\mathcal{C}l(X)$; that is, $W \preccurlyeq Z$ implies $W \subseteq Z$, but not necessarily the opposite. A *partial pre-cosheaf* G with values in $\mathbf{C}$ is a map G defined on $\mathcal{C}l(X)$, satisfying conditions (PC$_1$) and (pPC$_2$) if $Z \succcurlyeq W$, there is given an *extension morphism* $\delta_W^Z : \mathsf{G}(W) \to \mathsf{G}(Z)$ in the category $\mathbf{C}$, such that δ_W^W is the identity for every closed set W and $\delta_Z^T \circ \delta_W^Z = \delta_W^T$ whenever $T \succcurlyeq Z \succcurlyeq W$.

We may use the term "partial," since here not all pairs $w \leqslant z$ admit extensions, but only certain pairs, namely, those that satisfy $w \preccurlyeq z$. This is analogous to the familiar situation of a compatibility.

Definition 5.4. Let G be a partial pre-cosheaf on the topological space X. We say that G is a *partial cosheaf* if, for every nonempty closed set $W = \bigcap_{j\in J} W_j$, $J \subseteq I$ and for every family of sections $\{t_j \in \mathsf{G}(W_j)\}_{j\in J}$, the following conditions are fulfilled:

(pCS$_1$) given $t, t' \in \mathsf{G}(W)$ such that $\delta_W^{W_j}(t)$ and $\delta_W^{W_j}(t')$ exist and are equal, for every $j \in J$, then $t = t'$;

(pCS$_2$) if $\delta_{W_j}^{W_j\cup W_k}(t_j)$ and $\delta_{W_k}^{W_j\cup W_k}(t_k)$ exist and are equal, for every $j, k \in J$, then $\delta_W^{W_j}$ exists for every $j \in J$ and there exists a unique section $t \in \mathsf{G}(W)$ such that $\delta_W^{W_j}(t) = t_j$, for every $j \in J$.

5.2. Cosheaves of Operators on Indexed PIP*-Spaces*

Let again $V_I = \{V_r,\ r \in I\}$ be an indexed PIP-space. If we put on I the discrete topology, then the assaying subspaces may be taken as closed sets in V. In order to build a cosheaf, we consider the same map as before in (4.1):

$$\mathsf{F} : V_r \longmapsto \mathrm{Op}_r := \{A_{wr} : V_r \longrightarrow V_w,\ \text{for some } w \in I,\ A \in \mathrm{Op}(V_I)\}. \tag{5.1}$$

Then we immediately face a problem. Given V_q, there is no V_p, with $V_q \subseteq V_p$, such that $\delta_q^p : \mathrm{Op}_p \to \mathrm{Op}_q$ exists. One can always find an operator A such that $q \in \mathsf{d}(A)$ and $p \notin \mathsf{d}(A)$; in other words, A_{*q} exists, but it cannot be extended to V_p. This happens, for instance, each time $\mathsf{d}(A)$ has a maximal element (see Figure 3.1 in [5]). There are three ways out of this situation.

5.2.1. General Operators: The Additive Case

We still consider the set of all operators $\mathrm{Op}(V_I)$. Take two assaying subspaces V_r, V_s and two operators $A^{(r)} \in \mathrm{Op}_r$, $B^{(s)} \in \mathrm{Op}_s$, and assume that they have a common extension $C^{(r\vee s)}$ to $\mathrm{Op}_{r\vee s}$. This means that, for any suitable w, $C^{(r\vee s)} : V_{r\vee s} \to V_w$ is the $(r\vee s, w)$-representative $C_{w,r\vee s}$ of a unique operator $C \in \mathrm{Op}(V_I)$, and $C = A = B$. *A fortiori*, A, and B coincide on $V_{r\wedge s} = V_r \cap V_s$; that is, condition (pCS$_2$) is satisfied.

Assume now that V_I is additive; that is, $V_{r\vee s}$ coincides with $V_r + V_s$, with its inductive topology. Then we can proceed as in the case of a sheaf, by defining the extension C of A and B by linearity. In other words, since $C^{(r\vee s)} f_r = A_{*r} f_r$ and $C^{(r\vee s)} f_s = B_{*s} f_s$, we may write, for any $f = f_r + f_s \in V_r + V_s$,

$$C^{(r\vee s)} f = C_{*r\vee s} f_{r\vee s} = A_{*r} f_r + B_{*s} f_s. \tag{5.2}$$

Here too, this operator is well defined. Let indeed $f = f_r + f_s = f'_r + f'_s$ be two decompositions of $f \in V_r + V_s$, so that $f_r - f'_r = f'_s - f_s \in V_r \cap V_s$. Then necessarily $C^{(r \vee s)} f = A_{*r} f_r + B_{*s} f_s = A_{*r} f'_r + B_{*s} f'_s$. Hence, $A_{*r}(f_r - f'_r) = B_{*s}(f'_s - f_s)$. Taking the restriction to $V_r \cap V_s = V_{r \wedge s}$, this relation becomes $A_{*r\wedge s}(f - f')_{r\wedge s} = B_{*r\wedge s}(f' - f)_{r\wedge s}$, and this equals 0, since $A = B$ on $V_{r\wedge s}$, by the condition (pcS$_2$).

The conclusion is that, for any pair of assaying subspaces V_r, V_s, the extension maps $\delta_r^{r\vee s}$ and $\delta_s^{r\vee s}$ always exist, and the condition (pCS$_2$) is satisfied for that pair. However, this is not necessarily true for any comparable pair, and this motivates the coarsening of the order given in Proposition 5.5 below. Condition (pPC$_2$) is also satisfied, as one can see by taking supremums (= sums) of successive pairs within three assaying subspaces V_r, V_s, V_t (and using the associativity of $\vee$). Thus, me may state the following.

Proposition 5.5. *Let $V_I = \{V_r,\ r \in I\}$ be an additive indexed* PIP*-space. Then the map* F *given in* (4.1) *is a partial cosheaf with respect to the partial order on I:*

$$r \preccurlyeq s \Longleftrightarrow \exists w \in I \quad \text{such that } s = r \vee w. \tag{5.3}$$

5.2.2. Universal Left Multipliers

We consider only the operators A which are everywhere defined; that is, $\mathsf{d}(A) = I$. This is precisely the set of universal left multipliers:

$$\begin{aligned} L\mathrm{Op}(V_I) &= \left\{C \in \mathrm{Op}(V_I) : \forall r, \exists w \text{ such that } C_{wr} \text{ exists}\right\} \\ &= \left\{C \in \mathrm{Op}(V_I) : \mathsf{d}(C) = I\right\} = \bigcap_{r\in I} L_r, \end{aligned} \tag{5.4}$$

where $L_r := \{C \in \mathrm{Op}(V_I) : r \in \mathsf{d}(C)\}$. Correspondingly, we consider the map

$$\mathsf{G} : V_r \longmapsto L\mathrm{Op}_r := \left\{A \upharpoonright V_r : A \in L\mathrm{Op}(V_I)\right\}. \tag{5.5}$$

Here again, elements of $L\mathrm{Op}_r$ are representatives A_{*r}. In that case, extensions always exist. When $V_q \subseteq V_p$, define $\delta_q^p : L\mathrm{Op}_q \to L\mathrm{Op}_p$ by $\delta_q^p(A_{*q}) := A_{*p}$ for $A_{*q} \in L\mathrm{Op}_q$. Clearly, $\delta_p^p = id_{L\mathrm{Op}_p}$ and $\delta_q^p \circ \delta_r^q = \delta_r^p$ if $V_r \subset V_q \subset V_p$. Hence, G is a pre-cosheaf. Moreover, if $\delta_q^{p\vee q}(A_{*q}) = \delta_p^{p\vee q}(A'_{*p})$; that is, $A_{*,p\vee q} = A'_{*,p\vee q}$, these two operators are the $(p \vee q, *)$ representative of a unique operator $A \in L\mathrm{Op}(V_I)$. Then, of course, one has $\delta_{p\wedge q}^q(A_{\wp\wedge q}) = A_{*q}$ and $\delta_{p\wedge q}^p(A_{*p\wedge q}) = A_{*p}$; thus, G is a cosheaf. Therefore,

Proposition 5.6. *Let $V_I = \{V_r,\ r \in I\}$ be an arbitrary indexed* PIP*-space. Then the map given in* (5.5) *is a cosheaf on V_I with values in $L\mathrm{Op}(V_I)$, the set of universal left multipliers, with extensions given by $\delta_q^p(A_{*q}) = A_{*p}, q \leqslant p, A_{*q} \in L\mathrm{Op}_q$.*

5.2.3. General Operators: The Projective Case

We consider again all operators in $\mathrm{Op}(V_I)$. The fact is that the set $\bigcup_{r\in I} \mathrm{Op}_r$ is an initial set in $\mathrm{Op}(V_I)$, like every $\mathsf{d}(A)$; that is, it contains all predecessors of any of its elements, and this is natural for constructing sheaves into it. But it is *not* when considering cosheaves. By duality, one should rather take a final set, that is, containing all successors of any of its elements, just like any $\mathsf{i}(A)$. Hence, we define the map:

$$\widehat{\mathsf{G}} : V_r \longmapsto \widehat{\mathrm{Op}}_r := \{A_{r*} : A \in \mathrm{Op}(V_I),\ r \in \mathsf{i}(A)\}. \tag{5.6}$$

Now indeed $\bigvee_{r\in I}\widehat{\mathrm{Op}}_r$ is a final set. Elements of $\widehat{\mathrm{Op}}_r$ have been denoted as A_{r*}, but, for definiteness, we could also replace A_{r*} by $A_{r\overline{\infty}}$ where $V_{\overline{\infty}} := V^{\#}$.

We claim that the map $\widehat{\mathsf{G}}$ defines a cosheaf, with extensions $\delta^p_q(A_{q*}) := A_{p*}$, for $A_{q*} \in \widehat{\mathrm{Op}}_q$, which exist whenever $q \leqslant p$.

As in the case of Section 5.2.1, assume that $A_{r*} \in \widehat{\mathrm{Op}}_r$, $B_{s*} \in \widehat{\mathrm{Op}}_s$ have a common extension belonging to $\widehat{\mathrm{Op}}_{r\vee s}$; that is, $A_{r\vee s*} := E_{r\vee s,r}A_{r*} = B_{r\vee s*} := E_{r\vee s,s}B_{s*}$. Call this operator $C_{r\vee s*}$. Thus, for any suitable w, $C_{r\vee s,w}$ is the $(w, r\vee s)$ representative of a unique operator $C \in \mathrm{Op}(V_I)$, which extends A and B. Thus $C = A = B \in \mathrm{Op}(V_I)$. Assume now that V_I is projective; that is, $V_{r\wedge s} = V_r \cap V_s$, for all $r, s \in I$, with the projective topology. Then, $C_{r\wedge s*} = A_{r\wedge s*} = B_{r\wedge s*}$; that is, condition (CS$_2$) is satisfied with extensions $\delta^r_{r\wedge s}(C_{r\wedge s*}) = A_{r*}$ and $\delta^s_{r\wedge s}(C_{r\wedge s*}) = B_{s*}$. The rest is obvious. As in the case of sheaves, extensions from V_r, V_s to $V_{r\vee s}$ exist by linearity, since V_I is projective, hence, additive. Thus, we may state the following.

Proposition 5.7. *Let the indexed* PIP*-space V_I be projective that is, $V_{r\wedge s} = V_r \cap V_s$, for all $r, s \in I$, with the projective topology. Then the map $\widehat{\mathsf{G}}$ given in* (5.6) *is a cosheaf of operators on V_I.*

Corollary 5.8. *Let $V_I = \{V_r, r \in I\}$ be a projective indexed* PIP*-space. Then V_I generates in a natural fashion a sheaf and a cosheaf of operators. If V_I is additive, but not projective; it generates a sheaf and a partial cosheaf of operators.*

6. Cohomology of an Indexed PIP-Space

6.1. Cohomology of Operator Sheaves

It is possible to introduce a concept of sheaf cohomology group defined on an indexed PIP-space according to the usual definition of sheaf cohomology [10]. Let V_I be an indexed PIP-space. The set of its assaying spaces, endowed with the discrete topology, defines an open covering of V. Endowed with the Mackey topology, $\tau(V_r, V_{\overline{r}})$, each V_r is a Hausdorff vector subspace of V.

Definition 6.1. Given the indexed PIP-space V_I, let $\{V_j,\ j \in I\}$ be the open covering of V by its assaying subspaces, and let $\mathsf{F} : V_r \mapsto \mathrm{Op}_r = \{A \upharpoonright V_r : A \in \mathrm{Op}(V_I),\ r \in \mathsf{d}(A)\}$ be the sheaf of operators defined in (4.1) above. We call *p-cochain with values in* F the map which associates, to each intersection of open sets $V_{j_0} \cap V_{j_1} \cap \cdots \cap V_{j_p}$, $j_k \in I$, $j_0 < j_1 < \cdots < j_p$, an operator $A_{j_0,j_1,\dots,j_p}$ belonging to $\mathsf{F}(V_{j_0} \cap V_{j_1} \cap \cdots \cap V_{j_p})$.

Thus, a p-cochain with value in F is a set $\mathcal{A} = \{A_{j_0,j_1,\dots,j_p},\ j_0 < j_1 < \cdots < j_p\}$. The set of p-cochains is denoted by $C^p(I, \mathsf{F})$. For example, a 0-cochain is a set $\{A_j,\ j \in I\}$. A 1-cochain is a set $\{A_{j_0,j_1},\ j_0 < j_1\}$, $A_{j_0,j_1} \in \mathsf{F}(V_{j_0} \cap V_{j_1}), \dots$.

Definition 6.2. Given the notion of p-cochain, the corresponding *coboundary operator D* is defined by

$$
\begin{gathered}
D : C^{p}(I,\mathsf{F}) \longrightarrow C^{p+1}(I,\mathsf{F}) : \left\{A_{j_0,j_1,\dots,j_p}\right\} \longmapsto \left\{(D\mathcal{A})_{j_0,j_1,\dots,j_p,j_{p+1}}\right\}, \\
(D\mathcal{A})_{j_0,j_1,\dots,j_p,j_{p+1}} = \sum_{k=0}^{p+1}(-1)^{k}\rho_{V_{j_0}\cap V_{j_1}\cap\cdots\cap V_{j_{p+1}}}^{V_{j_0}\cap V_{j_1}\cap\cdots\cap\widehat{V}_{j_k}\cap\cdots\cap V_{j_{p+1}}} A_{j_0,j_1,\dots,\widehat{j}_k,\dots,j_{p+1}},
\end{gathered} \tag{6.1}
$$

where the hat corresponds to the omission of the corresponding symbol. One checks that D is a coboundary operator; that is, $DD = 0$.

If we compute the action of the coboundary operator on 0-cochains, we get

$$
(D\mathcal{A})_{j_0,j_1} = \rho_{V_{j_0}\cap V_{j_1}}^{V_{j_1}} A_{j_1} - \rho_{V_{j_0}\cap V_{j_1}}^{V_{j_0}} A_{j_0}. \tag{6.2}
$$

This equation is nothing but the condition (S_1) in Definition 4.4, that is, the necessary condition for getting a sheaf.

We can also compute the action of the coboundary operator on 1-cochains. We get

$$
(D\mathcal{A})_{j_0,j_1,j_2} = \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_1}\cap V_{j_2}} A_{j_1,j_2} - \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_0}\cap V_{j_2}} A_{j_0,j_2} + \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_0}\cap V_{j_1}} A_{j_0,j_1}. \tag{6.3}
$$

Now, let us suppose that a 1-cochain is defined by $\mathcal{A} = D\mathcal{B}$, where $\mathcal{B}$ is a 0-cochain on V. The previous formula becomes:

$$
\begin{aligned}
(D\mathcal{A})_{j_0,j_1,j_2} = {} & \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_1}\cap V_{j_2}}\left(\rho_{V_{j_1}\cap V_{j_2}}^{V_{j_2}} B_{j_2} - \rho_{V_{j_1}\cap V_{j_2}}^{V_{j_1}} B_{j_1}\right) - \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_0}\cap V_{j_2}}\left(\rho_{V_{j_0}\cap V_{j_2}}^{V_{j_2}} B_{j_2} - \rho_{V_{j_0}\cap V_{j_2}}^{V_{j_0}} B_{j_0}\right) \\
& + \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_0}\cap V_{j_1}}\left(\rho_{V_{j_0}\cap V_{j_1}}^{V_{j_1}} B_{j_1} - \rho_{V_{j_0}\cap V_{j_1}}^{V_{j_0}} B_{j_0}\right).
\end{aligned} \tag{6.4}
$$

Using the properties of restrictions we get

$$
\begin{aligned}
(D\mathcal{A})_{j_0,j_1,j_2} = {} & \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_2}} B_{j_2} - \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_1}} B_{j_1} - \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_2}} B_{j_2} \\
& + \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_0}} B_{j_0} + \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_1}} B_{j_1} - \rho_{V_{j_0}\cap V_{j_1}\cap V_{j_2}}^{V_{j_0}} B_{j_0} = 0,
\end{aligned} \tag{6.5}
$$

which shows that indeed $DD\mathcal{B} = 0$.

Now it is possible to define cohomology groups of the sheaf F on a indexed PIP-space V.

Definition 6.3. Let V_I be an indexed PIP-space, endowed with the open covering of its assaying spaces, $\{V_j,\ j \in I\}$, and let F be the corresponding sheaf of operators on V_I. Then the pth cohomology group is defined as $H^p(I,\mathsf{F}) := Z^p(I,\mathsf{F})/B^p(I,\mathsf{F})$, where $Z^p(I,\mathsf{F})$ is the set of p-cocycles, that is, p-cochains $\mathcal{A}$ such that $D\mathcal{A} = 0$, and $B^p(I,\mathsf{F})$ is the set of p-coboundaries, that is, p-cochains $\mathcal{B}$ for which there exists a $(p-1)$-cochain $\mathcal{C}$ with $\mathcal{B} = D\mathcal{C}$.

The definition of the group $H^p(I, \mathsf{F})$ is motivated by the fact that the p-cocycle $\mathcal{A}$ is given only up to a coboundary $\mathcal{B} = D\mathcal{C} : D\mathcal{A} = 0 = D(\mathcal{A} + \mathcal{B}) = D(\mathcal{A} + D\mathcal{C})$.

Our definition of cohomology groups of sheaves on indexed PIP-spaces depends on the particular open covering we are choosing. So far we have used the open covering given by all assaying subspaces, but there might be other ones, typically consisting of unions of assaying subspaces. We say that an open covering $\{V_j,\ j \in J\}$ of an indexed PIP-space V_I is finer than another one, $\{U_k,\ k \in K\}$, if there exists an application $t : J \to K$ such that $V_j \subset U_{t(j)}$, for all $j \in J$. For instance, $U_{t(j)}$ could be a union $\cup_l V_l$ containing V_j. According to the general theory of sheaf cohomology [9], this induces group homomorphisms $t_J^K : H^p(K, \mathsf{F}) \to H^p(J, \mathsf{F})$, for all $p \geqslant 0$. It is then possible to introduce cohomology groups that do not depend on the particular open covering, namely, by defining $H^p(V, \mathsf{F})$ as the inductive limit of the groups $H^p(J, \mathsf{F})$ with respect to the homomorphisms t_J^K.

Using a famous result of Cartan and Leray and an unpublished work of J. Shabani, we give now a theorem which characterizes the cohomology of sheaves on indexed PIP-spaces. We collect in Appendix B the definitions and the results needed for this discussion. We know that an indexed PIP-space V endowed with the Mackey topology is a separated (Hausdorff) locally convex space, but it is not necessarily paracompact, unless V is metrizable, in particular, a Banach or a Hilbert space. In that case, it is possible to define a fine sheaf (see Definition B.3) of operators on V and apply the Cartan-Leray Theorem B.4. This justifies the restriction of metrizability in the following theorem.

Theorem 6.4. *Given an indexed* PIP*-space V_I, define the sheaf $\mathsf{F} : V_r \to F(V_r) := \mathrm{Op}_r = \{A_r : V_r \to V\}$. Then, if V is metrizable for its Mackey topology, the sheaf F is acyclic; that is, the cohomology groups $H^p(V, \mathsf{F})$ are trivial for all $p \geqslant 1$; that is, $H^p(V, \mathsf{F}) = 0$.*

Proof. First of all, V is a paracompact space, since it is metrizable.

Next, we check that the sheaf $\mathsf{F} : V_r \to \mathsf{F}(V_r) = \mathrm{Op}_r = \{A_r : V_r \to V\}$ on the paracompact space V is fine. Indeed, let $\{V_j,\ j \in J\}$ be a locally finite open covering of V. We can associate to the latter a partition of unity $\{\varphi_j,\ j \in J\}$. Then we can use $\{\varphi_j,\ j \in J\}$ to define the homomorphisms $h_j : \mathsf{F} \to \mathsf{F}$. Let $V_r \subset \cup_{s \in K} V_s$ for some index set K. For each $s \in K$, we consider the set of operators $\mathrm{Op}_s = \{A_{*s} : V_s \to V\}$, and we define $h_j : A_s \mapsto h_j(A_s) := \varphi_j A_s$. This defines a homomorphism $\mathrm{Op}_s \to \mathrm{Op}_s$ and then a homomorphism $\mathsf{F} \to \mathsf{F}$. One can check that $h_j : \mathsf{F} \to \mathsf{F}$ satisfies conditions (1) and (2) in Definition B.3, and, thus, F is a fine sheaf.

Then the result follows from Theorem B.4. □

The crucial point of the proof here is the Cartan-Leray theorem. Let us give a flavor of the proof of a particular case of this classical result. Let $[\mathcal{C}]$ be an element of $H^1(V, \mathsf{F})$. We want to show that $[\mathcal{C}] = 0$. As an equivalence class, $[\mathcal{C}]$ can be represented by an element $\mathcal{C} = (C_{j_0,j_1},\ j_0 < j_1) \in I$ of $C^1(I, \mathsf{F})$, such that $D\mathcal{C} = 0$; that is,

$$(D\mathcal{C})_{j_0,j_1,j_2} = \rho^{V_{j_1} \cap V_{j_2}}_{V_{j_0} \cap V_{j_1} \cap V_{j_2}} C_{j_1,j_2} - \rho^{V_{j_0} \cap V_{j_2}}_{V_{j_0} \cap V_{j_1} \cap V_{j_2}} C_{j_0,j_2} + \rho^{V_{j_0} \cap V_{j_1}}_{V_{j_0} \cap V_{j_1} \cap V_{j_2}} C_{j_0,j_1} = 0. \tag{6.6}$$

Using the fact that F is a fine sheaf, we can choose the partition of unity $\{\varphi_i,\ i \in I\}$ to define the 0-cochain $\mathcal{E} = \{E_j\}$, where $E_j = -\sum_{k \in I} \varphi_k(C_{jk})$. This sum is well defined since the covering is locally finite. Applying the coboundary operator, we get

$$(D\mathcal{E})_{j_0,j_1} = \rho^{V_{j_1}}_{V_{j_0}\cap V_{j_1}} E_{j_1} - \rho^{V_{j_0}}_{V_{j_0}\cap V_{j_1}} E_{j_0} = -\sum_{k\in I}\varphi_k\left(\rho^{V_{j_1}}_{V_{j_0}\cap V_{j_1}} C_{j_1 k}\right) + \sum_{k\in I}\varphi_k\left(\rho^{V_{j_0}}_{V_{j_0}\cap V_{j_1}} C_{j_0 k}\right)$$
$$= \sum_{k\in I}\varphi_k\left(\rho^{V_{j_0}}_{V_{j_0}\cap V_{j_1}} C_{j_0 k} - \rho^{V_{j_1}}_{V_{j_0}\cap V_{j_1}} C_{j_1 k}\right) \tag{6.7}$$
$$= \sum_{k\in I}\varphi_k\left(\rho^{V_{j_0}\cap V_k}_{V_{j_0}\cap V_{j_1}\cap V_k} C_{j_0 k} - \rho^{V_{j_1}\cap V_k}_{V_{j_0}\cap V_{j_1}\cap V_k} C_{j_1 k}\right).$$

Putting $k = j_2$ and using the equation $(D\mathcal{C})_{j_0,j_1,j_2} = 0$, we find

$$(D\mathcal{E})_{j_0,j_1} = \sum_{j_2\in I}\varphi_{j_2}\left(\rho^{V_{j_0}\cap V_{j_1}}_{V_{j_0}\cap V_{j_1}\cap V_{j_2}} C_{j_0 j_1}\right). \tag{6.8}$$

But since $\{\varphi_i,\ i\in I\}$ is a partition of unity, we get $(D\mathcal{E})_{j_0,j_1} = C_{j_0 j_1}$. This means that $D\mathcal{E} = \mathcal{C}$, and; thus, $[\mathcal{C}] = 0$.

Of course, the restriction that V is to be metrizable is quite strong, but still the result applies to a significant number of interesting situations, for instance,

(1) a *finite* chain of reflexive Banach spaces or Hilbert spaces, for instance, a triplet of Hilbert spaces $\mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \hookrightarrow \mathcal{H}_{\overline{1}}$ or any of its refinements, as discussed in [5, Section 5.2.2],

(2) an indexed PIP-space V_I whose extreme space V is itself a Hilbert space, like the LHS of functions analytic in a sector described in [5, Section 4.6.3],

(3) a Banach Gel'fand triple, in the sense of Feichtinger (see [11]), that is, an RHS (or LBS) in which the extreme spaces are (nonreflexive) Banach spaces. A nice example, extremely useful in Gabor analysis, is the so-called *Feichtinger algebra* $\mathcal{S}_0(\mathbb{R}^d)$, which generates the triplet

$$\mathcal{S}_0\left(\mathbb{R}^d\right) \hookrightarrow L^2\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{S}_0^{\times}\left(\mathbb{R}^d\right). \tag{6.9}$$

The latter can often replace the familiar Schwartz triplet of tempered distributions. Of course, one can design all sorts of LHSs of LBSs interpolating between the extreme spaces, as explained in [5, Sections 5.3 and 5.4].

In fact, what is really needed for the Cartan-Leray theorem is not that V becomes metrizable, but that it becomes paracompact. Indeed, without that condition, the situation becomes totally unmanageable. However, except for metrizable spaces, we could not find interesting examples of indexed PIP-spaces with V paracompact.

6.2. Cohomology of Operator Cosheaves

It is also possible to get cohomological concepts on cosheaves defined from indexed PIP-spaces. The assaying spaces can also be considered as closed sets. Let, thus, $\{W_j,\ j\in I\}$ be such a closed covering of V, and let G be a cosheaf of operators defined in (5.5), namely,

$\mathsf{G} : W_j \mapsto LOp_j = \{A \upharpoonright W_j : A \in LOp(V_I)\}$ together with maps $\delta_{W_i}^{W_j} : \mathsf{G}(W_i) \to \mathsf{G}(W_j)$, $W_i \subset W_j$ as above. In the sequel, $\mathsf{G}(W_{j_0} \cup W_{j_1} \cup \cdots \cup W_{j_p})$ denotes the set of operators $W_{j_0} \cup W_{j_1} \cup \cdots \cup W_{j_p} \to V$. Alternatively, one could consider the cosheaf defined in (5.6), $\widehat{\mathsf{G}} : W_j \mapsto \widehat{\mathrm{Op}}_j := \{A_{j*} : A \in \mathrm{Op}(V_I), j \in \mathsf{i}(A)\}$ and proceed in the same way.

In this setup, we may introduce the necessary cohomological concepts, as in Definitions 6.1 and 6.2.

Definition 6.5. A *p-cochain* with values in the cosheaf G is a map which associates to each union of closed sets $W_{j_0} \cup W_{j_1} \cup \cdots \cup W_{j_p}$, $j_k \in I$, $j_0 < j_1 < \cdots < j_p$, an operator $A_{j_0,j_1,\dots,j_p}$ of $\mathsf{G}(W_{j_0} \cup W_{j_1} \cup \cdots \cup W_{j_p})$. A p-cochain is, thus, a set $\mathfrak{A} = \{A_{j_0,j_1,\dots,j_p},\ j_0 < j_1 < \cdots < j_p\}$. The set of such p-cochains will be denoted by $\widehat{C}^p(I, \mathsf{G})$.

Definition 6.6. One can then introduce the coboundary operator $\widehat{D}$ as follows:

$$
\begin{gathered}
\widehat{D} : \widehat{C}^p(I,\mathsf{G}) \longrightarrow \widehat{C}^{p+1}(I,\mathsf{G}) : \mathfrak{A} = \left\{A_{j_0,j_1,\dots,j_p}\right\} \longmapsto \left\{\left(\widehat{D}\mathfrak{A}\right)_{j_0,j_1,\dots,j_p,j_{p+1}}\right\}, \\
\left(\widehat{D}\mathfrak{A}\right)_{j_0,j_1,\dots,j_p,j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \delta_{W_{j_0} \cup W_{j_1} \cup \cdots \cup \widehat{W}_{j_k} \cup \cdots \cap W_{j_{p+1}}}^{W_{j_0} \cup W_{j_1} \cup \cdots \cup W_{j_{p+1}}} A_{j_0,j_1,\dots,\widehat{j}_k,\dots,j_{p+1}}.
\end{gathered}
\tag{6.10}
$$

In view of the properties of the maps δ_*^*, we check that $\widehat{D}\widehat{D} = 0$.

In the case of 0-cochains, a straightforward application of the formula leads to

$$
\left(\widehat{D}\mathfrak{A}\right)_{j_0,j_1} = \delta_{W_{j_1}}^{W_{j_0} \cup W_{j_1}} A_{j_1} - \delta_{W_{j_0}}^{W_{j_0} \cup W_{j_1}} A_{j_0}. \tag{6.11}
$$

And if we put this to zero, we get the constraint $\delta_{W_{j_1}}^{W_{j_0} \cup W_{j_1}} A_{j_1} = \delta_{W_{j_0}}^{W_{j_0} \cup W_{j_1}} A_{j_0}$, which has to be satisfied in order to build a cosheaf, according to the condition (CS_2). On 1-cochains, the coboundary action gives

$$
\left(\widehat{D}\mathfrak{A}\right)_{j_0,j_1,j_2} = \delta_{W_{j_1} \cup W_{j_2}}^{W_{j_0} \cup W_{j_1} \cup W_{j_2}} A_{j_1,j_2} - \delta_{W_{j_0} \cup W_{j_2}}^{W_{j_0} \cup W_{j_1} \cup W_{j_2}} A_{j_0,j_2} + \delta_{W_{j_0} \cup W_{j_1}}^{W_{j_0} \cup W_{j_1} \cup W_{j_2}} A_{j_0,j_1}. \tag{6.12}
$$

The cohomology groups of the cosheaf $\widehat{H^p}(I,\mathsf{G}) := \widehat{Z}^p(I,\mathsf{G})/\widehat{B}^p(I,\mathsf{G})$, with obvious notations, can then be defined in a natural way. Similarly for $\widehat{H^p}(V,\mathsf{G})$ and $\widehat{H^p}(I,\widehat{\mathsf{G}})$, $\widehat{H^p}(V,\widehat{\mathsf{G}})$.

Now it is tempting to proceed as in the case of sheaves and define the analog of a fine cosheaf, in such a way that one can apply a result similar to the Cartan-Leray theorem. But this is largely unexplored territory, so we will not venture into it.

7. Outcome

The analysis so far shows that several aspects of the theory of indexed PIP-spaces and operators on them have a natural formulation in categorical terms. Of course, this is only a first step, many questions remain open. For instance, does there exist a simple characterisation of the dual category $\mathbf{PIP}^{\mathrm{op}}$? Could it be somehow linked to the category of partial *algebras, in the same way as $\mathbf{Set}^{\mathrm{op}}$ is isomorphic to the category of complete atomic Boolean algebras (this is the so-called Lindenbaum-Tarski duality [12, Section VI.4.6]?

In addition, our constructions yield new concrete examples of sheaves and cosheaves, namely, (co)sheaves of operators on an indexed PIP-space, and this is probably the most important result of this paper. Then, another open question concerns the cosheaf cohomology groups. Can one find conditions under which the cosheaf is acyclic, that is, $\widehat{H}^p(V, \mathsf{G}) = 0$, for all $p \geqslant 1$, or, similarly, $\widehat{H}^p(V, \widehat{\mathsf{G}}) = 0$, for all $p \geqslant 1$?

In guise of conclusion, let us note that cosheaf is a new concept which was introduced in a logical framework in order to dualize the sheaf concept [13]. In fact one knows that the category of sheaves (which is in fact a topos) is related to Intuitionistic logic and Heyting algebras, in the same way as the category of sets has deep relations with the classical proposition logic and Boolean algebras [12, Section I.1.10].

More precisely, Classical logic satisfies the noncontradiction principle $\mathsf{NCP}\neg(p \wedge \neg p)$ and the excluded middle principle EMP $(p \vee \neg p)$ (i.e., NCP: = Not-(p And Not-p) and EMP: = p Or Not-p.). Intuitionistic logic satisfies NCP but not EMP. Finally, we know that Paraconsistent logic, satisfying EMP but not NCP, is related to Brouwer algebras, also called co-Heyting algebras [14, 15]. Then, it is natural to wonder what is the category, if any, (mimicking the category of sets for the classical case and the topos of sheaves for the intuitionistic case), corresponding to Paraconsistent logic? The category of cosheaves can be a natural candidate for this. And this is the reason why it was tentatively introduced in a formal logic context. The category of closed sets of a topological space happens to be a cosheaf. But up to now we did not know any other examples of cosheaves in other areas of mathematics. Therefore, it is interesting to find here additional concrete examples of cosheaves in the field of functional analysis.

In a completely different field, the search for a quantum gravity theory, Shahn Majid [16, page 294] (see also the diagram in [17, page 122]) has proposed to unify quantum field theory and general relativity using a self-duality principle expressed in categorical terms. His approach shows deep connections, on the one hand, between quantum concepts and Heyting algebras (the relations between quantum physics and Intuitionistic logic are well known) and, on the other hand, following a suggestion of Lawvere [16], between general relativity (Riemannian geometry and uniform spaces) and co-Heyting algebras (Brouwer algebras). Therefore, it is very interesting to shed light on concepts arising naturally from Brouwer algebras, and this is precisely the case of cosheaves. Sheaves of operators on PIP-spaces are connected to quantum physics. Is there any hope to connect cosheaves of operators on some PIP-spaces to (pseudo-)Riemannian geometry, uniform spaces, or to theories describing gravitation? This is an open question suggested by Majid's idea of a self-duality principle (let us note that classical logic is the prototype of a self-dual structure, self-duality being given by the de Morgan rule, which transforms NCP into EMP!).

Appendices

A. Partial Inner Product Spaces

A.1. PIP*-Spaces and Indexed* PIP*-Spaces*

For the convenience of the reader, we have collected here the main features of partial inner product spaces and operators on them, keeping only what is needed for reading the paper. Further information may be found in our review paper [6] or our monograph [5].

The general framework is that of a PIP-space V, corresponding to the linear compatibility $\#$, that is, a symmetric binary relation $f\#g$ which preserves linearity. We call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$, and we denote by $\mathcal{F}(V,\#)$ the family of all assaying subspaces of V, ordered by inclusion. The assaying subspaces are denoted by V_r, $V_q, \ldots$, and the index set is F. By definition, $q \leqslant r$ if and only if $V_q \subseteq V_r$. Thus, we may write

$$f\#g \Longleftrightarrow \exists r \in F \quad \text{such that } f \in V_r,\ g \in V_{\overline{r}}. \tag{A.1}$$

General considerations imply that the family $\mathcal{F}(V,\#) := \{V_r,\ r \in F\}$, ordered by inclusion, is a complete involutive lattice; that is, it is stable under the following operations, arbitrarily iterated:

(i) involution: $V_r \leftrightarrow V_{\overline{r}} = (V_r)^{\#}$,

(ii) infimum: $V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q, (p,q,r \in F)$,

(iii) supremum: $V_{p\vee q} := V_p \vee V_q = (V_p + V_q)^{\#\#}$.

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$, and the greatest element is $V = \bigcup_r V_r$.

By definition, the index set F is also a complete involutive lattice; for instance,

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}. \tag{A.2}$$

Given a vector space V equipped with a linear compatibility $\#$, a *partial inner product* on $(V,\#)$ is a Hermitian form $\langle \cdot \mid \cdot \rangle$ defined exactly on compatible pairs of vectors. A *partial inner product space* (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.

From now on, we will assume that our PIP-space $(V,\#,\langle \cdot \mid \cdot \rangle)$ is *nondegenerate*; that is, $\langle f \mid g \rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in F$, are a dual pair in the sense of topological vector spaces [18]. Next, we assume that every V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$, so that its conjugate dual is $(V_r)^{\times} = V_{\overline{r}}$, for all $r \in F$. Then, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

As a matter of fact, the whole structure can be reconstructed from a fairly small subset of $\mathcal{F}$, namely, a *generating* involutive sublattice $\mathcal{I}$ of $\mathcal{F}(V,\#)$, indexed by I, which means that

$$f\#g \Longleftrightarrow \exists r \in I \quad \text{such that } f \in V_r,\ g \in V_{\overline{r}}. \tag{A.3}$$

The resulting structure is called an *indexed* PIP-*space* and denoted simply by $V_I := (V, \mathcal{I}, \langle \cdot \mid \cdot \rangle)$.

Then an indexed PIP-space V_I is said to be

(i) *additive* if $V_{p\vee q} = V_p + V_q$, for all $p,q \in I$,

(ii) *projective* if $V_{p\wedge q}|_{\tau} \simeq (V_p \cap V_q)_{\text{proj}}$, for all $p,q \in I$; here $V_{p\wedge q}|_{\tau}$ denotes $V_{p\wedge q}$ equipped with the Mackey topology $\tau(V_{p\wedge q}, V_{\overline{p}\vee\overline{q}})$, the right-hand side denotes $V_p \cap V_q$ with the topology of the projective limit from V_p and V_q, and $\simeq$ denotes an isomorphism of locally convex topological spaces.

For practical applications, it is essentially sufficient to restrict oneself to the case of an indexed PIP-space satisfying the following conditions:

(i) every $V_r,\ r \in I$, is a Hilbert space or a reflexive Banach space, so that the Mackey topology $\tau(V_r, V_{\bar{r}})$ coincides with the norm topology;

(ii) there is a unique self-dual, Hilbert, assaying subspace $V_o = V_{\bar{o}}$.

In that case, the structure $V_I := (V, \mathcal{I}, \langle \cdot \mid \cdot \rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS) (see [5] for more precise definitions, including explicit requirements on norms). The important facts here are that

(i) every projective indexed PIP-space is additive;

(ii) an LBS or an LHS is projective if and only if it is additive.

Note that $V^{\#}, V$ themselves usually do *not* belong to the family $\{V_r,\ r \in I\}$, but they can be recovered as

$$V^{\#} = \bigcap_{r \in I} V_r, \quad V = \sum_{r \in I} V_r. \tag{A.4}$$

A standard, albeit trivial, example is that of a Rigged Hilbert space (RHS) $\Phi \subset \mathcal{H} \subset \Phi^{\#}$ (it is trivial because the lattice $\mathcal{F}$ contains only three elements). One should note that the construction of an RHS from a directed family of Hilbert spaces, via projective and inductive limits, has been investigated recently by Bellomonte and Trapani [19]. Similar constructions, in the language of categories, may be found in the work of Mityagin and Shvarts [20] and that of Semadeni and Zidenberg [21].

Let us give some concrete examples.

(i) Sequence Spaces

Let V be the space ω of *all* complex sequences $x = (x_n)$ and define on it (i) a compatibility relation by $x \# y \Leftrightarrow \sum_{n=1}^{\infty} |x_n y_n| < \infty$, (ii) a partial inner product $\langle x \mid y \rangle = \sum_{n=1}^{\infty} \overline{x_n} y_n$. Then $\omega^{\#} = \varphi$, the space of finite sequences, and the complete lattice $\mathcal{F}(\omega, \#)$ consists of Köthe's perfect sequence spaces [18, § 30]. Among these, a nice example is the lattice of the so-called ℓ_{ϕ} spaces associated to symmetric norming functions or, more generally, the Banach sequence ideals discussed in [5, Section 4.3.2] and previously in [20, § 6] (in this example, the extreme spaces are, resp., ℓ^1 and ℓ^{∞}).

(ii) Spaces of Locally Integrable Functions

Let V be $L^1_{\text{loc}}(\mathbb{R},\ dx)$, the space of Lebesgue measurable functions, integrable over compact subsets, and define a compatibility relation on it by $f \# g \Leftrightarrow \int_{\mathbb{R}} |f(x) g(x)|\, dx < \infty$ and a partial inner product $\langle f \mid g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) dx$. Then $V^{\#} = L^{\infty}_c(\mathbb{R},\ dx)$, the space of bounded measurable functions of compact support. The complete lattice $\mathcal{F}(L^1_{\text{loc}}, \#)$ consists of the so-called Köthe function spaces. Here again, normed ideals of measurable functions in $L^1([0,1], dx)$ are described in [20, § 8].

A.2. Operators on Indexed PIP*-Spaces*

Let V_I and Y_K be two nondegenerate indexed PIP-spaces (in particular, two LHSs or LBSs). Then an *operator* from V_I to Y_K is a map A from a subset $\mathfrak{D}(A) \subset V$ into Y, such that

(i) $\mathfrak{D}(A) = \bigcup_{q \in \mathsf{d}(A)} V_q$, where $\mathsf{d}(A)$ is a nonempty subset of I.

(ii) For every $r \in \mathsf{d}(A)$, there exists $u \in K$ such that the restriction of A to V_r is a continuous linear map into Y_u (we denote this restriction by A_{ur}).

(iii) A has no proper extension satisfying (i) and (ii).

We denote by $\mathrm{Op}(V_I, Y_K)$ the set of all operators from V_I to Y_K and, in particular, $\mathrm{Op}(V_I) := \mathrm{Op}(V_I, V_I)$. The continuous linear operator $A_{ur} : V_r \to Y_u$ is called a *representative* of A. Thus, the operator A may be identified with the collection of its representatives, $A \simeq \{A_{ur} : V_r \to Y_u \text{ exists as a bounded operator}\}$. We will also need the following sets:

$$\begin{aligned} \mathsf{d}(A) &= \{r \in I : \text{there is a } u \text{ such that } A_{ur} \text{ exists}\}, \\ \mathsf{i}(A) &= \{u \in K : \text{there is a } r \text{ such that } A_{ur} \text{ exists}\}. \end{aligned} \tag{A.5}$$

The following properties are immediate:

(i) $\mathsf{d}(A)$ is an initial subset of I: if $r \in \mathsf{d}(A)$ and $r' < r$, then $r \in \mathsf{d}(A)$, and $A_{ur'} = A_{ur}\, E_{rr'}$, where $E_{rr'}$ is a representative of the unit operator.

(ii) $\mathsf{i}(A)$ is a final subset of K: if $u \in \mathsf{i}(A)$ and $u' > u$, then $u' \in \mathsf{i}(A)$, and $A_{u'r} = E_{u'u}\, A_{ur}$.

Although an operator may be identified with a separately continuous sesquilinear form on $V^{\#} \times V^{\#}$, it is more useful to keep also the *algebraic operations* on operators, namely,

(i) *adjoint:* every $A \in \mathrm{Op}(V_I, Y_K)$ has a unique adjoint $A^{\times} \in \mathrm{Op}(Y_K, V_I)$ and one has $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I, Y_K)$: no extension is allowed, by the maximality condition (iii) of the definition.

(ii) *partial multiplication:* let V_I, W_L, and Y_K be nondegenerate indexed PIP-spaces (some, or all, may coincide). Let $A \in \mathrm{Op}(V_I, W_L)$, and, $B \in \mathrm{Op}(W_L, Y_K)$. We say that the product BA is defined if and only if there is a $t \in \mathsf{i}(A) \cap \mathsf{d}(B)$; that is, if and only if there is continuous factorization through some W_t:

$$V_r \xrightarrow{A} W_t \xrightarrow{B} Y_u, \quad \text{that is, } (BA)_{ur} = B_{ut} A_{tr}, \quad \text{for some } r \in \mathsf{d}(A),\ u \in \mathsf{i}(B). \tag{A.6}$$

Among operators on indexed PIP-spaces, a special role is played by morphisms.

An operator $A \in \mathrm{Op}(V_I, Y_K)$ is called a *homomorphism* if

(i) for every $r \in I$, there exists $u \in K$ such that both A_{ur} and $A_{\overline{ur}}$ exist;

(ii) for every $u \in K$, there exists $r \in I$ such that both A_{ur} and $A_{\overline{ur}}$ exist.

We denote by $\mathrm{Hom}(V_I, Y_K)$ the set of all homomorphisms from V_I into Y_K and by $\mathrm{Hom}(V_I)$ those from V_I into itself. The following properties are immediate.

Proposition A.1. *Let $V_I, Y_K, \ldots$ be indexed* PIP*-spaces. Then,*

(i) $A \in \mathrm{Hom}(V_I, Y_K)$ *if and only if* $A^{\times} \in \mathrm{Hom}(Y_K, V_I)$*;*

(ii) *the product of any number of homomorphisms (between successive* PIP*-spaces) is defined and is a homomorphism;*

(iii) *if* $A \in \mathrm{Hom}(V_I, Y_K)$*, then* $f \# g$ *implies* $Af \# Ag$.

The definition of homomorphisms just given is tailored in such a way that one may consider the category **PIP** of all indexed PIP-spaces, with the homomorphisms as morphisms (arrows), as we have done in Section 3.3 above. In the same language, we may define particular classes of morphisms, such as monomorphisms, epimorphisms, and isomorphisms.

(i) Let $M \in \mathrm{Hom}(W_L, Y_K)$. Then M is called a *monomorphism* if $MA = MB$ implies $A = B$, for any two elements of $A, B \in \mathrm{Hom}(V_I, W_L)$, where V_I is any indexed PIP-space.

(ii) Let $N \in \mathrm{Hom}(W_L, Y_K)$. Then N is called an *epimorphism* if $AN = BN$ implies $A = B$, for any two elements $A, B \in \mathrm{Hom}(Y_K, V_I)$, where V_I is any indexed PIP-space.

(iii) An operator $A \in \mathrm{Op}(V_I, Y_K)$ is an *isomorphism* if $A \in \mathrm{Hom}(V_I, Y_K)$, and there is a homomorphism $B \in \mathrm{Hom}(Y_K, V_I)$ such that $BA = 1_V$, $AB = 1_Y$, the identity operators on V, Y, respectively.

Typical examples of monomorphisms are the inclusion maps resulting from the restriction of a support, for instance, the natural injection $M^{(\Omega)} : L^1_{\mathrm{loc}}(\Omega, \mathrm{d}x) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x)$, where $\mathbb{R} = \Omega \cup \Omega'$ is the partition of $\mathbb{R}$ in two measurable subsets of nonzero measure. More examples and further properties of morphisms may be found in [5, Section 3.3] and in [7].

Finally, an *orthogonal projection* on a nondegenerate indexed PIP-space V_I, in particular, an LBS or an LHS, is a homomorphism $P \in \mathrm{Hom}(V_I)$ such that $P^2 = P^\times = P$.

A PIP-subspace W of a PIP-space V is defined in [5, Section 3.4.2] as an *orthocomplemented* subspace of V, that is, a vector subspace W for which there exists a vector subspace $Z \subseteq V$ such that $V = W \oplus Z$ and

(i) $\{f\}^\# = \{f_W\}^\# \cap \{f_Z\}^\#$ for every $f \in V$, where $f = f_W + f_Z$, $f_W \in W$, $f_Z \in Z$;

(ii) if $f \in W$, $g \in Z$ and $f \# g$, then $\langle f \mid g \rangle = 0$.

In the same Section 3.4.2 of [5], it is shown that a vector subspace W of a nondegenerate PIP-space is orthocomplemented if and only if it is *topologically regular*, which means that it satisfies the following two conditions:

(i) for every assaying subset $V_r \subseteq V$, the intersections $W_r = W \cap V_r$ and $W_{\overline{r}} = W \cap V_{\overline{r}}$ are a dual pair in V;

(ii) the intrinsic Mackey topology $\tau(W_r, W_{\overline{r}})$ coincides with the Mackey topology $\tau(V_r, V_{\overline{r}})|_{W_r}$ induced by V_r.

Then the fundamental result, which is the analogue to the similar statement for a Hilbert space, says that a vector subspace W of the nondegenerate PIP-space V is orthocomplemented if and only if it is the range of an orthogonal projection:

$$W = PV, \quad V = W \oplus W^\perp = PV \oplus (1 - P)V. \tag{A.7}$$

Clearly, this raises the question, discussed in Section 3.3.1, of identifying the subobjects of any category consisting of PIP-spaces.

B. Fine Sheaves

We collect here some classical notions and results used in Section 6. We recall first that the support of the continuous function $\varphi : X \to \mathbb{R}$ on the topological space X is the closed set $\mathrm{supp}\,\varphi := \mathrm{closure}\{x \in X : \varphi \neq 0\}$, which is the smallest closed set outside which φ is zero. The same definition applies to a distribution. Then we recall the standard notion of a partition of unity.

Definition B.1. Let $U = \{U_i,\ i \in I\}$ be an open covering of the topological space X. A set of real and continuous functions $\{\varphi_i,\ i \in I\}$ defined on X is called a *partition of unity* with respect to U if one has the following.

(i) $\varphi_i(x) \geqslant 0$, for all $x \in X$.

(ii) supp $\varphi_i \subset U_i$, for all $i \in I$.

(iii) Each point $x \in X$ has an open neighborhood that meets supp φ for a finite number of $i \in I$ only.

(iv) $\sum_{i\in I}\varphi_i(x) = 1$, for all $i \in I$. This sum is well defined by (iii).

We recall that a topological space is *paracompact* if it is separated (Hausdorff), and every open covering admits a locally finite open covering that is finer [22, § 6]. Every metrizable locally convex space is paracompact, but there are nonmetrizable paracompact spaces as well. The following result is standard.

Theorem B.2. *X is paracompact if and only if X is a separated topological space and each open covering of X admits a partition of unity.*

The main use of paracompact spaces is for the definition of a fine sheaf, to which the Cartan-Leray theorem applies (see below).

Definition B.3. Let F be a sheaf on a paracompact topological space X. One says that F is *fine* if, for every locally finite open covering $U = \{U_i,\ i \in I\}$ of X, there exists a set of homomorphisms $\{h_i : \mathsf{F} \to \mathsf{F},\ i \in I\}$ such that

(1) for each $i \in I$, there exists a closed set M_i of X such that $M_i \subset U_i$ and $h_i(\mathsf{F}_x) = 0$ for $x \notin M_i$, where F_x is the stalk of the sheaf F at the point x (the stalk is defined by the inductive limit $\overrightarrow{\lim}_{V_k \ni x}\mathsf{F}(V_k)$);

(2) $\sum_{i\in I} h_i = 1$. This sum is well defined since the covering U is locally finite.

Then the basic result is the following standard theorem. (A good introduction to the cohomology of sheaves and to the Cartan-Leray theorem can be found in [23, pages 166–197].)

Theorem B.4 (Cartan-Leray). *Let F be a fine sheaf on a paracompact topological space X. Then F is acyclic; that is, the higher-order sheaf cohomology groups are trivial; $H^p(X, \mathsf{F}) = 0$ for all $p \geq 1$.*

Acknowledgments

J.-P. Antoine thanks Juma Shabani (UNESCO) for communicating his unpublished results. D. Lambert thanks Bertrand Hespel (FUNDP) for inspiring discussions. The authors all thank the anonymous referee for his constructive remarks that have definitely improved the paper.

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