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Research Article

Existence of Positive Solutions for Fourth-Order Three-Point Boundary Value Problems

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We are concerned with the nonlinear fourth-order three-point boundary value problem $u^{(4)}(t) = a(t)f(u(t))$, $0 < t < 1$, $u(0) = u(1) = 0$, $\alpha u''(\eta) - \beta u'''(\eta) = 0$, $\gamma u''(1) + \delta u'''(1) = 0$. By using Krasnoselskii's fixed point theorem in a cone, we get some existence results of positive solutions.

1. Introduction

As is pointed out in [1, 2], boundary value problems for second- and higher-order differential equations play a very important role in both theory and applications. Recently, an increasing interest in studying the existence of solutions and positive solutions to boundary value problems for fourth-order differential equations is observed; see, for example, [3–8].

In this paper, we are concerned with the existence of positive solutions for the following fourth-order three-point boundary value problem (BVP):

$$\begin{gathered} u^{(4)}(t) = a(t)f(u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = 0, \\ \alpha u''(\eta) - \beta u'''(\eta) = 0, \qquad \gamma u''(1) + \delta u'''(1) = 0, \end{gathered} \tag{1.1}$$

where α, β, γ, and δ are nonnegative constants satisfying $\alpha\delta + \beta\gamma + \alpha\gamma > 0$; $0 < \eta < 1$, $a \in C[0,1]$, and $f \in C([0,\infty),[0,\infty))$. We use Krasnoselskii's fixed point theorem in cones to establish some simple criteria for the existence of at least one positive solution to BVP (1.1). To the best of our knowledge, no paper in the literature has investigated the existence of positive solutions for BVP (1.1).

The paper is formulated as follows. In Section 2, some definitions and lemmas are given. In Section 3, we prove some existence theorems of the positive solutions for BVP (1.1).

2. Preliminaries and lemmas

In this section, we introduce some necessary definitions and preliminary results that will be used to prove our main results.

First, we list the following notations and assumptions:

$$f_0 = \lim_{u\to 0}\frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty}\frac{f(u)}{u}. \tag{2.1}$$

(H_1) $f : [0,\infty)\to[0,\infty)$ is continuous;

(H_2) $a \in C[0,1]$, $a(t) \le 0$, for all $t \in [0,\eta]$, $a(t) \ge 0$, for all $t \in [\eta,1]$, and $a(t) \not\equiv 0$, for all $t \in (p,\eta) \cup (\eta,q)$ $(0 < p < \eta < q < 1)$.

By routine calculation, we easily obtain the following lemma.

Lemma 2.1. *Suppose that α, β, γ, δ are nonnegative constants satisfying $\alpha\delta+\beta\gamma+\alpha\gamma > 0$. If $h \in C[0,1]$, then the boundary value problem*

$$\begin{gathered} v''(t) = h(t), \quad t \in [0,1], \\ \alpha v(\eta) - \beta v'(\eta) = 0, \qquad \gamma v(1) + \delta v'(1) = 0 \end{gathered} \tag{2.2}$$

has a unique solution

$$v(t) = \int_\eta^t (t-s)h(s)ds + \frac{1}{\sigma}\int_\eta^1 \big(\alpha(\eta-t)-\beta\big)\big(\gamma(1-s)+\delta\big)h(s)ds, \tag{2.3}$$

where $\sigma = \alpha\delta+\beta\gamma+\alpha\gamma(1-\eta) > 0$.

Let $G(t,s)$ be the Green's function of the differential equation

$$-u''(t) = 0, \quad t \in (0,1), \tag{2.4}$$

subject to the boundary condition

$$u(0) = u(1) = 0. \tag{2.5}$$

In particular,

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t < s \le 1. \end{cases} \tag{2.6}$$

It is rather straightforward that

$$0 \le G(t,s) \le G(s,s), \quad 0 \le t,\, s \le 1, \tag{2.7}$$

$$G(t,s) \ge mG(s,s), \quad t \in [p,q],\, s \in [0,1], \tag{2.8}$$

where $0 < p < q < 1$, and $0 < m = \min\{p, 1-q\} < 1$.

Let X be the Banach space $C[0,1]$ endowed with the norm

$$\|u\| = \max_{0 \le t \le 1} |u(t)|. \tag{2.9}$$

We define the operator $T : X \to X$ by

$$\begin{aligned} Tu(t) = \int_0^1 G(t,s)\Big[\int_\eta^s (\tau - s)a(\tau)f(u(\tau))\,d\tau \\ + \frac{1}{\sigma}\int_\eta^1 (\beta - \alpha(\eta - s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\Big]ds, \end{aligned} \tag{2.10}$$

where $G(t,s)$ as in (2.6). From Lemma 2.1, we easily know that $u(t)$ is a solution of the fourth-order three-point boundary value problem (1.1) if and only if $u(t)$ is a fixed point of the operator T.

Define the cone K in X by

$$K = \Big\{u \in X \mid u \ge 0,\, \min_{t \in [p,q]} u(t) \ge m\|u\|\Big\}, \tag{2.11}$$

where $0 < p < \eta < q < 1$, and

$$0 < m = \max\{p, 1-q\} < 1. \tag{2.12}$$

Lemma 2.2. *Assume that* (H_1) *and* (H_2) *hold. If* $\beta \ge \alpha\eta$, *then* $T : K \to K$ *is completely continuous.*

Proof. For any $u \in K$, we know from (2.10), (H_1), (H_2), and $\beta \ge \alpha\eta$ that

$$\begin{aligned} (Tu)(t) = \int_0^\eta G(t,s)\Big[\int_s^\eta (s-\tau)a(\tau)f(u(\tau))\,d\tau \\ + \frac{1}{\sigma}\int_\eta^1 (\beta - \alpha(\eta - s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\Big]ds \\ + \int_\eta^1 G(t,s)\Big[\int_\eta^s (\tau - s)a(\tau)f(u(\tau))\,d\tau \\ + \frac{1}{\sigma}\int_\eta^s (\beta - \alpha(\eta - s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau \\ + \frac{1}{\sigma}\int_s^1 (\beta - \alpha(\eta - s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\Big]ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\eta G(t,s)\bigg[\int_s^\eta (s-\tau)a(\tau)f(u(\tau))\,d\tau \\
&\qquad\qquad + \frac{1}{\sigma}\int_\eta^1 (\beta-\alpha(\eta-s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\bigg]ds \\
&\quad + \int_\eta^1 G(t,s)\bigg[\frac{1}{\sigma}\int_\eta^s [\alpha\delta(\tau-\eta)+\beta\gamma(1-s)+\alpha\gamma(1-s)(\tau-\eta)+\beta\delta]a(\tau)f(u(\tau))\,d\tau \\
&\qquad\qquad + \frac{1}{\sigma}\int_s^1 (\beta+\alpha(s-\eta))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\bigg]ds \\
&\geq 0, \quad t\in[0,1].
\end{aligned}
\tag{2.13}
$$

Hence, in view of (2.13) and (2.7), we have

$$
\begin{aligned}
\|Tu\| = \max_{t\in[0,1]}(Tu)(t) \leq &\int_0^\eta G(s,s)\bigg[\int_s^\eta (s-\tau)a(\tau)f(u(\tau))\,d\tau \\
&\qquad + \frac{1}{\sigma}\int_\eta^1 (\beta-\alpha(\eta-s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\bigg]ds \\
&+ \int_\eta^1 G(s,s)\bigg[\frac{1}{\sigma}\int_\eta^s [\alpha\delta(\tau-\eta)+\beta\gamma(1-s)+\alpha\gamma(1-s)(\tau-\eta)+\beta\delta]a(\tau)f(u(\tau))\,d\tau \\
&\qquad + \frac{1}{\sigma}\int_s^1 (\beta+\alpha(s-\eta))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\bigg]ds.
\end{aligned}
\tag{2.14}
$$

Thus from (2.8), (2.13), and (2.14), we get

$$
\begin{aligned}
&\min_{t\in[p,q]}(Tu)(t) \\
&\quad \geq m\int_0^\eta G(s,s)\bigg[\int_s^\eta (s-\tau)a(\tau)f(u(\tau))\,d\tau \\
&\quad + \frac{1}{\sigma}\int_\eta^1 (\beta-\alpha(\eta-s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\bigg]ds \\
&\quad + m\int_\eta^1 G(s,s)\bigg[\frac{1}{\sigma}\int_\eta^s [\alpha\delta(\tau-\eta)+\beta\gamma(1-s)+\alpha\gamma(1-s)(\tau-\eta)+\beta\delta]a(\tau)f(u(\tau))\,d\tau \\
&\qquad\qquad + \frac{1}{\sigma}\int_s^1 (\beta+\alpha(s-\eta))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))\,d\tau\bigg]ds = m\|Tu\|,
\end{aligned}
\tag{2.15}
$$

where m as in (2.12). So $T: K\to K$. Moreover, it is easy to check by the Arzela-Ascoli theorem that the operator T is completely continuous. □

Remark 2.3. By $\sigma = \alpha\delta+\beta\gamma+\alpha\gamma(1-\eta) > 0$ and $\beta \geq \alpha\eta$, we have $\beta > 0$.

Recently, Krasnoselskii's theorem of cone expansion/compression type has been used to study the existence of positive solutions of boundary value problems in many papers;

see, for example, Liu [7], Ma [9], Torres [10], and the references contained therein. The following lemma (Krasnoselskii's fixed point theorem) will play an important role in the proof of our theorem.

Lemma 2.4 [11]. *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely operator such that either*

(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main result

We are now in a position to present and prove our main result.

Theorem 3.1. *Let $\beta \ge \alpha\eta$. Assume that (H_1)-(H_2) hold. If $f_0 = \infty$ and $f_\infty = 0$, then (1.1) has at least a positive solution.*

Proof. Since $f_0 = \infty$, we can choose $r > 0$ sufficiently small so that

$$f(u) \ge \varepsilon u \quad \text{for } 0 \le u \le r, \tag{3.1}$$

where ε satisfies

$$\varepsilon \ge \begin{cases} \dfrac{6}{m(1-\eta)\int_0^\eta -a(\tau)\tau^3 d\tau}, & \text{if } a(t_0) < 0, \text{ for some } t_0 \in (p,\eta), \\ \dfrac{\sigma}{\beta m\eta \int_\eta^1 (\tau-\eta)(1-\tau)(\gamma(1-\tau)+\delta)a(\tau)d\tau}, & \text{if } a(t_1) > 0, \text{ for some } t_1 \in (\eta,q). \end{cases} \tag{3.2}$$

Set $\Omega_r = \{u \in K \mid \|u\| < r\}$. From condition (H_2), we consider two cases as follows.

Case 1. If $a(t_0) < 0$ for some $t_0 \in (p,\eta)$, then, for $u \in \partial\Omega_r$, we have from (2.13), (3.1), and (3.2) that

$$\begin{aligned}(Tu)(\eta) \ge{}& \int_0^\eta G(\eta,s)\Big[\int_s^\eta (s-\tau)a(\tau)f(u(\tau))d\tau \\ &+ \frac{1}{\sigma}\int_\eta^1 (\beta-\alpha(\eta-s))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))d\tau\Big]ds \\ \ge{}& \int_0^\eta G(\eta,s)\int_s^\eta (s-\tau)a(\tau)f(u(\tau))d\tau\, ds \ge \varepsilon\int_0^\eta G(\eta,s)\int_s^\eta (s-\tau)a(\tau)u(\tau)d\tau\, ds \\ \ge{}& m\varepsilon\|u\|\int_0^\eta G(\eta,s)\int_s^\eta (s-\tau)a(\tau)d\tau\, ds = m\varepsilon\|u\|\int_0^\eta a(\tau)d\tau\int_0^\tau G(\eta,s)(s-\tau)ds \\ ={}& m\varepsilon\|u\|\int_0^\eta a(\tau)d\tau\int_0^\tau (1-\eta)s(s-\tau)ds = m\varepsilon\|u\|\frac{1-\eta}{6}\int_0^\eta -a(\tau)\tau^3 d\tau \ge \|u\|,\end{aligned} \tag{3.3}$$

which implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_r. \tag{3.4}$$

Case 2. If $a(t_1) > 0$ for some $t_1 \in (\eta, q)$, then, for $u \in \partial\Omega_r$, we have from (2.13), (3.1), and (3.2) that

$$\begin{aligned}
(Tu)(\eta) \geq & \int_\eta^1 G(\eta,s)\bigg[\frac{1}{\sigma}\int_\eta^s [\alpha\delta(\tau-\eta)+\beta\gamma(1-s)+\alpha\gamma(1-s)(\tau-\eta)+\beta\delta]a(\tau)f(u(\tau))d\tau \\
& \qquad + \frac{1}{\sigma}\int_s^1 (\beta+\alpha(s-\eta))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))d\tau\bigg]ds \\
\geq & \frac{1}{\sigma}\int_\eta^1 G(\eta,s)\int_s^1 (\beta+\alpha(s-\eta))(\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))d\tau\, ds \\
\geq & \frac{\beta}{\sigma}\int_\eta^1 G(\eta,s)\int_s^1 (\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))d\tau\, ds \\
\geq & \frac{\varepsilon\beta m}{\sigma}\|u\|\int_\eta^1 G(\eta,s)ds\int_s^1 (\gamma(1-\tau)+\delta)a(\tau)d\tau \\
= & \frac{\varepsilon\beta m}{\sigma}\|u\|\int_\eta^1 (\gamma(1-\tau)+\delta)a(\tau)d\tau\int_\eta^\tau \eta(1-s)ds \\
= & \frac{\varepsilon\beta m}{\sigma}\|u\|\int_\eta^1 \eta(\tau-\eta)\left(1-\frac{1}{2}(\tau+\eta)\right)(\gamma(1-\tau)+\delta)a(\tau)d\tau \\
\geq & \frac{\varepsilon\beta\eta m}{\sigma}\|u\|\int_\eta^1 (\tau-\eta)(1-\tau)(\gamma(1-\tau)+\delta)a(\tau)d\tau \geq \|u\|,
\end{aligned} \tag{3.5}$$

that is,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_r. \tag{3.6}$$

Next, define a function $f^*(v) : [0,\infty) \to [0,\infty)$ by

$$f^*(v) = \max_{0\leq u\leq v} f(u). \tag{3.7}$$

It is easy to see that $f^*(v)$ is nondecreasing. Since $f_\infty = 0$, we have $\lim_{v\to\infty} f^*(v)/v = 0$. Thus, there exists $R > r$ such that

$$f^*(R) \leq \theta R, \tag{3.8}$$

where θ satisfies

$$\begin{aligned}
\theta\bigg[&\frac{1}{12}\int_0^\eta -a(\tau)\tau^3 d\tau + \frac{1}{6\sigma}[(1-\eta)\sigma+\beta\delta](1-\eta^2)\int_\eta^1 a(\tau)d\tau \\
&+ \frac{1}{6\sigma}(\beta+\alpha(1-\eta))\int_\eta^1 (\gamma(1-\tau)+\delta)a(\tau)d\tau\bigg] \leq 1.
\end{aligned} \tag{3.9}$$

Hence, we obtain

$$f(u) \le f^*(R) \le \theta R, \quad 0 \le u \le R. \tag{3.10}$$

Thus from (2.14) and (3.10), for all $u \in \partial\Omega_R$, we have

$$\begin{aligned}
\|Tu\| \le \theta R\bigg[&\int_0^\eta G(s,s)\bigg[\int_s^\eta (s-\tau)a(\tau)d\tau + \frac{1}{\sigma}\int_\eta^1 (\beta-\alpha\eta+\alpha s)\big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau\bigg]ds \\
&+\int_\eta^1 G(s,s)\bigg[\frac{1}{\sigma}\int_\eta^s [\alpha\delta(\tau-\eta)+\beta\gamma(1-s)+\alpha\gamma(1-s)(\tau-\eta)+\beta\delta]a(\tau)d\tau \\
&\qquad+\frac{1}{\sigma}\int_s^1 \big(\beta+\alpha(s-\eta)\big)\big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau\bigg]ds\bigg] \\
\le \theta R\bigg[&\int_0^\eta a(\tau)d\tau\int_0^\tau s(1-s)(s-\tau)ds + \frac{\beta}{\sigma}\int_\eta^1 \big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau\int_0^\eta s(1-s)ds \\
&+\frac{1}{\sigma}\int_\eta^1 s(1-s)ds\int_\eta^1 [\alpha\delta(1-\eta)+\beta\gamma(1-\eta)+\alpha\gamma(1-\eta)^2+\beta\delta]a(\tau)d\tau \\
&+\frac{1}{\sigma}\int_\eta^1 s(1-s)ds\int_\eta^1 \big(\beta+\alpha(1-\eta)\big)\big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau\bigg] \\
= \theta R\bigg[&\frac{1}{12}\int_0^\eta -a(\tau)\tau^3 d\tau + \frac{\beta}{6\sigma}(3\eta^2-2\eta^3)\int_\eta^1 \big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau \\
&+\frac{1}{6\sigma}[(1-\eta)\sigma+\beta\delta](1-3\eta^2+2\eta^3)\int_\eta^1 a(\tau)d\tau \\
&+\frac{1}{6\sigma}\big(\beta+\alpha(1-\eta)\big)(1-3\eta^2+2\eta^3)\int_\eta^1 \big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau\bigg] \\
\le \theta R\bigg[&\frac{1}{12}\int_0^\eta -a(\tau)\tau^3 d\tau + \frac{1}{6\sigma}[(1-\eta)\sigma+\beta\delta](1-\eta^2)\int_\eta^1 a(\tau)d\tau \\
&+\frac{1}{6\sigma}\big(\beta+\alpha(1-\eta)\big)\int_\eta^1 \big(\gamma(1-\tau)+\delta\big)a(\tau)d\tau\bigg] \le R = \|u\|,
\end{aligned} \tag{3.11}$$

that is,

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \partial\Omega_R. \tag{3.12}$$

Hence, from (3.6), (3.12), and Lemma 2.4, T has a fixed point $u \in \overline{\Omega}_R \setminus \Omega_r$, which means that u is a positive solution of BVP (1.1). □

THEOREM 3.2. *Let $\beta \ge \alpha\eta$. Assume that (H_1)-(H_2) hold. If $f_0 = 0$ and $f_\infty = \infty$, then (1.1) has at least a positive solution.*

Proof. Since $f_\infty = \infty$, we can choose $R_1 > 0$ sufficiently large so that

$$f(u) \geq Au, \quad u \geq R_1, \tag{3.13}$$

where A satisfies

$$A \geq \begin{cases} \dfrac{6}{m(1-\eta)\int_p^\eta -a(\tau)(\tau-p)(\tau^2+\tau p-\tau p^2)d\tau}, & \text{if } a(t_0)<0, \text{ for some } t_0 \in (p,\eta), \\ \dfrac{\sigma}{\beta m\eta \int_\eta^q (\tau-\eta)(1-\tau)(\gamma(1-\tau)+\delta)a(\tau)d\tau}, & \text{if } a(t_1)>0, \text{ for some } t_1 \in (\eta,q). \end{cases} \tag{3.14}$$

Choose

$$R \geq \frac{R_1}{m}, \tag{3.15}$$

where $m > 0$ as in (2.12). Let $u \in \partial\Omega_R$. Since $u(t) \geq m\|u\| = mR \geq R_1$ for $t \in [p,q]$, from (3.13), we see that

$$f(u(t)) \geq Au(t) \geq AmR, \quad \forall t \in [p,q], \ u \in \partial\Omega_R. \tag{3.16}$$

For $u \in \partial\Omega_R$, we consider two cases as follows.

Case 1. If $a(t_0) < 0$ for some $t_0 \in (p,\eta)$, then we have from (3.3), (3.14), and (3.16) that

$$\begin{aligned} (Tu)(\eta) &\geq \int_0^\eta G(\eta,s)\int_s^\eta (s-\tau)a(\tau)f(u(\tau))d\tau\, ds \\ &\geq \int_p^\eta G(\eta,s)\int_s^\eta (s-\tau)a(\tau)f(u(\tau))d\tau\, ds \\ &\geq AmR\int_p^\eta G(\eta,s)\int_s^\eta (s-\tau)a(\tau)d\tau\, ds \\ &= AmR(1-\eta)\int_\eta^p a(\tau)d\tau\int_p^\tau s(s-\tau)ds \\ &= \frac{1}{6}AmR(1-\eta)\int_\eta^p -a(\tau)(\tau-p)(\tau^2+\tau p-2p^2)d\tau \\ &\geq R = \|u\|, \end{aligned} \tag{3.17}$$

which implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_R. \tag{3.18}$$

Case 2. If $a(t_1) > 0$ for some $t_1 \in (\eta, q)$, then we have from (3.5), (3.14), and (3.16) that

$$\begin{aligned}
(Tu)(\eta) &\geq \frac{\beta}{\sigma}\int_\eta^1 G(\eta,s)\int_s^1 (\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))d\tau\, ds \\
&\geq \frac{\beta}{\sigma}\int_\eta^q G(\eta,s)\int_s^q (\gamma(1-\tau)+\delta)a(\tau)f(u(\tau))d\tau\, ds \\
&\geq AmR\frac{\beta}{\sigma}\int_\eta^q G(\eta,s)\int_s^q (\gamma(1-\tau)+\delta)a(\tau)d\tau\, ds \\
&= AmR\frac{\beta}{\sigma}\int_\eta^q (\gamma(1-\tau)+\delta)a(\tau)d\tau\int_\eta^\tau \eta(1-s)ds \\
&\geq AmR\frac{\beta\eta}{\sigma}\int_\eta^q (\tau-\eta)(1-\tau)(\gamma(1-\tau)+\delta)a(\tau)d\tau \geq R = \|u\|,
\end{aligned} \tag{3.19}$$

which implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_R. \tag{3.20}$$

Since $f_0 = 0$, we can choose $0 < r < R$ such that

$$f(u) \leq \theta u, \quad 0 \leq u \leq r, \tag{3.21}$$

where θ as in (3.9). For $u \in \partial\Omega_r$, we have from (3.11) and (3.21) that

$$\begin{aligned}
\|Tu\| \leq \theta\|u\|\bigg[\frac{1}{12}\int_0^\eta -a(\tau)\tau^3 d\tau + \frac{1}{6\sigma}[(1-\eta)\sigma+\beta\delta](1-\eta^2)\int_\eta^1 a(\tau)d\tau \\
+ \frac{1}{6\sigma}(\beta+\alpha(1-\eta))\int_\eta^1 (\gamma(1-\tau)+\delta)a(\tau)d\tau\bigg] \leq \|u\|.
\end{aligned} \tag{3.22}$$

So,

$$\|Tu\| \leq \|u\|, \quad u \in \partial\Omega_r. \tag{3.23}$$

Therefore, from (3.20), (3.23), and Lemma 2.4, T has a fixed point $u \in \overline{\Omega}_R \setminus \Omega_r$, which means u is a positive solution of BVP (1.1). □

Finally, we conclude this paper with the following example.

Example 3.3. Consider the following fourth-order three-point boundary value problem:

$$\begin{gathered}
u^{(4)}(t) = \sin\pi(1+2t)u^r(t), \quad 0 < t < 1, \\
u(0) = u(1) = 0, \\
\alpha u''\left(\frac{1}{2}\right) - \beta u'''\left(\frac{1}{2}\right) = 0, \qquad \gamma u''(1) + \delta u'''(1) = 0,
\end{gathered} \tag{3.24}$$

where $0 < r < 1$, α, β, γ, and δ are nonnegative constants satisfying $\alpha\delta + \beta\gamma + \alpha\gamma > 0$ and $\beta \geq (1/2)\alpha$. Then BVP (3.24) has at least one positive solution.

To see this, we will apply Theorem 3.1. Set

$$f(u) = u^r, \qquad a(t) = \sin\pi(1+2t), \qquad \eta = \frac{1}{2}. \tag{3.25}$$

With the above functions f and a, we see that (H_1) and (H_2) hold. Moreover, it is easy to see that

$$f_0 = \infty, \qquad f_\infty = 0, \qquad \beta \geq \alpha\eta. \tag{3.26}$$

The result now follows from Theorem 3.1.

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