

*Research Article*

# Multiplicity Results via Topological Degree for Impulsive Boundary Value Problems under Non-Well-Ordered Upper and Lower Solution Conditions

Xu Xian,<sup>1</sup> Donal O'Regan,<sup>2</sup> and R. P. Agarwal<sup>3</sup>

<sup>1</sup> Department of Mathematics, Xuzhou Normal University, Xuzhou, Jiangsu 221116, China

<sup>2</sup> Department of Mathematics, National University of Ireland, Galway, Ireland

<sup>3</sup> Department of Mathematical Science, Florida Institute of Technology, Melbourne, FL 32901, USA

Correspondence should be addressed to R. P. Agarwal, agarwal@fit.edu

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Some multiplicity results for solutions of an impulsive boundary value problem are obtained under the condition of non-well-ordered upper and lower solutions. The main ideas of this paper are to associate a Leray-Schauder degree with the lower or upper solution.

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## 1. Introduction

In this paper, we study multiplicity of solutions of the impulsive boundary value problem

$$\begin{aligned}y'' + f(t, y(t), y'(t)) &= 0, \quad t \neq t_k, \\ \Delta y|_{t=t_k} &= I_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta y'|_{t=t_k} &= \bar{I}_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ y(0) = 0 &= y(1) - \alpha y(\eta),\end{aligned}\tag{1.1}$$

where  $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ ,  $J = [0, 1]$ ,  $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$ ,  $k = 1, 2, \dots, m$ ,  $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ ,  $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$ ,  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ ,  $\alpha \in [0, 1]$ .

Impulsive differential equations arise naturally in a wide variety of applications, such as spacecraft control, inspection processes in operations research, drug administration, and threshold theory in biology. In the past twenty years, a significant development in the

theory of impulsive differential equations was seen. Many authors have studied impulsive differential equations using a variety of methods (see [1–5] and the references therein).

The purpose of this paper is to study the multiplicity of solutions of the impulsive boundary value problems (1.1) by the method of upper and lower solutions. The method of lower and upper solutions has a very long history. Some of the ideas can be traced back to Picard [6]. This method deals mainly with existence results for various boundary value problems. For an overview of this method for ordinary differential equations, the reader is referred to [7]. Usually, when one uses the method of upper and lower solutions to study the existence and multiplicity of solutions of impulsive differential equations, one assumes that the upper solution is larger than the lower solution, that is, the condition that upper and lower solutions are well ordered. For example, Guo [1] studied the PBVP for second-order integrodifferential equations of mixed type in real Banach space  $E$ :

$$\begin{aligned} -u'' &= f(t, u, Tu, Su) \quad \forall t \in [0, 2\pi], t \neq t_i, \\ \Delta u|_{t=t_i} &= L_i u'(t_i), \\ \Delta u'|_{t=t_i} &= L_i^* u(t_i), \quad i = 1, 2, \dots, m, \\ u(0) &= u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned} \tag{1.2}$$

where  $f \in C([0, 2\pi] \times E \times E \times E, E)$ ,  $T$  and  $S : E \mapsto E$  are two linear operators,  $0 < t_1 < t_2 < \dots < t_m < 1$ ,  $L_i, L_i^*$  ( $i = 1, 2, \dots, m$ ) are constants. In [1] Guo first obtained a comparison result, and then, by establishing two increasing and decreasing sequences, he proved an existence result for maximal and minimal solutions of the PBVP (1.2) in the ordered interval defined by the lower and upper solutions.

However, to the best of our knowledge, only in the last few years, it was shown that existence and multiplicity for impulsive differential equation under the condition that the upper solution is not larger than the lower solution, that is, the condition of non-well-ordered upper and lower solutions. In [8], Rachůnková and Tvrdý studied the existence of solutions of the nonlinear impulsive periodic boundary value problem

$$\begin{aligned} u'' &= f(t, u, u'), \quad t \neq t_i \\ u(t_i^+) &= \mathcal{J}_i(u(t_i)), \quad i = 1, 2, \dots, m, \\ u'(t_i^+) &= \mathcal{M}_i(u'(t_i)), \quad i = 1, 2, \dots, m, \\ u(0) &= u(T), \quad u'(0) = u'(T), \end{aligned} \tag{1.3}$$

where  $f \in C([0, T] \times \mathbb{R}^2)$ ,  $\mathcal{J}_i, \mathcal{M}_i \in C(\mathbb{R})$ . Using Leray-Schauder degree, the authors of [8] showed some existence results for (1.3) under the non-well-ordered upper and lower solutions condition. For other results related to non-well-ordered upper and lower solutions, the reader is referred to [7, 9–14]. Also, here we mention the main results of a very recent paper [15]. In that paper, we studied the second-order three-point boundary value problem

$$\begin{aligned} y''(t) + f(t, y) &= 0, \quad 0 \leq t \leq 1, \\ y(0) &= 0, \quad y(1) - \alpha y(\eta) = 0, \end{aligned} \tag{1.4}$$

where  $0 < \eta < 1$ ,  $0 < \alpha < 1$ ,  $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ . In [15], we made the following assumption.

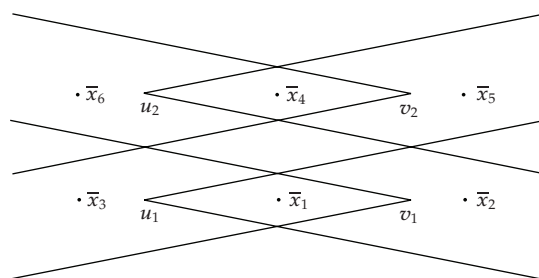


Figure 1: The positions of  $u_1, u_2, v_1, v_2$  and six solutions  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_6$  in Theorem 1.1.

(A<sub>0</sub>) There exists  $M > 0$  such that

$$f(t, x_2) - f(t, x_1) \geq -M(x_2 - x_1), \quad t \in [0, 1], x_2 \geq x_1. \tag{1.5}$$

Let the function  $e$  be  $e = e(t) = t$  for  $t \in [0, 1]$ . In [15], we proved the following theorem (see, [15, Theorem 3.4]).

**Theorem 1.1.** *Suppose that (A<sub>0</sub>) holds,  $u_1$  and  $u_2$  are two strict lower solutions of (1.4),  $v_1$  and  $v_2$  are two strict upper solutions of (1.4), and  $u_1 < v_1, u_2 < v_2, u_2 \not\leq v_1, u_1 \not\leq v_2$ . Moreover, assume*

$$-\zeta_0 e \leq u_2 - u_1 \leq \zeta_0 e, \quad -\zeta_0 e \leq v_2 - v_1 \leq \zeta_0 e \tag{1.6}$$

for some  $\zeta_0 > 0$ . Then the three-point boundary value problem (1.4) has at least six solutions  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_6$ .

Theorem 1.1 establishes the existence of at least six solutions of the three-point boundary value problem (1.4) only under the condition of two pairs of strict lower and upper solutions. The positions of  $u_1, u_2, v_1, v_2$  and six solutions  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_6$  in Theorem 1.1 can be illustrated roughly by Figure 1.

In some sense, we can say that these two pairs of lower and upper solutions are parallel to each other. The position of these two pairs of lower and upper solutions is sharply different from that of the lower and upper solutions of the main results in [14, 16, 17]. The technique to prove our main results of [15] is to use the fixed-point index of some increasing operator with respect to some closed convex sets, which are translations of some special cones (see  $Q_c, Q_{\bar{c}}$  of [15]).

This paper is a continuation of the paper [15]. The aim of this paper is to study the multiplicity of solutions of the impulsive boundary value problem (1.1) under the conditions of non-well-ordered upper and lower solutions. In this paper, we will permit the presence of impulses and the first derivative. The main ideas of this paper are to associate a Leray-Schauder degree with the lower or upper solution. We will give some multiplicity results for at least eight solutions. To obtain this multiplicity result, an additional pair of lower and upper solutions is needed, that is, we will employ a condition of three pairs of lower and upper solutions. The position of these three pairs of lower and upper solutions will be illustrated in Remark 2.16.

## 2. Results for at least eight solutions

Let  $J' = J \setminus \{t_1, t_2, \dots, t_m\}$ ,  $PC[J, \mathbb{R}] = \{x \mid x \text{ is a map from } J \text{ into } \mathbb{R} \text{ such that } x(t) \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k \text{ and its right-hand limit } x(t_k^+) \text{ at } t = t_k \text{ exists}\}$ , and  $PC^1[J, \mathbb{R}] = \{x \mid x \text{ is a map from } J \text{ into } \mathbb{R} \text{ such that } x(t) \text{ and } x'(t) \text{ are continuous at } t \neq t_k, \text{ left continuous at } t = t_k \text{ and their right-hand limits } x(t_k^+) \text{ and } x'(t_k^+) \text{ at } t = t_k \text{ exists}\}$ . For each  $x \in PC^1[J, \mathbb{R}]$ , let

$$\|x\|_{PC^1} = \max \{ \|x\|_{PC}, \|x'\|_{PC} \}, \quad (2.1)$$

where  $\|x\|_{PC} = \sup_{t \in J} |x(t)|$  and  $\|x'\|_{PC} = \sup_{t \in J} |x'(t)|$ . Then,  $PC^1[J, \mathbb{R}]$  is a real Banach space with the norm  $\|\cdot\|_{PC^1}$ . The function  $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$  is called a solution of the boundary value problem (1.1) if it satisfies all the equalities of (1.1).

Now, for convenience, we make the following assumptions.

( $H_0$ )  $0 = t_0 < t_1 < \dots < t_m < \eta < t_{m+1} = 1$ ,  $\alpha \in [0, 1)$ .

( $H_1$ )  $I_k$  ( $k = 1, 2, \dots, m$ ) is increasing on  $\mathbb{R}$ .

Let  $x, y \in PC[J, \mathbb{R}]$ . Now, we define the ordering  $<$  by

$$x < y \quad \text{iff } x(t) < y(t) \quad \forall t \in J, \quad x(t_k^+) < y(t_k^+) \quad \text{for each } k = 1, 2, \dots, m. \quad (2.2)$$

*Definition 2.1.* The function  $u \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$  is called a strict lower solution of (1.1) if

$$\begin{aligned} u''(t) + f(t, u(t), u'(t)) &> 0, \quad t \neq t_k, \\ u(0) < 0, \quad u(1) - \alpha u(\eta) &< 0, \end{aligned} \quad (2.3)$$

whenever  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$

$$\begin{aligned} \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta u'|_{t=t_k} &> \bar{I}_k(u(t_k)), \quad k = 1, 2, \dots, m, \end{aligned} \quad (2.4)$$

whenever  $I_k(x) = \bar{I}_k(x) = 0$  for each  $x \in \mathbb{R}$  and  $k \in \{1, 2, \dots, m\}$ ,  $\Delta u|_{t=t_k} = \Delta u'|_{t=t_k} = 0$  for each  $k \in \{1, 2, \dots, m\}$ .

The function  $v \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$  is called a strict upper solution of (1.1) if

$$\begin{aligned} v''(t) + f(t, v(t), v'(t)) &< 0, \quad t \neq t_k, \\ v(0) > 0, \quad v(1) - \alpha v(\eta) &> 0, \end{aligned} \quad (2.5)$$

whenever  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$

$$\begin{aligned} \Delta v|_{t=t_k} &= I_k(v(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta v'|_{t=t_k} &< \bar{I}_k(v(t_k)), \quad k = 1, 2, \dots, m, \end{aligned} \quad (2.6)$$

and whenever  $I_k(x) = \bar{I}_k(x) = 0$  for each  $x \in \mathbb{R}$  and  $k \in \{1, 2, \dots, m\}$ ,  $\Delta v|_{t=t_k} = \Delta v'|_{t=t_k} = 0$  for each  $k \in \{1, 2, \dots, m\}$ .

*Definition 2.2.* Let  $u(t), v(t) \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$ ,  $u(t) \leq v(t)$  for all  $t \in J$ . We say that  $f$  satisfies Nagumo condition with respect to  $[u, v]$  if there exists function  $\phi \in C([0, \infty), (0, \infty))$  such that

$$|f(t, x, y)| \leq \phi(|y|), \quad \forall (t, x, y) \in J \times [u(t), v(t)] \times \mathbb{R},$$

$$\int_0^\infty \frac{s}{\phi(s)} ds = \infty. \quad (2.7)$$

*Definition 2.3.* Let  $r_1(t), r_2(t), \dots, r_n(t)$  be strict upper solutions of (1.1) and  $r(t) = \min\{r_1(t), r_2(t), \dots, r_n(t)\}$  for each  $t \in J$ . Then, we say the upper solutions  $r_1(t), r_2(t), \dots, r_n(t)$  are well ordered if for each  $k \in \{1, 2, \dots, m\}$ , there exist  $i_0, j_0 \in \{1, 2, \dots, n\}$  and  $\delta_0 > 0$  small enough such that

$$r(t) = \begin{cases} r_{i_0}(t), & t \in (t_k - \delta_0, t_k], \\ r_{j_0}(t), & t \in (t_k, t_k + \delta_0]. \end{cases} \quad (2.8)$$

*Definition 2.4.* Let  $l_1(t), l_2(t), \dots, l_n(t)$  be strict lower solutions of (1.1) and  $l(t) = \max\{l_1(t), l_2(t), \dots, l_n(t)\}$  for each  $t \in J$ . Then, we say the lower solutions  $l_1(t), l_2(t), \dots, l_n(t)$  are well ordered if for each  $k \in \{1, 2, \dots, m\}$ , there exist  $i_0, j_0 \in \{1, 2, \dots, n\}$  and  $\delta_0 > 0$  small enough such that

$$l(t) = \begin{cases} l_{i_0}(t), & t \in (t_k - \delta_0, t_k], \\ l_{j_0}(t), & t \in (t_k, t_k + \delta_0]. \end{cases} \quad (2.9)$$

From [18, Lemma 5.4.1], we have the following lemma.

**Lemma 2.5.**  $H \subset PC^1[J, \mathbb{R}]$  is a relative compact set if and only if for all  $x \in H$ ,  $x(t)$  and  $x'(t)$  are uniformly bounded on  $J$  and equicontinuous on each  $J_k$  ( $k = 1, 2, \dots, m$ ), where  $J_1 = [0, t_1]$ ,  $J_i = (t_{i-1}, t_i]$ ,  $i = 2, 3, \dots, m + 1$ .

The following lemma can be easily proved.

**Lemma 2.6.** Suppose that  $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$  satisfies

$$-x''(t) = f(t, x(t), x'(t)), \quad t \neq t_k \quad (k = 1, 2, \dots, m). \quad (2.10)$$

Then

$$x'(t) = x'(0) - \int_0^t f(s, x(s), x'(s)) ds + \sum_{0 < t_k < t} [x'(t_k^+) - x'(t_k)] \quad \forall t \in J,$$

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)f(s, x(s), x'(s)) ds + \sum_{0 < t_k < t} [x(t_k^+) - x(t_k)]$$

$$+ \sum_{0 < t_k < t} [x'(t_k^+) - x'(t_k)](t - t_k) \quad \forall t \in J. \quad (2.11)$$

**Lemma 2.7.** Let  $g \in PC[J, \mathbb{R}]$  and  $a_k, b_k \in \mathbb{R}$  ( $k = 0, 1, 2, \dots, m$ ). Then,  $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$  is a solution of

$$\begin{aligned} -x''(t) &= g(t), \quad t \neq t_k, \quad k = 1, 2, \dots, m, \\ \Delta x|_{t=t_k} &= a_k, \quad k = 1, 2, \dots, m, \\ \Delta x'|_{t=t_k} &= b_k, \quad k = 1, 2, \dots, m, \\ x(0) &= a_0, \quad x(1) - \alpha x(\eta) = b_0 \end{aligned} \quad (2.12)$$

if and only if  $x \in PC[J, \mathbb{R}]$  satisfies

$$\begin{aligned} x(t) &= a_0 \left[ 1 - \frac{1-\alpha}{1-\alpha\eta} t \right] + \frac{b_0 t}{1-\alpha\eta} + \frac{t}{1-\alpha\eta} \int_0^1 (1-s)g(s)ds - \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta-s)g(s)ds \\ &\quad - \frac{t}{1-\alpha\eta} \sum_{k=1}^m \{ (1-\alpha)a_k + [1-t_k - \alpha(\eta-t_k)]b_k \} - \int_0^t (t-s)g(s)ds \\ &\quad + \sum_{0 < t_k < t} [a_k + b_k(t-t_k)], \quad t \in J. \end{aligned} \quad (2.13)$$

*Proof.* Let  $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$  be a solution of (2.12). From Lemma 2.6, we have

$$\begin{aligned} x(t) &= x(0) + x'(0)t - \int_0^t (t-s)g(s)ds + \sum_{0 < t_k < t} \Delta x|_{t=t_k} + \sum_{0 < t_k < t} \Delta x'|_{t=t_k} (t-t_k) \\ &= a_0 + x'(0)t - \int_0^t (t-s)g(s)ds + \sum_{0 < t_k < t} a_k + \sum_{0 < t_k < t} b_k(t-t_k). \end{aligned} \quad (2.14)$$

Thus,

$$\begin{aligned} x(1) &= a_0 + x'(0) - \int_0^1 (1-s)g(s)ds + \sum_{k=1}^m a_k + \sum_{k=1}^m b_k(1-t_k), \\ x(\eta) &= a_0 + x'(0)\eta - \int_0^\eta (\eta-s)g(s)ds + \sum_{k=1}^m a_k + \sum_{k=1}^m b_k(\eta-t_k). \end{aligned} \quad (2.15)$$

Using the boundary value condition  $x(1) - \alpha x(\eta) = b_0$ , we have

$$\begin{aligned} x'(0) &= \frac{1}{1-\alpha\eta} b_0 - \frac{1-\alpha}{1-\alpha\eta} a_0 + \frac{1}{1-\alpha\eta} \int_0^1 (1-s)g(s)ds - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta-s)g(s)ds \\ &\quad - \frac{1-\alpha}{1-\alpha\eta} \sum_{k=1}^m a_k - \frac{1}{1-\alpha\eta} \sum_{k=1}^m b_k(1-t_k) + \frac{\alpha}{1-\alpha\eta} \sum_{k=1}^m b_k(\eta-t_k). \end{aligned} \quad (2.16)$$

The equality (2.13) now follows from (2.14) and (2.16).

On the other hand, if  $x \in PC[J, \mathbb{R}]$  satisfies (2.13), by direct computation, we can easily show that  $x$  satisfies (2.12). The proof is complete.  $\square$

Let us define the operator  $A : PC^1[J, \mathbb{R}] \mapsto PC^1[J, \mathbb{R}]$  by

$$\begin{aligned} (Ax)(t) &= \frac{t}{1-\alpha\eta} \int_0^1 (1-s)f(s, x(s), x'(s))ds - \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta-s)f(s, x(s), x'(s))ds \\ &\quad - \int_0^t (t-s)f(s, x(s), x'(s))ds + \sum_{0 < t_k < t} [I_k(x(t_k)) + \bar{I}_k(x(t_k))(t-t_k)] \\ &\quad - \frac{t}{1-\alpha\eta} \sum_{k=1}^m \{(1-\alpha)I_k(x(t_k)) + [1-t_k-\alpha(\eta-t_k)]\bar{I}_k(x(t_k))\}. \end{aligned} \quad (2.17)$$

From Lemma 2.5,  $A : PC^1[J, \mathbb{R}] \mapsto PC^1[J, \mathbb{R}]$  is a completely continuous operator.

**Theorem 2.8.** *Suppose that  $(H_0)$  and  $(H_1)$  hold. Let  $\alpha_i, \beta_i$  ( $i = 1, 2, \dots, n$ ) be  $n$  pairs of strict lower and upper solution, and*

$$\begin{aligned} \bar{\alpha}(t) &= \max \{ \alpha_1(t), \alpha_2(t), \dots, \alpha_n(t) \}, \quad t \in J, \\ \bar{\beta}(t) &= \min \{ \beta_1(t), \beta_2(t), \dots, \beta_n(t) \}, \quad t \in J. \end{aligned} \quad (2.18)$$

*Suppose that  $\alpha_i < \beta_i$  ( $i = 1, 2, \dots, n$ ),  $\bar{\alpha} < \bar{\beta}$ ,  $f$  satisfies Nagumo condition with respect to  $[\alpha_1, \beta_1]$ . Moreover, the strict lower solutions  $\alpha_1, \alpha_2, \dots, \alpha_n$  and the strict upper solutions  $\beta_1, \beta_2, \dots, \beta_n$  are well ordered whenever  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$ . Then, there exist  $R_0 > 0$  and  $L_0 > 0$  sufficiently large such that for each  $R \geq R_0$  and  $L > L_0$*

$$\deg(I - A, \Omega, \theta) = 1, \quad (2.19)$$

where

$$\begin{aligned} \Omega &= \{x \in B(\theta, R) \mid \bar{\alpha} < x < \bar{\beta}, -L < x' < L\}, \\ B(\theta, R) &= \{x \in PC^1[J, \mathbb{R}] \mid \|x\|_{PC^1} < R\}. \end{aligned} \quad (2.20)$$

*Proof.* We only prove the case when  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$ . The conclusion is achieved in four steps.

*Step 1.* Since  $f$  satisfies Nagumo condition with respect to  $[\alpha_1, \beta_1]$ , then there exists  $\phi \in C([0, \infty), (0, \infty))$  such that

$$\begin{aligned} |f(t, x, y)| &\leq \phi(|y|), \quad (t, x, y) \in J \times [\alpha(t), \beta(t)] \times \mathbb{R}, \\ \int_0^\infty \frac{s}{\phi(s)} ds &= \infty. \end{aligned} \quad (2.21)$$

Let  $\mu_0 = \min_{1 \leq k \leq m+1} (t_k - t_{k-1})$ . Take  $\lambda > 0$  such that

$$\lambda > \frac{\max_{1 \leq i \leq n} \sup_{t \in J} \beta_i(t) - \min_{1 \leq i \leq n} \inf_{t \in J} \alpha_i(t)}{\mu_0}, \quad (2.22)$$

and  $N > 0$  such that

$$\int_{\lambda}^N \frac{s}{\phi(s)} ds > 2\lambda. \quad (2.23)$$

Let  $L_0 = \max\{N, 2\lambda, \max_{1 \leq i \leq n} \|\alpha'_i\|_{PC}, \max_{1 \leq i \leq n} \|\beta'_i\|_{PC}\}$ . Define the functions  $g, h : J \times \mathbb{R}^2 \mapsto \mathbb{R}$  by

$$g(t, x, y) = \begin{cases} f(t, x, L), & y > L, \\ f(t, x, y), & -L \leq y \leq L, \\ f(t, x, -L), & y < -L, \end{cases} \quad (2.24)$$

$$h(t, x, y) = \begin{cases} g(t, \bar{\beta}(t), y), & x > \bar{\beta}(t), \\ g(t, x, y), & \bar{\alpha}(t) \leq x \leq \bar{\beta}(t), \\ g(t, \bar{\alpha}(t), y), & x < \bar{\alpha}(t). \end{cases}$$

For each  $k \in \{1, 2, \dots, m\}$ , let us define the functions  $J_k, \bar{J}_k : \mathbb{R} \mapsto \mathbb{R}$  by

$$J_k(x) = \begin{cases} I_k(\bar{\beta}(t_k)), & x > \bar{\beta}(t_k), \\ I_k(x), & \bar{\alpha}(t_k) \leq x \leq \bar{\beta}(t_k), \\ I_k(\bar{\alpha}(t_k)), & x < \bar{\alpha}(t_k), \end{cases} \quad (2.25)$$

$$\bar{J}_k(x) = \begin{cases} \bar{I}_k(\bar{\beta}(t_k)), & x > \bar{\beta}(t_k), \\ \bar{I}_k(x), & \bar{\alpha}(t_k) \leq x \leq \bar{\beta}(t_k), \\ \bar{I}_k(\bar{\alpha}(t_k)), & x < \bar{\alpha}(t_k). \end{cases}$$

It is easy to see that there exists  $M_1 > 0$  such that

$$\begin{aligned} |h(t, x, y)| &\leq M_1, \quad (t, x, y) \in J \times \mathbb{R}^2, \\ |J_k(x)| &\leq M_1, \quad |\bar{J}_k(x)| \leq M_1, \quad x \in \mathbb{R}, k = 1, 2, \dots, m. \end{aligned} \quad (2.26)$$



Let us define the operator  $A^* : PC^1[J, \mathbb{R}] \mapsto PC^1[J, \mathbb{R}]$  by

$$\begin{aligned}
(A^*x)(t) &= \frac{t}{1-\alpha\eta} \int_0^1 (1-s)h(s, x(s), x'(s))ds - \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta-s)h(s, x(s), x'(s))ds \\
&\quad - \int_0^t (t-s)h(s, x(s), x'(s))ds + \sum_{0 < t_k < t} [J_k(x(t_k)) + (t-t_k)\bar{J}_k(x(t_k))] \\
&\quad - \frac{t}{1-\alpha\eta} \sum_{k=1}^m \{(1-\alpha)J_k(x(t_k)) + [(1-t_k) - \alpha(\eta-t_k)]\bar{J}_k(x(t_k))\}.
\end{aligned} \tag{2.27}$$

By (2.26), we have

$$\begin{aligned}
(A^*x)(t) &\leq \frac{1}{1-\alpha\eta} \int_0^1 (1-s)|h(s, x(s), x'(s))|ds + \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta-s)|h(s, x(s), x'(s))|ds \\
&\quad + \int_0^1 (1-s)|h(s, x(s), x'(s))|ds + \sum_{k=1}^m [ |J_k(x(t_k))| + (1-t_k)|\bar{J}_k(x(t_k))| ] \\
&\quad + \frac{2}{1-\alpha\eta} \sum_{k=1}^m [ |J_k(x(t_k))| + |\bar{J}_k(x(t_k))| ] \\
&\leq \frac{M_1}{1-\alpha\eta} \left[ \frac{1}{2} + \frac{1}{2}\alpha\eta^2 + \frac{1}{2} + 2m + 4m \right] \\
&\leq \frac{M_1}{1-\alpha\eta} [3 + 6m], \quad t \in J, \\
(A^*x)'(t) &\leq \frac{1}{1-\alpha\eta} \int_0^1 (1-s)|h(s, x(s), x'(s))|ds + \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta-s)|h(s, x(s), x'(s))|ds \\
&\quad + \int_0^1 |h(s, x(s), x'(s))|ds + \sum_{k=1}^m |\bar{J}_k(x(t_k))| + \frac{2}{1-\alpha\eta} \sum_{k=1}^m [ |J_k(x(t_k))| + |\bar{J}_k(x(t_k))| ] \\
&\leq \frac{M_1}{1-\alpha\eta} \left[ \frac{1}{2} + \frac{1}{2}\alpha\eta^2 + 1 + 5m \right] \\
&\leq \frac{M_1}{1-\alpha\eta} [5m + 3], \quad t \in J.
\end{aligned} \tag{2.28}$$

From (2.28), we have  $\|A^*x\|_{PC^1} \leq (M_1/(1-\alpha\eta))[11m+6]$  for each  $x \in PC^1[J, \mathbb{R}]$ . Let  $R_0 = (M_1/(1-\alpha\eta))(11m+6) + 1$ . Then,  $A^*(PC^1[J, \mathbb{R}]) \subset B(\theta, R_0)$ . By the properties of the Leray-Schauder degree, we have

$$\deg(I - A^*, B(\theta, R), \theta) = 1. \tag{2.29}$$

Thus,  $A^*$  has at least one fixed point  $x_0$ . From Lemma 2.7,  $x_0$  satisfies

$$\begin{aligned}x_0''(t) + h(t, x_0(t), x_0'(t)) &= 0, \quad t \neq t_k, \\ \Delta x_0|_{t=t_k} &= J_k(x_0(t_k)), \quad k = 1, 2, \dots, m \\ \Delta x_0'|_{t=t_k} &= \bar{J}_k(x_0(t_k)), \quad k = 1, 2, \dots, m, \\ x_0(0) &= 0 = x_0(1) - \alpha x_0(\eta),\end{aligned}\tag{2.30}$$

*Step 2.* Next, we will show that

$$\bar{\alpha} < x_0 < \bar{\beta},\tag{2.31}$$

$$-L < x_0' < L.\tag{2.32}$$

We first show that

$$\bar{\alpha}(t) \leq x_0(t) \leq \bar{\beta}(t) \quad \forall t \in J.\tag{2.33}$$

To begin, we show that  $x_0(t) \leq \bar{\beta}(t)$  for all  $t \in J$ . Suppose not, then there exists  $t' \in J$  such that  $x_0(t') > \bar{\beta}(t')$ . Set  $w(t) = x_0(t) - \bar{\beta}(t)$  for  $t \in J$ . There are a number of cases to consider.

(1)  $w(0) = \sup_{t \in J} w(t) > 0$ , then, we have

$$0 < w(0) = x_0(0) - \bar{\beta}(0) = -\bar{\beta}(0) < 0,\tag{2.34}$$

which is a contradiction.

(2)  $w(1) = \sup_{t \in J} w(t) > 0$ ; assume without loss of generality that  $\alpha > 0$  and  $\bar{\beta}(1) = \beta_{i_0}(1)$  for some  $i_0 \in \{1, 2, \dots, n\}$ , then, we have

$$0 < w(1) = x_0(1) - \beta_{i_0}(1) \leq \alpha x_0(\eta) - \alpha \beta_{i_0}(\eta) \leq \alpha x_0(\eta) - \alpha \bar{\beta}(\eta) = \alpha w(\eta) \leq \alpha w(1),\tag{2.35}$$

which is a contradiction.

(3) There exist  $k_0 \in \{1, 2, \dots, m, m+1\}$  and  $\tau_0 \in (t_{k_0-1}, t_{k_0})$  such that  $w(\tau_0) = \sup_{t \in J} w(t) > 0$ . Assume without loss of generality that  $\bar{\beta}(\tau_0) = \beta_{i_0}(\tau_0)$  for some  $i_0 \in \{1, 2, \dots, n\}$ . We have the following two cases:

(3A)  $\beta_j(\tau_0) > \beta_{i_0}(\tau_0)$  for each  $j \in \{1, 2, \dots, n\}$  and  $j \neq i_0$ ;

(3B) there exists  $j_0 \in \{1, 2, \dots, n\}$ ,  $j_0 \neq i_0$  such that  $\beta_{j_0}(\tau_0) = \beta_{i_0}(\tau_0)$ .

For case (3A), there exists  $\delta_0 > 0$  small enough such that  $[\tau_0 - \delta_0, \tau_0 + \delta_0] \subset (t_{k_0-1}, t_{k_0})$  and

$$w(t) = x_0(t) - \beta_{i_0}(t), \quad t \in [\tau_0 - \delta_0, \tau_0 + \delta_0].\tag{2.36}$$

Then,  $w \in C^2[\tau_0 - \delta_0, \tau_0 + \delta_0]$ ,  $w(\tau_0)$  is the maximum of  $w$  on  $[\tau_0 - \delta_0, \tau_0 + \delta_0]$ . Thus,  $w'(\tau_0) = 0$ ,  $w''(\tau_0) \leq 0$ . By (2.30), we have

$$\begin{aligned} 0 &\geq w''(\tau_0) = x_0''(\tau_0) - \beta_{i_0}''(\tau_0) = -h(\tau_0, x_0(\tau_0), x_0'(\tau_0)) - \beta_{i_0}''(\tau_0) \\ &= -f(\tau_0, \beta_{i_0}(\tau_0), \beta_{i_0}'(\tau_0)) - \beta_{i_0}''(\tau_0), \end{aligned} \quad (2.37)$$

which is a contradiction.

For case (3B), set  $w_1(t) = x_0(t) - \beta_{j_0}(t)$  for  $t \in (t_{k_0-1}, t_{k_0})$ . For any  $t' \in (t_{k_0-1}, t_{k_0})$ , we have

$$\begin{aligned} w_1(\tau_0) &= x_0(\tau_0) - \beta_{j_0}(\tau_0) = x_0(\tau_0) - \beta_{i_0}(\tau_0) = w(\tau_0) \\ &\geq w(t') = x_0(t') - \bar{\beta}(t') \geq x_0(t') - \beta_{j_0}(t') = w_1(t'). \end{aligned} \quad (2.38)$$

This implies that  $w_1(\tau_0)$  is a local maximum. Since  $w_1 \in C^2(t_{k_0-1}, t_{k_0})$ , then  $w_1'(\tau_0) = 0$ ,  $w_1''(\tau_0) \leq 0$ . Therefore,

$$0 \geq w_1''(\tau_0) = x_0''(\tau_0) - \beta_{j_0}''(\tau_0) = -f(\tau_0, \beta_{j_0}(\tau_0), \beta_{j_0}'(\tau_0)) - \beta_{j_0}''(\tau_0) > 0, \quad (2.39)$$

which is a contradiction.

(4) There exists  $k_0 \in \{1, 2, \dots, m\}$  such that  $w(t_{k_0}) = \sup_{t \in J} w(t) > 0$ . Without loss of generality, we may assume  $w(\tau) < \sup_{t \in J} w(t)$  for each  $\tau \in (t_{k-1}, t_k)$  and  $k \in \{1, 2, \dots, m, m+1\}$ . (Otherwise, if there exists  $\tau_0 \in (t_{k_0-1}, t_{k_0})$  for some  $k_0 \in \{1, 2, \dots, m, m+1\}$  such that  $w(\tau_0) = \sup_{t \in J} w(t)$ , then we can get a contradiction as in case (3)). In this case, we have the following two subcases:

- (4A) there exists  $i_0 \in \{1, 2, \dots, n\}$  such that  $\beta_{i_0}(t_{k_0}) < \beta_j(t_{k_0})$  for  $j = 1, 2, \dots, n$  and  $j \neq i_0$ ;
- (4B) there exists a subset  $\{n_1, n_2, \dots, n_s\} \subset \{1, 2, \dots, n\}$  such that

$$\bar{\beta}(t_{k_0}) = \beta_{n_1}(t_{k_0}) = \beta_{n_2}(t_{k_0}) = \dots = \beta_{n_s}(t_{k_0}), \quad (2.40)$$

while  $\beta_l(t_{k_0}) > \bar{\beta}(t_{k_0})$  for each  $l \in \{1, 2, \dots, n\} \setminus \{n_1, n_2, \dots, n_s\}$ ,  $s \geq 2$ .

First, we consider case (4A). Since  $I_{k_0}$  is increasing on  $\mathbb{R}$ , then

$$\beta_{i_0}(t_{k_0}^+) = \beta_{i_0}(t_{k_0}) + I_{k_0}(\beta_{i_0}(t_{k_0})) < \beta_j(t_{k_0}) + I_{k_0}(\beta_j(t_{k_0})) = \beta_j(t_{k_0}^+), \quad j \neq i_0. \quad (2.41)$$

Then, there exists  $\delta_0 > 0$  small enough such that  $\bar{\beta}(t) = \beta_{i_0}(t)$  for  $t \in [t_{k_0} - \delta_0, t_{k_0} + \delta_0]$  and so  $w(t) = x_0(t) - \beta_{i_0}(t)$  for  $t \in [t_{k_0} - \delta_0, t_{k_0} + \delta_0]$ . Since  $\beta_{i_0}(t)$  is a strict upper solution, we

have

$$\begin{aligned}
 w(t_{k_0}^+) &= x_0(t_{k_0}^+) - \beta_{i_0}(t_{k_0}^+) \\
 &= [x_0(t_{k_0}) + J_{k_0}(x_0(t_{k_0}))] - [\beta_{i_0}(t_{k_0}) + I_{k_0}(\beta_{i_0}(t_{k_0}))] \\
 &= [x_0(t_{k_0}) - \beta_{i_0}(t_{k_0})] + [J_{k_0}(x_0(t_{k_0})) - I_{k_0}(\beta_{i_0}(t_{k_0}))] \\
 &= w(t_{k_0}) + [I_{k_0}(\beta_{i_0}(t_{k_0})) - I_{k_0}(\beta_{i_0}(t_{k_0}))] \\
 &= w(t_{k_0}).
 \end{aligned} \tag{2.42}$$

Since  $w(\tau) < w(t_{k_0})$  for each  $\tau \in (t_{k_0-1}, t_{k_0})$ , then we have  $w'(t_{k_0}) \geq 0$ . Similarly, we have  $w'(t_{k_0}^+) \leq 0$ . Therefore,

$$\begin{aligned}
 0 &\geq w'(t_{k_0}^+) = x'(t_{k_0}^+) - \beta'_{i_0}(t_{k_0}^+) \\
 &> [x'_0(t_{k_0}) + \bar{J}_{k_0}(x_0(t_{k_0}))] - [\beta'_{i_0}(t_{k_0}) + \bar{I}_{k_0}(\beta_{i_0}(t_{k_0}))] \\
 &= w'(t_{k_0}) + [\bar{I}_{k_0}(\beta_{i_0}(t_{k_0})) - \bar{I}_{k_0}(\beta_{i_0}(t_{k_0}))] = w'(t_{k_0}) \geq 0,
 \end{aligned} \tag{2.43}$$

which is contradiction.

Now we consider case (4B). Since  $I_{k_0}$  is increasing, then we have

$$\bar{\beta}(t_{k_0}^+) = \beta_{n_1}(t_{k_0}^+) = \beta_{n_2}(t_{k_0}^+) = \cdots = \beta_{n_s}(t_{k_0}^+), \tag{2.44}$$

while  $\beta_l(t_{k_0}^+) > \beta(t_{k_0}^+)$  for each  $l \in \{1, 2, \dots, n\} \setminus \{n_1, n_2, \dots, n_s\}$ . For case (4B), we have two subcases:

(4Ba) there exists  $\delta_0 > 0$  small enough and  $i_0 \in \{n_1, n_2, \dots, n_s\}$  such that  $\beta_{i_0}(t) = \beta(t)$  for  $t \in [t_{k_0} - \delta_0, t_{k_0} + \delta_0]$ ;

(4Bb) there exists  $\delta_0 > 0$  small enough and  $i_0 \neq j_0, i_0, j_0 \in \{n_1, n_2, \dots, n_s\}$  such that

$$\bar{\beta}(t) = \begin{cases} \beta_{i_0}(t), & t \in [t_{k_0} - \delta_0, t_{k_0}], \\ \beta_{j_0}(t), & t \in (t_{k_0}, t_{k_0} + \delta_0]. \end{cases} \tag{2.45}$$

For case (4Ba) as in case (4A), we can easily obtain a contradiction. For case (4Bb), we have

$$w(t) = \begin{cases} x_0(t) - \beta_{i_0}(t), & t \in [t_{k_0} - \delta_0, t_{k_0}], \\ x_0(t) - \beta_{j_0}(t), & t \in (t_{k_0}, t_{k_0} + \delta_0]. \end{cases} \tag{2.46}$$

In the same way as in the proof of case (4A), we see that  $w(t_{k_0}^+) = w(t_{k_0})$ ,  $w'(t_{k_0}) \geq 0$  and we have  $w'(t_{k_0}^+) \leq 0$ . Note that  $\beta'_{j_0}(t_{k_0}) \leq \beta'_{i_0}(t_{k_0})$ , and we have

$$\begin{aligned}
 0 &\geq w'(t_{k_0}^+) = x'_0(t_{k_0}^+) - \beta'_{j_0}(t_{k_0}^+) \\
 &= [x'_0(t_{k_0}) + \Delta x'_0|_{t=t_{k_0}}] - [\beta'_{j_0}(t_{k_0}) + \Delta \beta'_{j_0}|_{t=t_{k_0}}] \\
 &> [x'_0(t_{k_0}) - \beta'_{j_0}(t_{k_0})] + [\bar{J}_{k_0}(x_0(t_{k_0})) - \bar{I}_{k_0}(\beta_{j_0}(t_{k_0}))] \\
 &\geq x'_0(t_{k_0}) - \beta'_{i_0}(t_{k_0}) + [\bar{I}_{k_0}(\beta_{j_0}(t_{k_0})) - \bar{I}_{k_0}(\beta_{j_0}(t_{k_0}))] \\
 &= w'(t_{k_0}) \geq 0,
 \end{aligned} \tag{2.47}$$

which is a contradiction.

(5) There exists a  $k_0 \in \{1, 2, \dots, m\}$  such that  $w(t_{k_0}^+) = \sup_{t \in J} w(t) > 0$ . Without loss of generality, we may assume that  $w(\tau) < w(t_{k_0}^+)$  for each  $k \in \{1, 2, \dots, m, m+1\}$  and  $\tau \in (t_{k-1}, t_k)$ . We have two subcases:

(5A) there exists  $i_0 \in \{1, 2, \dots, n\}$  such that  $\beta_{i_0}(t_{k_0}^+) < \beta_j(t_{k_0}^+)$  for each  $j \neq i_0$ ;

(5B) there exists a subset  $\{n_1, n_2, \dots, n_s\} \subset \{1, 2, \dots, n\}$  such that

$$\bar{\beta}(t_{k_0}^+) = \beta_{n_1}(t_{k_0}^+) = \beta_{n_2}(t_{k_0}^+) = \dots = \beta_{n_s}(t_{k_0}^+), \tag{2.48}$$

while  $\beta_l(t_{k_0}^+) > \bar{\beta}(t_{k_0}^+)$  for each  $l \in \{1, 2, \dots, n\} \setminus \{n_1, n_2, \dots, n_s\}$ ,  $s \geq 2$ .

Since  $I_{k_0}$  is increasing, then for case (5A), we have

$$\beta_{i_0}(t_{k_0}) < \beta_j(t_{k_0}) \quad (j \neq i_0), \quad x_0(t_{k_0}) > \beta_{i_0}(t_{k_0}), \tag{2.49}$$

and for case (5B), we have  $x_0(t_{k_0}) > \beta_{i_0}(t_{k_0})$  and

$$\bar{\beta}(t_{k_0}) = \beta_{n_1}(t_{k_0}) = \beta_{n_2}(t_{k_0}) = \dots = \beta_{n_s}(t_{k_0}), \tag{2.50}$$

while  $\beta_l(t_{k_0}) > \bar{\beta}(t_{k_0})$  for each  $l \in \{1, 2, \dots, n\} \setminus \{n_1, n_2, \dots, n_s\}$ . Therefore, we can use the same method as in case (4) to obtain a contradiction.

From the discussions of (1)–(5), we see that  $x_0(t) \leq \bar{\beta}(t)$  for  $t \in J$ . Similarly, we can prove that  $\bar{\alpha}(t) \leq x_0(t)$  for  $t \in J$ . Thus, (2.33) holds.

Next, we prove that  $\bar{\alpha} < x_0 < \bar{\beta}$ . If the inequality  $x_0 < \bar{\beta}$  does not hold, then either there exists  $\tau_0 \in J$  such that  $x_0(\tau_0) = \bar{\beta}(\tau_0)$  or there exists  $k_0 \in \{1, 2, \dots, m\}$  such that  $x_0(t_{k_0}^+) = \bar{\beta}(t_{k_0}^+)$ . Set  $w(t) = x_0(t) - \bar{\beta}(t)$  for  $t \in J$ . Then, we have either  $w(\tau_0) = \sup_{t \in J} w(t)$  or  $w(t_{k_0}^+) = \sup_{t \in J} w(t)$  for some  $k_0 \in \{1, 2, \dots, m\}$ . Essentially the same reasoning as in (1)–(5) above yields a contradiction. Thus,  $x_0 < \bar{\beta}$ . Similarly,  $\bar{\alpha} < x_0$ . Consequently, (2.31) holds.

*Step 3.* Now, we show (2.32). Suppose not, then we have the following two subcases:

(I) there exists  $s_1 \in J$  such that  $|x'_0(s_1)| \geq L$ ;

(II) there exists  $k_0 \in \{1, 2, \dots, m\}$  such that  $|x'_0(t_{k_0}^+)| \geq L$ .

We only consider case (II). A similar argument works for case (I). We may assume without loss of generality that  $x'_0(t_{k_0}^+) \geq L$ . By the mean-value theorem, there exists  $s_2 \in (t_{k_0}, t_{k_0+1})$  such that

$$x'_0(s_2) = \frac{x_0(t_{k_0+1}) - x_0(t_{k_0})}{t_{k_0+1} - t_{k_0}} \leq \frac{\max_{1 \leq i \leq n} \sup_{t \in J} \beta_i(t) - \min_{1 \leq i \leq n} \inf_{t \in J} \alpha_i(t)}{\mu_0} < \lambda < L. \quad (2.51)$$

Let  $L_1$  be such that  $L_0 < L_1 < L$ , then, there exist  $s_3, s_4 \in (t_{k_0}, s_2]$  such that  $s_3 < s_4$ ,  $x'_0(s_3) = L_1$ ,  $x'_0(s_4) = \lambda$ , and  $\lambda \leq x'_0(s) \leq L_1$  for  $s \in [s_3, s_4]$ . Therefore,

$$|x''_0(s)| = |h(s, x_0(s), x'_0(s))| = |f(s, x_0(s), x'_0(s))| \leq \phi(x'_0(s)), \quad s \in [s_3, s_4]. \quad (2.52)$$

Consequently,

$$\left| \int_{s_3}^{s_4} \frac{x'_0(s)x''_0(s)}{\phi(x'_0(s))} ds \right| \leq \int_{s_3}^{s_4} |x'_0(s)| ds = \int_{s_3}^{s_4} x'_0(s) ds = x_0(s_4) - x_0(s_3) < \lambda. \quad (2.53)$$

On the other hand,

$$\left| \int_{s_3}^{s_4} \frac{x'_0(s)x''_0(s)}{\phi(x'_0(s))} ds \right| = \left| \int_{\lambda}^{L_1} \frac{s}{\phi(s)} ds \right| \geq \int_{\lambda}^N \frac{s}{\phi(s)} ds > \lambda, \quad (2.54)$$

which is a contradiction. Thus, (2.32) holds.

*Step 4.* From the excision property of Leray-Schauder degree and (2.29), we have

$$\deg(I - A^*, \Omega, \theta) = \deg(I - A^*, B(\theta, R), \theta) = 1. \quad (2.55)$$

From (2.31) and (2.32), we see that  $Ax = A^*x$  for each  $x \in \overline{\Omega}$ , and so

$$\deg(I - A, \Omega, \theta) = 1. \quad (2.56)$$

The proof is complete.  $\square$

*Remark 2.9.* From the proof of Theorem 2.8, we see that  $A$  has no fixed point on  $\partial\Omega$ .

**Theorem 2.10.** *Suppose that  $(H_0)$ ,  $(H_1)$  hold,  $u_1(t)$ ,  $u_2(t)$  are strict lower solutions,  $v_1(t)$ ,  $v_0(t)$  are strict upper solutions,  $u_1 < v_1 < v_0$ ,  $u_2 < v_0$ ,  $u_2(t') > v_1(t')$  for some  $t' \in J$ , and  $f$  satisfies Nagumo condition with respect to  $[u_1, v_0]$ . Moreover, the strict lower solutions  $u_1(t)$ ,  $u_2(t)$  are well ordered whenever  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$ . Then, (1.1) has at least three solutions  $x_1, x_2$ , and  $x_3$ , such that*

$$u_1 < x_1 < v_1, \quad u_1 < x_2 < v_0, \quad u_2 < x_2 < v_0, \quad u_1 < x_3 < v_0, \quad (2.57)$$

and  $v_1(s_1) < x_3(s_1)$ ,  $x_3(s_2) < u_2(s_2)$  for some  $s_1, s_2 \in J$ .

*Proof.* Set  $\bar{\alpha}(t) = \max\{u_1(t), u_2(t)\}$  for  $t \in J$ , and  $\bar{\alpha}(t_k^+) = \max\{u_1(t_k^+), u_2(t_k^+)\}$  for each  $k \in \{1, 2, \dots, m\}$ . From Theorem 2.8, we see that there exist  $R > 0$  and  $L > 0$  large enough such that

$$\begin{aligned} \deg(I - A, G_0, \theta) &= 1, \\ \deg(I - A, G_1, \theta) &= 1, \\ \deg(I - A, G_2, \theta) &= 1, \end{aligned} \tag{2.58}$$

where  $G_0 = \{x \in B(\theta, R) \mid u_1 < x < v_0, -L < x' < L\}$ ,  $G_1 = \{x \in B(\theta, R) \mid u_1 < x < v_1, -L < x' < L\}$ , and  $G_2 = \{x \in B(\theta, R) \mid \bar{\alpha} < x < v_0, -L < x' < L\}$ . Then,  $A$  has fixed points  $x_1 \in G_1$  and  $x_2 \in G_2$ , respectively. From the conditions of Theorem 2.10, we see that  $\bar{G}_1 \cap \bar{G}_2 = \emptyset$ . Let  $\omega_0(t)$  be a continuous function on  $J$  such that its graph passes the points  $(0, (v_0(0) + v_1(0))/2)$  and  $(t', (v_1(t') + u_1(t'))/2)$ , and satisfies  $u_1 < \omega_0 < v_0$ . By the well-known Weierstrass approximation theorem, there exists  $\omega_1 \in C^1[0, 1]$  such that

$$|\omega_1(t) - \omega_0(t)| < \min \left\{ \frac{v_1(t') - u_1(t')}{4}, \frac{v_0(0) - v_1(0)}{4}, \frac{1}{4} \|\omega_0 - v_0\|_{PC}, \frac{1}{4} \|\omega_0 - u_1\|_{PC} \right\}, \quad t \in J. \tag{2.59}$$

It is easy to see that  $\omega_1 \in G_0 \setminus (\bar{G}_1 \cup \bar{G}_2)$ , and so  $G_0 \setminus (\bar{G}_1 \cup \bar{G}_2)$  is a nonempty open set. Note  $A$  has no fixed point on  $\partial G_0, \partial G_1$ , and  $\partial G_2$ . From (2.58), we have

$$\deg(I - A, G_0 \setminus (\bar{G}_1 \cup \bar{G}_2), \theta) = \deg(I - A, G_0, \theta) - \deg(I - A, G_1, \theta) - \deg(I - A, G_2, \theta) = -1. \tag{2.60}$$

Thus,  $A$  has at least one fixed point  $x_3 \in G_0 \setminus (\bar{G}_1 \cup \bar{G}_2)$ . Since  $x_3 \notin \bar{G}_1 \cup \bar{G}_2$ , then there exist  $s_1, s_2 \in J$  such that  $v_1(s_1) < x_3(s_1)$  and  $x_3(s_2) < u_2(s_2)$ . The proof is complete.  $\square$

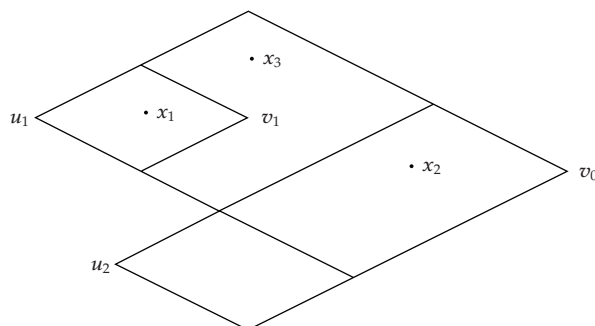
*Remark 2.11.* Theorem 2.10 is a partial generalization of the main results of [16, Theorem 2.2]. Here, we do not need to assume that  $u_2$  satisfies  $u_1 \leq u_2 \leq v_0$ .

*Remark 2.12.* The position of  $u_1, u_2, v_1, v_0$  in Theorem 2.10 can be illustrated roughly by Figure 2.

*Remark 2.13.* The relationship of  $u_1, u_2, v_1, v_0$  is different from that of [12, Theorems 9 and 10].

Similarly, we have the following result.

**Theorem 2.14.** *Suppose that  $(H_0), (H_1)$  hold,  $u_0, u_1$  are strict lower solutions of (1.1),  $v_1$  and  $v_2$  are strict upper solutions of (1.1),  $u_0 < u_1 < v_1, u_0 < v_2, v_2(t') < u_1(t')$  for some  $t' \in J$ , and  $f$  satisfies Nagumo condition with respect to  $[u_0, v_1]$ . Moreover, the strict upper solutions  $v_1(t), v_2(t)$  are well ordered whenever  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$ .*



**Figure 2:** The position of  $u_1, u_2, v_1, v_0$  in Theorem 2.10.

Then, (1.1) has at least three solutions  $x_1, x_2, x_3$  such that

$$u_1 < x_1 < v_1, \quad u_0 < x_2 < v_1, \quad u_0 < x_2 < v_2, \quad u_0 < x_3 < v_1, \quad (2.61)$$

and  $v_2(s_1) < x_3(s_1), x_3(s_2) < u_1(s_2)$  for some  $s_1, s_2 \in J$ .

From Theorems 2.10 and 2.14, we have the following Theorem 2.15.

**Theorem 2.15.** Suppose that  $(H_0), (H_1)$  hold,  $u_0, u_1, u_2$  are three strict lower solutions of (1.1),  $v_0, v_1, v_2$  are three strict upper solutions of (1.1),  $u_0 < u_1 < v_1 < v_0, u_0 < u_2 < v_2 < v_0, u_2(t') > v_1(t'), v_2(t'') < u_1(t'')$  for some  $t', t'' \in J$ , and  $f$  satisfies Nagumo conditions with respect to  $[u_0, v_0]$ . Moreover, the strict lower solutions  $u_0, u_1, u_2$  and the strict upper solutions  $v_0, v_1, v_2$  are well ordered whenever  $I_{i_0}(x) \neq 0$  or  $\bar{I}_{j_0}(x) \neq 0$  for some  $i_0, j_0 \in \{1, 2, \dots, m\}$  and some  $x \in \mathbb{R}$ . Then, (1.1) has at least eight solutions.

*Proof.* Now Theorem 2.10 guarantees that (1.1) has at least three solutions  $x_1, x_2, x_3$  such that

$$u_1 < x_1 < v_1, \quad u_1 < x_2 < v_0, \quad u_2 < x_2 < v_0, \quad u_1 < x_3 < v_0, \quad (2.62)$$

and  $v_1(s_1) < x_3(s_1), x_3(s_2) < u_2(s_2)$  for some  $s_1, s_2 \in J$ .

Also (1.1) has at least two solutions  $x_4$  and  $x_5$  such that

$$u_2 < x_4 < v_2, \quad u_2 < x_5 < v_0, \quad (2.63)$$

and  $v_2(s_3) < x_5(s_3), x_5(s_4) < u_1(s_4)$ .

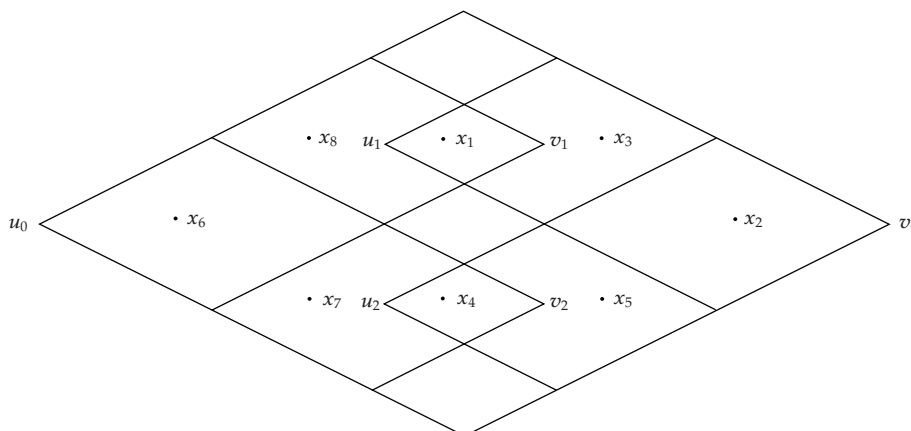
Now Theorem 2.14 guarantees that (1.1) has at least two solutions  $x_6, x_7$  such that

$$u_0 < x_6 < v_2, \quad u_0 < x_6 < v_1, \quad u_0 < x_7 < v_2, \quad (2.64)$$

and  $v_1(s_5) < x_7(s_5), x_7(s_6) < u_2(s_6)$ .

Also (1.1) has at least one solution  $x_8$  such that  $u_0 < x_8 < v_1$  and  $v_2(s_7) < x_8(s_7), x_8(s_8) < u_1(s_8)$  for some  $s_7, s_8 \in J$ . It is easy to see that  $x_1, x_2, \dots, x_8$  are distinct eight solutions of (1.1). The proof is complete.  $\square$





**Figure 3:** The position of  $u_0, u_1, u_2, v_0, v_1, v_2$  in Theorem 2.15.

*Remark 2.16.* The position of  $u_0, u_1, u_2, v_0, v_1, v_2$  in Theorem 2.15 can be illustrated roughly by Figure 3.

### 3. Further discussions

For simplicity, in this section, we will always assume that

$$I_k(x) = \bar{I}_k(x) = 0, \quad x \in \mathbb{R}, k \in \{1, 2, \dots, m\}. \quad (3.1)$$

In this case, (1.1) can be reduced to the following three-point boundary value problem

$$\begin{aligned} y'' + f(t, y(t), y'(t)) &= 0, \quad t \in (0, 1), \\ y(0) = 0 &= y(1) - \alpha y(\eta), \end{aligned} \quad (3.2)$$

where  $0 < \eta < 1$  and  $0 \leq \alpha < 1$ .

In this section, we will use the following assumptions.

(A<sub>1</sub>) Suppose that  $u_1, u_2$  are two strict lower solutions,  $v_1, v_2$  are two strict upper solutions of (1.1),  $u_1 < v_1, u_2 < v_2$ , and  $u_2(s_1) > v_1(s_1), u_1(s_2) > v_2(s_2)$  for some  $s_1, s_2 \in J$ .

Recently, this multipoint boundary value problem has been studied by many authors, see [16, 17, 19–21] and the references therein. The goal of this section is to prove some multiplicity results for (3.2) using the condition of two pairs of strict upper and lower solutions. As we can see from [13], some bounding condition on the nonlinear term is needed. Instead of the space  $PC^1[J, \mathbb{R}]$ , in this section we will use the space  $C^1(J)$ . First, we have the following theorem.

**Theorem 3.1.** *Suppose that (A<sub>1</sub>) holds, and*

$$|f(t, x, y)| < D_0, \quad (t, x, y) \in J \times \mathbb{R}^2 \quad (3.3)$$

*for some  $D_0 > 0$ . Then, (3.2) has at least eight solutions.*

*Proof.* First, we show that there exist strict lower and upper solutions  $u_0, v_0$  such that

$$u_0 < u_i < v_i < v_0, \quad i = 1, 2. \quad (3.4)$$

Let  $a_0 > \|u_1\|_C + \|u_2\|_C + \|v_1\|_C + \|v_2\|_C + 1$ . Now, we consider the following boundary value problem:

$$\begin{aligned} v_0''(t) + D_0 &= 0, \quad t \in (0, 1), \\ v_0(0) &= a_0 = v_0(1) - \alpha v_0(\eta). \end{aligned} \quad (3.5)$$

Let

$$\begin{aligned} g_1(t) &= \frac{1 - \alpha\eta + \alpha t}{1 - \alpha\eta}, \quad t \in J, \\ g_2(t) &= \frac{t}{2(1 - \alpha\eta)} - \frac{\alpha\eta^2 t}{2(1 - \alpha\eta)} - \frac{1}{2}t^2, \quad t \in J. \end{aligned} \quad (3.6)$$

By Lemma 2.7, we have

$$\begin{aligned} v_0(t) &= g_1(t)a_0 + \frac{D_0 t}{1 - \alpha\eta} \int_0^1 (1 - s) ds - \frac{\alpha D_0 t}{1 - \alpha\eta} \int_0^\eta (\eta - s) ds - D_0 \int_0^t (t - s) ds \\ &= g_1(t)a_0 + g_2(t)D_0, \quad t \in J. \end{aligned} \quad (3.7)$$

It is easy to see that  $g_1(t) \geq 1$  and  $g_2(t) \geq 0$  for each  $t \in J$ . Thus,  $v_0(t) \geq a_0$  for each  $t \in J$ , and therefore,  $u_i < v_0$ ,  $v_i < v_0$  for  $i = 1, 2$ . On the other hand, from (3.5), it is easy to see that  $v_0$  is a strict upper solution of (1.1). Similarly, we can show the existence of  $u_0$ . Then, by Theorem 2.15, the conclusion holds.  $\square$

*Remark 3.2.* Obviously, the condition (3.3) is restrictive. In the following, we will make use of a weaker condition. We study the multiplicity of solutions of (3.2) under a Nagumo-Knobloch-Schmitt condition. For this kind of bounding condition, the reader is referred to [13].

**Theorem 3.3.** *Suppose  $(A_1)$  holds, and there exists function  $\beta_1, \beta_2 \in C^1(J)$ ,  $\beta_1 \leq \beta_2$  such that*

$$\beta_1(t) \leq u_i'(t) \leq \beta_2(t), \quad \beta_1(t) \leq v_i'(t) \leq \beta_2(t), \quad t \in J, \quad (3.8)$$

$$-f(t, x, \beta_1(t)) < \beta_1'(t), \quad -f(t, x, \beta_2(t)) > \beta_2'(t), \quad (t, x) \in J \times [s_2, s_1], \quad (3.9)$$

$$\beta_1(1) \leq -\frac{2}{1 - \alpha\eta} D_0 \leq \frac{2}{1 - \alpha\eta} D_0 \leq \beta_2(1), \quad (3.10)$$

where  $D_0 = \max_{(t,x,y) \in J \times [s_2, s_1] \times \mathbb{R}} |f(t, y, \phi(t, y))|$ ,  $\gamma = \|u_1\|_C + \|v_1\|_C + \|u_2\|_C + \|v_2\|_C$ ,

$$s_2 = -\int_0^1 |\beta_1(t)| dt - \gamma, \quad s_1 = \int_0^1 |\beta_2(t)| dt + \gamma,$$

$$\phi(t, y) = \begin{cases} \beta_2(t), & y > \beta_2(t), \\ y, & \beta_1(t) \leq y \leq \beta_2(t), \\ \beta_1(t), & y < \beta_1(t). \end{cases} \quad (3.11)$$

Then, (3.2) has at least eight solutions.

*Proof.* Let  $\rho(x) = \max\{s_2, \min\{x, s_1\}\}$  for each  $x \in \mathbb{R}$ , and

$$f^*(t, x, y) = f(t, \rho(x), \phi(t, y)), \quad (t, x, y) \in J \times \mathbb{R}^2. \quad (3.12)$$

Now, we consider the following boundary value problem:

$$\begin{aligned} y''(t) + f^*(t, y(t), y'(t)) &= 0, \quad t \in (0, 1), \\ y(0) = 0 &= y(1) - \alpha y(\eta). \end{aligned} \quad (3.13)$$

From  $(A_1)$  and (3.8), we see that  $u_1, u_2$  are strict lower solutions of (3.13), and  $v_1$  and  $v_2$  are two strict upper solutions of (3.13). By Theorem 3.1, (3.13) has at least eight solutions  $x_1, x_2, \dots, x_8$ . We need only to show that  $x_1, x_2, \dots, x_8$  are solutions of (3.2). We claim that

$$\beta_1(t) \leq x'_i(t) \leq \beta_2(t), \quad t \in J, i = 1, 2, \dots, 8. \quad (3.14)$$

We only show that  $x'_1(t) \leq \beta_2(t)$  for  $t \in J$ . If  $x'_1(t') > \beta_2(t')$  for some  $t' \in J$ , then  $\max_{t \in J} z(t) = z(t_0) > 0$  for some  $t_0 \in J$ , where  $z(t) = x'_1(t) - \beta_2(t)$  for  $t \in J$ . If  $t_0 \in [0, 1)$ , then  $z'(t_0) \leq 0$ , and so

$$\begin{aligned} 0 \geq z'(t_0) &= x''_1(t_0) - \beta'_2(t_0) = -f^*(t_0, x_1(t_0), x'_1(t_0)) - \beta'_2(t_0) \\ &= -f(t_0, \rho(x_1(t_0)), \phi(t_0, x'_1(t_0))) - \beta'_2(t_0) \\ &= -f(t_0, \rho(x_1(t_0)), \beta_2(t_0)) - \beta'_2(t_0), \end{aligned} \quad (3.15)$$

which contradicts (3.9).

From Lemma 2.6, we have

$$\begin{aligned} x_1(t) &= \frac{t}{1 - \alpha\eta} \int_0^1 (1-s) f^*(s, x(s), x'(s)) ds - \frac{\alpha t}{1 - \alpha\eta} \int_0^\eta (\eta-s) f^*(s, x(s), x'_1(s)) ds \\ &\quad - \int_0^t (t-s) f^*(s, x(s), x'_1(s)) ds, \end{aligned} \quad (3.16)$$

and so

$$x_1'(t) \leq |x_1'(t)| \leq \left[ \frac{1}{1-\alpha\eta} \int_0^1 (1-s)ds + \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta-s)ds + 1 \right] \leq \frac{2}{1-\alpha\eta} D_0 \leq \beta_2(1). \quad (3.17)$$

This implies that  $t_0 \neq 1$ . Therefore, (3.14) holds. Integrating (3.14), we have

$$s_2 \leq -\int_0^1 |\beta_1(t)| dt \leq x_1(t) \leq \int_0^1 |\beta_2(t)| dt \leq s_1, \quad t \in J. \quad (3.18)$$

From (3.13)–(3.18), we see that  $x_1, x_2, \dots, x_8$  are eight solutions of (3.2). The proof is complete.  $\square$

*Remark 3.4.* We also can replace (3.3) by other bounding conditions, see [13].

*Remark 3.5.* To end this paper, we point out that the results of this paper can be applied to study the multiplicity of radial solutions of elliptic differential equation in an annulus with impulses at some radii.

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