Hindawi Publishing Corporation
Boundary Value Problems
Volume 2008, Article ID 293987, 17 pages
doi:10.1155/2008/293987

Research Article

Existence of Four Solutions of Some Nonlinear Hamiltonian System

Tacksun Jung[1] and Q-Heung Choi[2]

[1] *Department of Mathematics, Kunsan National University, Kunsan 573-701, South Korea*
[2] *Department of Mathematics Education, Inha University, Incheon 402-751, South Korea*

Correspondence should be addressed to Q-Heung Choi, qheung@inha.ac.kr

Received 25 August 2007; Accepted 3 December 2007

Recommended by Kanishka Perera

We show the existence of four 2π-periodic solutions of the nonlinear Hamiltonian system with some conditions. We prove this problem by investigating the geometry of the sublevels of the functional and two pairs of sphere-torus variational linking inequalities of the functional and applying the critical point theory induced from the limit relative category.

1. Introduction and statements of main results

Let $H(t,z)$ be a C^2 function defined on $R^1 \times R^{2n}$ which is 2π-periodic with respect to the first variable t.In this paper, we investigate the number of 2π-periodic nontrivial solutions of the following nonlinear Hamiltonian system

$$\dot{z} = J\big(H_z(t, z(t))\big), \tag{1.1}$$

where $z : R \to R^{2n}$, $\dot{z} = dz/dt$,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{1.2}$$

I_n is the identity matrix on R^n, $H : R^1 \times R^{2n} \to R$, and H_z is the gradient of H. Let $z = (p,q)$, $p = (z_1, \dots, z_n)$, $q = (z_{n+1}, \dots, z_{2n}) \in R^n$. Then (1.1) can be rewritten as

$$\begin{aligned} \dot{p} &= -H_q(t,p,q), \\ \dot{q} &= H_p(t,p,q). \end{aligned} \tag{1.3}$$

We assume that $H \in C^2(R^1 \times R^{2n}, R^1)$ satisfies the following conditions.

(H1) There exist constants $\alpha < \beta$ such that

$$\alpha I \leq d_z^2 H(t,z) \leq \beta I \quad \forall (t,z) \in R^1 \times R^{2n}. \tag{1.4}$$

(H2) Let $j_1, j_2 = j_1 + 1$ and $j_3 = j_2 + 1$ be integers and α, β be any numbers (without loss of generality, we may assume $\alpha, \beta \notin Z$) such that $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + 1 = j_3$. Suppose that there exist $\gamma > 0$ and $\tau > 0$ such that $j_2 < \gamma < \beta$ and

$$H(t,z) \geq \frac{1}{2}\gamma \|z\|_{L^2}^2 - \tau \quad \forall (t,z) \in R^1 \times R^{2n}. \tag{1.5}$$

(H3) $H(t,0) = 0$, $H_z(t,0) = 0$, and $j \in [j_1, j_2) \cap Z$ such that

$$jI < d_z^2 H(t,0) < (j+1) I \quad \forall t \in R^1. \tag{1.6}$$

(H4) H is 2π-periodic with respect to t.

We are looking for the weak solutions of (1.1). Let $E = W^{1/2,2}((0,2\pi), R^{2n})$. The 2π-periodic weak solution $z = (p,q) \in E$ of (1.3) satisfies

$$\int_0^{2\pi} \left[\left(\dot{p} + H_q(t, z(t)) \right) \cdot \psi - \left(\dot{q} - H_p(t, z(t)) \right) \cdot \phi \right] dt = 0 \quad \forall \zeta = (\phi, \psi) \in E \tag{1.7}$$

and coincides with the critical points of the induced functional

$$I(z) = \int_0^{2\pi} p\dot{q}\, dt - \int_0^{2\pi} H(t, z(t)) dt = A(z) - \int_0^{2\pi} H(t, z(t)) dt, \tag{1.8}$$

where $A(z) = (1/2) \int_0^{2\pi} \dot{z} \cdot Jz\, dt$.

Our main results are the following.

Theorem 1.1. *Assume that H satisfies conditions (H1)–(H4). Then there exists a number $\delta > 0$ such that for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta < j_2 + 1 = j_3$, $\alpha > 0$, system* (1.1) *has at least four nontrivial 2π-periodic solutions.*

Theorem 1.2. *Assume that H satisfies conditions (H1)–(H4). Then there exists a number $\delta > 0$ such that for any α and β, and $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta < j_2 + 1 = j_3$, $\beta < 0$,system* (1.1) *has at least four nontrivial 2π-periodic solutions.*

Chang proved in [1] that, under conditions (H1)–(H4), system (1.1) has at least two nontrivial 2π-periodic solutions. He proved this result by using the finite dimensional variational reduction method. He first investigate the critical points of the functional on the finite dimensional subspace and the $(P.S.)$ condition of the reduced functional and find one critical point of the mountain pass type. He also found another critical point by the shape of graph of the reduced functional.

For the proofs of Theorems 1.1 and 1.2, we first separate the whole space E into the four mutually disjoint four subspaces X_0, X_1, X_2, X_3 which are introduced in Section 3 and then we investigate two pairs of sphere-torus variational linking inequalities of the reduced functional $\widetilde{I}$ and $\check{I}$ of I on the submanifold with boundary $\widetilde{C}$ and $\check{C}$, respectively, and translate these two pairs of sphere-torus variational links of $\widetilde{I}$ and $\check{I}$ into the two pairs of torus-sphere variational links of $-\widetilde{I}$ and $-\check{I}$, where $\widetilde{I}$ and $\check{I}$ are the restricted functionals of I to the manifold with boundary $\widetilde{C}$ and $\check{C}$, respectively. Since $\widetilde{I}$ and $\check{I}$ are strongly indefinite functinals, we use the notion of the $(P.S.)_c^*$ condition and the limit relative category instead of the notion of $(P.S.)_c$ condition and the relative category, which are the useful tools for the proofs of the main theorems. We also investigate the limit relative category of torus in (torus, boundary of torus) on $\widetilde{C}$ and $\check{C}$, respectively. By the critical point theory induced from the limit relative category theory we obtain two nontrivial 2π-periodic solutions in each subspace X_1 and X_2, so we obtain at least four nontrivial 2π-periodic solutions of (1.1).

In Section 2, we introduce some notations and some notions of $(P.S.)_c^*$ condition and the limit relative category and recall the critical point theory on the manifold with boundary. We also prove some propositions. In Section 3, we prove Theorem 1.1 and in Section 4, we prove Theorem 1.2.

2. Recall of the critical point theory induced from the limit relative category

Let $E = W^{1/2,2}((0,2\pi), R^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-1/2,2}([0,2\pi], R^{2n})$. It is known that if $z \in C^\infty(R, R^{2n})$ is 2π-periodic, then it has a Fourier expansion $z(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{ikn}$ with $a_k \in C^{2n}$ and $a_{-k} = \overline{a_k}$: E is the closure of such functions with respect to the norm

$$\|z\| = \left(\sum_{k\in Z}(1+|k|)|a_k|^2\right)^{1/2}. \tag{2.1}$$

Let us set the functional

$$A(z) = \frac{1}{2}\int_0^{2\pi} \dot{z}\cdot Jz\,dt = \int_0^{2\pi} p\dot{q}\,dt, \quad z=(p,q)\in E,\ p,q\in R^n, \tag{2.2}$$

so that

$$I(z) = A(z) - \int_0^{2\pi} H(t,z(t))dt. \tag{2.3}$$

Let $e_1,\dots,e_{2n}$ denote the usual bases in R^{2n} and set

$$\begin{aligned}
E^0 &= \operatorname{span}\{e_1,\dots,e_{2n}\},\\
E^+ &= \operatorname{span}\{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid j\in N,\ 1\le k\le n\},\\
E^- &= \operatorname{span}\{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid j\in N,\ 1\le k\le n\}.
\end{aligned} \tag{2.4}$$

Then $E = E^0 \oplus E^+ \oplus E^-$ and $E^0,\ E^+,\ E^-$ are the subspaces of E on which A is null, positive definite and negative definite, and these spaces are orthogonal with respect to the bilinear form

$$B[z,\zeta] \equiv \int_0^{2\pi} p \cdot \dot{\psi} + \phi \cdot \dot{q}\, dt \tag{2.5}$$

associated with A. Here, $z = (p,q)$ and $\zeta = (\phi,\psi)$. If $z \in E^+$ and $\zeta \in E^-$, then the bilinear form is zero and $A(z+\zeta) = A(z) + A(\zeta)$. We also note that $E^0,\ E^+$, and E^- are mutually orthogonal in $L^2((0,2\pi), R^{2n})$. Let P^+ be the projection from E onto E^+ and P^- the one from E onto E^-. Then the norm in E is given by

$$\|z\|^2 = \left|z^0\right|^2 + A\left(z^+\right) - A\left(z^-\right) = \left|z^0\right|^2 + \left\|P^+ z\right\|^2 + \left\|P^- z\right\|^2 \tag{2.6}$$

which is equivalent to the usual one. The space E with this norm is a Hilbert space.

We need the following facts which are proved in [2].

Proposition 2.1. *For each $s \in [1,\infty)$, E is compactly embedded in $L^s((0,2\pi), R^{2n})$. In particular, there is an $\alpha_s > 0$ such that*

$$\|z\|_{L^s} \le \alpha_s \|z\| \tag{2.7}$$

for all $z \in E$.

Proposition 2.2. *Assume that $H(t,z) \in C^2(R^1 \times R^{2n}, R)$. Then $I(z)$ is C^1, that is, $I(z)$ is continuous and Fréchet differentiable in E with Fréchet derivative*

$$DI(z)\omega = \int_0^{2\pi} \left(\dot{z} - J(H_z(t,z))\right) \cdot J\omega = \int_0^{2\pi} \left[\left(\dot{p} + H_q(t,z)\right) \cdot \psi - \left(\dot{q} - H_p(t,z)\right) \cdot \phi\right] dt, \tag{2.8}$$

where $z = (p,q)$ and $\omega = (\phi,\psi) \in E$. Moreover, the functional $z \mapsto \int_0^{2\pi} H(t,z)dt$ is C^1.

Proof. For $z, w \in E$,

$$\begin{aligned}
&\left|I(z+w) - I(z) - DI(z)w\right| \\
&\quad = \left|\frac{1}{2}\int_0^{2\pi} (\dot{z}+\dot{w}) \cdot J(z+w) - \int_0^{2\pi} H(t,z+w) - \frac{1}{2}\int_0^{2\pi} \dot{z} \cdot Jz + \int_0^{2\pi} H(t,z) - \int_0^{2\pi} \left(\dot{z} - J\left(H_z(t,z)\right)\right) \cdot Jw\right| \\
&\quad = \left|\frac{1}{2}\int_0^{2\pi} \left[\dot{z} \cdot Jw + \dot{w} \cdot Jz + \dot{w} \cdot Jw\right] - \int_0^{2\pi} \left[H(t,z+w) - H(t,z)\right] - \int_0^{2\pi} \left[\dot{z} - J\left(H_z(t,z)\right) \cdot Jw\right]\right|.
\end{aligned} \tag{2.9}$$

We have

$$\left|\int_0^{2\pi} \left[H(t,z+w) - H(t,z)\right]\right| \le \left|\int_0^{2\pi} \left[H_z(t,z) \cdot w + o(|w|)\right] dt\right| = O(|w|). \tag{2.10}$$

Thus, we have

$$\left|I(z+w)-I(z)-DI(z)w\right| = O(|w|^2). \tag{2.11}$$

Next, we prove that $I(z)$ is continuous. For $z, w \in E$,

$$\begin{aligned}\left|I(z+w)-I(z)\right| &= \left|\frac{1}{2}\int_0^{2\pi}(\dot{z}+\dot{w})\cdot J(z+w)-\int_0^{2\pi}H(t,z+w)-\frac{1}{2}\int_0^{2\pi}\dot{z}\cdot Jz+\int_0^{2\pi}H(t,z)\right| \\ &= \left|\frac{1}{2}\int_0^{2\pi}\left[\dot{z}\cdot Jw+\dot{w}\cdot Jz+\dot{w}\cdot Jw\right]-\int_0^{2\pi}\left[H(t,z+w)-H(t,z)\right]\right| \\ &= O(|w|).\end{aligned} \tag{2.12}$$

Similarly, it is easily checked that I is C^1. □

Now, we consider the critical point theory on the manifold with boundary induced from the limit relative category. Let E be a Hilbert space and X be the closure of an open subset of E such that X can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing X. The $(P.S.)_c^*$ condition and the limit relative category (see [3]) are useful tools for the proof of the main theorem.

Let $(E_n)_n$ be a sequence of a closed finite dimensional subspace of E with the following assumptions: $E_n = E_n^- \oplus E_n^+$ where $E_n^+ \subset E^+$, $E_n^- \subset E^-$ for all n (E_n^+ and E_n^- are subspaces of E), $\dim E_n < +\infty$, $E_n \subset E_{n+1}$, $\bigcup_{n\in N} E_n$ are dense in E. Let $X_n = X \cap E_n$, for any n, be the closure of an open subset of E_n and has the structure of a C^2 manifold with boundary in E_n. We assume that for any n there exists a retraction $r_n : X \to X_n$. For a given $B \subset E$, we will write $B_n = B \cap E_n$. Let Y be a closed subspace of X.

Definition 2.3. Let B be a closed subset of X with $Y \subset B$. Let $\mathrm{cat}_{(X,Y)}(B)$ be the relative category of B in (X,Y). We define the limit relative category of B in (X,Y), with respect to $(X_n)_n$, by

$$\mathrm{cat}^*_{(X,Y)}(B) = \limsup_{n\to\infty}\mathrm{cat}_{(X_n,Y_n)}(B_n). \tag{2.13}$$

We set

$$\begin{aligned}\mathcal{B}_i &= \left\{B \subset X \mid \mathrm{cat}^*_{(X,Y)}(B) \ge i\right\}, \\ c_i &= \inf_{B\in\mathcal{B}_i}\sup_{x\in B} f(x).\end{aligned} \tag{2.14}$$

We have the following multiplicity theorem (for the proof, see [4]).

Theorem 2.4. *Let $i \in N$ and assume that*

(1) $c_i < +\infty$,

(2) $\sup_{x\in Y} f(x) < c_i$,

(3) *the $(P.S.)^*_{c_i}$ condition with respect to $(X_n)_n$ holds.*

Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \cdots = c_{i+k-1} = c, \tag{2.15}$$

then

$$\operatorname{cat}_X\left(\left\{x \in X \mid f(x) = c,\ \operatorname{grad}_X^- f(x) = 0\right\}\right) \geq k. \tag{2.16}$$

Now, we state the following multiplicity result (for the proof, see [4, Theorem 4.6]) which will be used in the proofs of our main theorems.

Theorem 2.5. *Let H be a Hilbert space and let $H = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of H with X_1, X_2 of finite dimension. For a given subspace X of H, let P_X be the orthogonal projection from H onto X. Set*

$$C = \left\{x \in H \mid \left\|P_{X_2}x\right\| \geq 1\right\}, \tag{2.17}$$

and let $f : W \to R$ be a $C^{1,1}$ function defined on a neighborhood W of C. Let $1 < \rho < R$, $R_1 > 0$. One defines

$$\begin{aligned}
\Delta &= \left\{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \left\|x_1\right\| \leq R_1,\ 1 \leq \left\|x_2\right\| \leq R\right\},\\
\Sigma &= \left\{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \left\|x_1\right\| \leq R_1,\ \left\|x_2\right\| = 1\right\}\\
&\quad \cup \left\{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \left\|x_1\right\| \leq R_1,\ \left\|x_2\right\| = R\right\}\\
&\quad \cup \left\{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \left\|x_1\right\| = R_1,\ 1 \leq \left\|x_2\right\| \leq R\right\},\\
S &= \left\{x \in X_2 \oplus X_3 \mid \|x\| = \rho\right\},\\
B &= \left\{x \in X_2 \oplus X_3 \mid \|x\| \leq \rho\right\}.
\end{aligned} \tag{2.18}$$

Assume that

$$\sup f(\Sigma) < \inf f(S) \tag{2.19}$$

and that the $(P.S.)_c$ condition holds for f on C, with respect to the sequrnce $(C_n)_n$, for all $c \in [a, b]$, where

$$a = \inf f(S), \qquad b = \sup f(\Delta). \tag{2.20}$$

Moreover, one assumes $b < +\infty$ and $f|_{X_1 \oplus X_3}$ has no critical points z in $X_1 \oplus X_3$ with $a \leq f(z) \leq b$. Then there exist two lower critical points z_1, z_2 for f on C such that $a \leq f(z_i) \leq b$, $i = 1.2$.

3. Proof of Theorem 1.1

We assume that $0 < \alpha < \beta$. Let $e_1, \ldots, e_{2n}$ denote the usual bases in R^{2n} and set

$$\begin{aligned}
X_0 \equiv \operatorname{span}&\left\{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n}, (\sin jt)e_k + (\cos jt)e_{k+n},\right.\\
&\quad \left.(\cos jt)e_k - (\sin jt)e_{k+n}, e_1, e_2, \ldots, e_{2n} \mid j \leq j_1 - 1,\ j \in N,\ 1 \leq k \leq n\right\},\\
X_1 \equiv \operatorname{span}&\left\{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid j = j_1,\ 1 \leq k \leq n\right\},\\
X_2 \equiv \operatorname{span}&\left\{\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid j = j_2,\ 1 \leq k \leq n\right\},\\
X_3 \equiv \operatorname{span}&\left\{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid j \geq j_2 + 1 = j_3,\ j \in N,\ 1 \leq k \leq n\right\}.
\end{aligned} \tag{3.1}$$

Then E is the topological direct sum of subspaces X_0, X_1, X_2, and X_3, where X_1 and X_2 are finite dimensional subspaces. We also set

$$\begin{aligned}
S_1(\rho) &= \{z \in X_1 \mid \|z\| = \rho\},\\
S_{r^{(1)}}(X_0 \oplus X_1) &= \{z \in X_0 \oplus X_1 \mid \|z\| = r^{(1)}\},\\
B_{r^{(1)}}(X_0 \oplus X_1) &= \{z \in X_0 \oplus X_1 \mid \|z\| \le r^{(1)}\},\\
\Sigma_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3) &= \{z = z_1 + z_2 + z_3 \in X_1 \oplus X_2 \oplus X_3 \mid z_1 \in S_1(\rho),\ \|z_1 + z_2 + z_3\| = R^{(1)}\},\\
\Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3) &= \{z = z_1 + z_2 + z_3 \in X_1 \oplus X_2 \oplus X_3 \mid z_1 \in S_1(\rho),\ \|z_1 + z_2 + z_3\| \le R^{(1)}\},\\
S_2(\rho) &= \{z \in X_2 \mid \|z\| = \rho\},\\
S_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2) &= \{z \in X_0 \oplus X_1 \oplus X_2 \mid \|z\| = r^{(2)}\},\\
B_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2) &= \{z \in X_0 \oplus X_1 \oplus X_2 \mid \|z\| \le r^{(2)}\},\\
\Sigma_{R^{(2)}}(S_2(\rho), X_3) &= \{z = z_2 + z_3 \in X_2 \oplus X_3 \mid z_2 \in S_2(\rho),\ \|z_2 + z_3\| = R^{(2)}\},\\
\Delta_{R^{(2)}}(S_2(\rho), X_3) &= \{z = z_2 + z_3 \in X_2 \oplus X_3 \mid z_2 \in S_2(\rho),\ \|z_2 + z_3\| \le R^{(2)}\}.
\end{aligned} \tag{3.2}$$

We have the following two pairs of the sphere-torus variational linking inequalities.

Lemma 3.1 (first sphere-torus variational linking). *Assume that H satisfies the conditions (H1), (H3), (H4), and the condition*

(H2)′ *suppose that there exist $\gamma > 0$ and $\tau > 0$ such that $j_1 < \gamma < \beta$ and*

$$H(t,z) \ge \frac{1}{2}\gamma\|z\|^2 - \tau \quad \forall (t,z) \in R^1 \times R^{2n}. \tag{3.3}$$

Then there exist $\delta_1 > 0$, $\rho > 0$, $r^{(1)} > 0$, and $R^{(1)} > 0$ such that $r^{(1)} < R^{(1)}$, and for any α and β with $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$ and $\alpha > 0$,

$$\begin{gathered}
\sup_{z \in S_{r^{(1)}}(X_0 \oplus X_1)} I(z) < 0 < \inf_{z \in \Sigma_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} I(z),\\
\inf_{z \in \Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} I(z) > -\infty, \qquad \sup_{z \in B_{r^{(1)}}(X_0 \oplus X_1)} I(z) < \infty.
\end{gathered} \tag{3.4}$$

Proof. Let $z = z_0 + z_1 \in X_0 \oplus X_1$. By $(H2)'$, we have

$$\begin{aligned}
I(z) &= \frac{1}{2}\int_0^{2\pi} \dot{z} \cdot Jz\, dt - \int_0^{2\pi} H(t, z(t))dt\\
&\le \frac{1}{2}\|z_0 + z_1\|^2 - \frac{\gamma}{2}\|z_0 + z_1\|_{L^2}^2 + \tau\\
&\le \frac{1}{2}(j_1 - \gamma)\|z_0 + z_1\|_{L^2}^2 + \tau
\end{aligned} \tag{3.5}$$

for some $\tau > 0$. Since $j_1 - \gamma < 0$, there exists $r^{(1)} > 0$ such that if $z_0 + z_1 \in S_{r^{(1)}}(X_0 \oplus X_1)$, then $I(z) < 0$. Thus, $\sup_{z \in S_{r^{(1)}}(X_0 \oplus X_1)} I(z) < 0$. Moreover, if $z \in B_{r^{(1)}}(X_0 \oplus X_1)$, then $I(z) \leq (1/2)(j_1 - \gamma)\|z_0 + z_1\|_{L^2}^2 + \tau < \tau < \infty$, so we have $\sup_{z \in B_{r^{(1)}}(X_0 \oplus X_1)} I(z) < \infty$. Next, we will show that there exist $\delta_1 > 0$, $\rho > 0$ and $R^{(1)} > 0$ such that if $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$, then $\inf_{z \in \Sigma_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} I(z) > 0$. Let $z = z_1 + z_2 + z_3 \in X_1 \oplus X_2 \oplus X_3$ with $z_1 \in S_1(\rho)$, $z_2 \in X_2$, $z_3 \in X_3$, where ρ is a small number. Let $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta < j_2 + 1 = j_3$ for some $\delta > 0$ and $\alpha > 0$. Then $X_1 \oplus X_2 \oplus X_3 \subset E^+$ and $P^-(z_1 + z_2 + z_3) = 0$. By (H1), there exists $d > 0$ such that

$$\begin{aligned} I(z) &= \frac{1}{2}\int_0^{2\pi} \dot{z} \cdot Jz\, dt - \int_0^{2\pi} H(t, z(t))dt \\ &\geq \frac{1}{2}\left\|P^+(z_1 + z_2 + z_3)\right\|^2 - \frac{\beta}{2}\left\|P^+(z_1 + z_2 + z_3)\right\|_{L^2}^2 - d \\ &\geq \frac{1}{2}(j_1 - \beta)\left\|P^+ z_1\right\|_{L^2}^2 + \frac{1}{2}(j_2 - \beta)\left\|P^+ z_2\right\|_{L^2}^2 + \frac{1}{2}(j_3 - \beta)\left\|P^+ z_3\right\|_{L^2}^2 - d \\ &= \frac{1}{2}(j_1 - \beta)\rho^2 - \frac{1}{2}\delta\left\|P^+ z_2\right\|_{L^2}^2 + \frac{1}{2}(j_3 - \beta)\left\|P^+ z_3\right\|_{L^2}^2 - d. \end{aligned} \tag{3.6}$$

Since $j_1 - \beta < 0$, $j_2 - \beta > -\delta$, and $j_3 - \beta > 0$, there exist a small number $\delta_1 > 0$ and $R^{(1)} > 0$ with $\delta_1 < \delta$ and $R^{(1)} > r^{(1)}$ such that if $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$ and $z \in \Sigma_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)$, then $I(z) > 0$. Thus, we have $\inf_{z \in \Sigma_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} I(z) > 0$. Moreover, if $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$ and $z \in \Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)$, then we have $I(z) > (1/2)(j_1 - \beta)\rho^2 - (1/2)\delta)1\|P^+ z_2\|_{L^2}^2 - d > -\infty$. Thus, $\inf_{\Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} I(z) > -\infty$. Thus, we prove the lemma. □

Lemma 3.2. *Let δ_1 be the number introduced in Lemma 3.1. Then for any α and β with $j_1 - 1 < \alpha < j_1 < \beta \leq j_2 < j_2 + 1 = j_3$ and $\alpha > 0$, if u is a critical point for $I|_{X_0 \oplus (X_2 \oplus X_3)}$, then $I(u) = 0$.*

Proof. We notice that from Lemma 3.1, for fixed $u_0 \in X_0$, the functional $u_{23} \mapsto I(u_0 + u_{23})$ is weakly convex in $X_2 \oplus X_3$, while, for fixed $u_{23} \in X_2 \oplus X_3$, the functional $u_0 \mapsto I(u_0 + u_{23})$ is strictly concave in X_0. Moreover, 0 is the critical point in $X_0 \oplus X_2 \oplus X_3$ with $I(0) = 0$. So if $u = u_0 + u_{23}$ is another critical point for $I|_{X_0 \oplus (X_2 \oplus X_3)}$, then we have

$$0 = I(0) \leq I(u_{23}) \leq I(u_0 + u_{23}) \leq I(u_0) \leq I(0) = 0. \tag{3.7}$$

So we have $I(u) = I(0) = 0$. □

Let P_{X_1} be the orthogonal projection from E onto X_1 and

$$\widetilde{C} = \left\{z \in E \mid \left\|P_{X_1} z\right\| \geq 1\right\}. \tag{3.8}$$

Then $\widetilde{C}$ is the smooth manifold with boundary. Let $\widetilde{C}_n = \widetilde{C} \cap E_n$. Let us define afunctional $\widetilde{\Psi} : E \setminus \{X_0 \oplus (X_2 \oplus X_3)\} \to E$ by

$$\widetilde{\Psi}(z) = z - \frac{P_{X_1} z}{\left\|P_{X_1} z\right\|} = P_{X_0 \oplus (X_2 \oplus X_3)} z + \left(1 - \frac{1}{\left\|P_{X_1} z\right\|}\right) P_{X_1} z. \tag{3.9}$$

We have

$$\nabla\widetilde{\Psi}(z)(w) = w - \frac{1}{\|P_{X_1}z\|}\left(P_{X_1}w - \left\langle \frac{P_{X_1}z}{\|P_{X_1}z\|}, w \right\rangle \frac{P_{X_1}z}{\|P_{X_1}z\|}\right). \tag{3.10}$$

Let us define the functional $\widetilde{I} : \widetilde{C} \to R$ by

$$\widetilde{I} = I \circ \widetilde{\Psi}. \tag{3.11}$$

Then $\widetilde{I} \in C^{1,1}_{\text{loc}}$. We note that if $\widetilde{z}$ is the critical point of $\widetilde{I}$ and lies in the interior of $\widetilde{C}$, then $z = \widetilde{\Psi}(\widetilde{z})$ is the critical point of I. We also note that

$$\left\|\text{grad}^{-}_{\widetilde{C}}\widetilde{I}(\widetilde{z})\right\| \geq \left\|P_{X_0\oplus(X_2\oplus X_3)}\nabla I(\widetilde{\Psi}(\widetilde{z}))\right\| \quad \forall \widetilde{z} \in \partial\widetilde{C}. \tag{3.12}$$

Let us set

$$\begin{aligned}
\widetilde{S_{r^{(1)}}} &= \widetilde{\Psi}^{-1}\left(S_{r^{(1)}}(X_0 \oplus X_1)\right),\\
\widetilde{B_{r^{(1)}}} &= \widetilde{\Psi}^{-1}\left(B_{r^{(1)}}(X_0 \oplus X_1)\right),\\
\widetilde{\Sigma_{R^{(1)}}} &= \widetilde{\Psi}^{-1}\left(\Sigma_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)\right),\\
\widetilde{\Delta_{R^{(1)}}} &= \widetilde{\Psi}^{-1}\left(\Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)\right).
\end{aligned} \tag{3.13}$$

We note that $\widetilde{S_{r^{(1)}}}, \widetilde{B_{r^{(1)}}}, \widetilde{\Sigma_{R^{(1)}}}$, and $\widetilde{\Delta_{R^{(1)}}}$ have the same topological structure as $S_{r^{(1)}}$, $B_{r^{(1)}}$, $\Sigma_{R^{(1)}}$, and $\Delta_{R^{(1)}}$, respectively.

Lemma 3.3. *$-\widetilde{I}$ satisfies the $(P.S.)^*_{\widetilde{c}}$ condition with respect to $(\widetilde{C}_n)_n$ for every real number $\widetilde{c}$ such that*

$$0 < \inf_{\widetilde{z}\in\widetilde{\Psi}^{-1}(S_{r^{(1)}}(X_0\oplus X_1))} (-\widetilde{I})(\widetilde{z}) \leq \widetilde{c} \leq \sup_{\widetilde{z}\in\widetilde{\Psi}^{-1}(\Delta_{R^{(1)}}(S_1(\rho),X_2\oplus X_3))} (-\widetilde{I})(\widetilde{z}). \tag{3.14}$$

Proof. Let $(k_n)_n$ be a sequence such that $k_n \to +\infty$, $(\widetilde{z_n})_n$ be a sequence in C such that $\widetilde{z_n} \in C_{k_n}$, for all n, $(-\widetilde{I})(\widetilde{z_n}) \to \widetilde{c}$ and $\text{grad}^{-}_{C}(-\widetilde{I})|_{E_{k_n}}(\widetilde{z_n}) \to 0$. Set $z_n = \Psi(\widetilde{z_n})$ (and hence $z_n \in E_{k_n}$) and $(-I)(z_n) \to \widetilde{c}$. We first consider the case in which $z_n \notin X_0 \oplus (X_2 \oplus X_3)$, for all n. Since for n large $P_{E_n} \circ P_{X_1} = P_{X_1} \circ P_{E_n} = P_{X_1}$, we have

$$P_{E_{k_n}}\nabla(-\widetilde{I})(\widetilde{z_n}) = P_{E_{k_n}}\Psi'(\widetilde{z_n})(\nabla(-I)(z_n)) = \Psi'(\widetilde{z_n})(P_{E_{k_n}}\nabla(-I)(z_n)) \longrightarrow 0. \tag{3.15}$$

By (3.9) and (3.10),

$$\begin{gathered}
P_{E_{k_n}}\nabla(-I)z_n \longrightarrow 0 \quad \text{or}\\
P_{X_0\oplus(X_2\oplus X_3)}P_{E_{k_n}}\nabla(-I)(z_n) \longrightarrow 0, \qquad P_{X_1}z_n \longrightarrow 0.
\end{gathered} \tag{3.16}$$

In the first case, the claim follows from the limit Palais-Smale condition for $-I$. In the second case, $P_{X_0\oplus(X_2\oplus X_3)}P_{E_{k_n}}\nabla(-I)(z_n) \to 0$. We claim that $(z_n)_n$ is bounded. By contradiction, we suppose that $\|z_n\| \to +\infty$ and set $w_n = z_n/\|z_n\|$. Up to a subsequence $w_n \rightharpoonup w_0$ weakly for some

$w_0 \in X_0 \oplus (X_2 \oplus X_3)$. By the asymptotically linearity of $\nabla(-I)(z_n)$ we have

$$\left\langle \frac{\nabla(-I)(z_n)}{\|z_n\|}, w_n \right\rangle = \left\langle P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \frac{\nabla(-I)(z_n)}{\|z_n\|}, w_n \right\rangle + \left\langle \frac{\nabla(-I)(z_n)}{\|z_n\|^2}, P_{X_1} z_n \right\rangle \longrightarrow 0. \quad (3.17)$$

We have

$$\left\langle \frac{\nabla(-I)(z_n)}{\|z_n\|}, w_n \right\rangle = \frac{2(-I)(z_n)}{\|z_n\|^2} + \int_0^{2\pi} \left[-\frac{2H(t, z_n)}{\|z_n\|^2} + \frac{H_z(t, z_n) \cdot w_n}{\|z_n\|} \right] dt, \quad (3.18)$$

where $z_n = ((z_n)_1, \dots, (z_n)_{2n})$. Passing to the limit, we get

$$\lim_{n\to\infty} \int_0^{2\pi} \left[\frac{2H(t, z_n)}{\|z_n\|^2} - \frac{H_z(t, z_n) \cdot w_n}{\|z_n\|} \right] dt = 0. \quad (3.19)$$

Since H and $H_z(t, z_n) \cdot z_n$ are bounded and $\|z_n\| \to \infty$ in Ω, $w_0 = 0$. On the other hand, we have

$$\left\langle P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \frac{\nabla(-I)(z_n)}{\|z_n\|}, w_n \right\rangle = \int_0^{2\pi} \left[-\dot{w}_n \cdot J w_n + \left(P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \frac{H_z(t, z_n)}{\|z_n\|} \right) \cdot w_n \right] dt. \quad (3.20)$$

Moreover, we have

$$\begin{aligned} &\left\langle P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \frac{\nabla(-I)(z_n)}{\|z_n\|}, P^+ w_n - P^- w_n \right\rangle \\ &\quad = -\left\| P_{X_2\oplus X_3} P^+ w_n \right\|^2 - \left\| P_{X_0} P^- w_n \right\|^2 - \int_0^{2\pi} P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \frac{H_z(t, z_n)}{\|z_n\|} \cdot (P^+ w_n - P^- w_n) dt. \end{aligned} \quad (3.21)$$

Since w_n converges to 0 weakly and $H_z(t, z_n) \cdot (P^+ w_n - P^- w_n)$ is bounded, $\|P_{X_2\oplus X_3} P^+ w_n\|^2 + \|P_{X_0} P^- w_n\|^2 \to 0$. Since $\|P_{X_1} w_n\|^2 \to 0$, w_n converges to 0 strongly, which is a contradiction. Hence, $(z_n)_n$ is bounded. Up to a subsequence, we can suppose that z_n converges to z_0 for some $z_0 \in X_0 \oplus (X_2 \oplus X_3)$. We claim that z_n converges to z_0 strongly. We have

$$\begin{aligned} &\left\langle P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \nabla(-I) z_n, P^+ z_n - P^- z_n \right\rangle \\ &\quad = -\left\| P_{X_2\oplus X_3} P_{E_{k_n}} P^+ z_n \right\|^2 - \left\| P_{X_0} P_{E_{k_n}} P^- z_n \right\|^2 + P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \int_0^{2\pi} H_z(t, z_n) \cdot (P^+ z_n - P^- z_n). \end{aligned} \quad (3.22)$$

By (H1) and the boundedness of $H_z(t, z_n)(P^+ z_n - P^- z_n)$,

$$\left\| P_{X_2\oplus X_3} P_{E_{k_n}} P^+ z_n \right\|^2 + \left\| P_{X_0} P_{E_{k_n}} P^- z_n \right\|^2 \longrightarrow P_{X_0\oplus(X_2\oplus X_3)} P_{E_{k_n}} \int_0^{2\pi} H_z(t, z) \cdot (P^+ z - P^- z). \quad (3.23)$$

That is, $\|P_{X_2\oplus X_3}P_{E_{k_n}}P^+z_n\|^2 + \|P_{X_0}P_{E_{k_n}}P^-z_n\|^2$ converges. Since $\|P_{X_1}z_n\|^2 \to 0$, $\|z_n\|^2$ converges, so z_n converges to z strongly. Therefore, we have

$$\mathrm{grad}_C^-(-\widetilde{I})(\widetilde{z}) = \mathrm{grad}_C^-(-I)(z) = \lim_{n\to\infty} P_{E_{k_n}}\mathrm{grad}_C^-(-I)(z_n) = \lim_{n\to\infty} P_{E_{k_n}}\mathrm{grad}_C^-(-\widetilde{I})(\widetilde{z_n}) = 0. \tag{3.24}$$

So we proved the first case.

We consider the case $P_{X_1}z_n = 0$, that is, $z_n \in X_0 \oplus (X_2 \oplus X_3)$. Then $\widetilde{z_n} \in \partial C$, for all n. In this case, $z_n = \Psi(\widetilde{z_n}) \in X_0 \oplus (X_2 \oplus X_3)$ and $P_{X_0\oplus(X_2\oplus X_3)}\nabla(-I)(z_n) \to 0$. Thus, by the same argument as the first case, we obtain the conclusion. So we prove the lemma. □

Proposition 3.4. *Assume that H satisfies the conditions (H1), $(H2)'$, (H3), (H4). Then there exists a number $\delta_1 > 0$ such that for any α and β with $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$ and $\alpha > 0$, there exist at least two nontrivial critical points z_i, $i = 1, 2$, in X_1 for the functional I such that*

$$\inf_{z\in\Delta_{R^{(1)}}(S_1(\rho),X_2\oplus X_3)} I(z) \le I(z_i) \le \sup_{z\in S_{r^{(1)}}(X_0\oplus X_1)} I(z) < 0 < \inf_{z\in\Sigma_{R^{(1)}}(S_1(\rho),X_2\oplus X_3)} I(z), \tag{3.25}$$

where ρ, $r^{(1)}$, and $R^{(1)}$ are introduced in Lemma 3.1.

Proof. First, we will find two nontrivial critical points for $-\widetilde{I}$. By Lemma 3.1, $-\widetilde{I}$ satisfies the torus-sphere variational linking inequality, that is, there exist $\delta_1 > 0$, $\rho > 0$, $r^{(1)} > 0$, and $R^{(1)} > 0$ such that $r^{(1)} < R^{(1)}$, and for any α and β with $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$ and $\alpha > 0$

$$\begin{aligned}
\sup_{\widetilde{z}\in\widetilde{\Sigma_{R^{(1)}}}}(-\widetilde{I})(\widetilde{z}) &= \sup_{z\in\Sigma_{R^{(1)}}(S_1(\rho),X_2\oplus X_3)}(-I)(z) < 0 < \inf_{z\in S_{r^{(1)}}(X_0\oplus X_1)}(-I)(z) = \inf_{\widetilde{z}\in\widetilde{S_{r^{(1)}}}}(-\widetilde{I})(\widetilde{z}),\\
\sup_{\widetilde{z}\in\widetilde{\Delta_{R^{(1)}}}}(-\widetilde{I})(\widetilde{z}) &= \sup_{z\in\Delta_{R^{(1)}}(S_1(\rho),X_2\oplus X_3)}(-I)(z) = -\inf_{z\in\Delta_{R^{(1)}}(S_1(\rho)X_2\oplus X_3)} I(z) < \infty,\\
\inf_{\widetilde{z}\in\widetilde{B_{r^{(1)}}}}(-\widetilde{I})(\widetilde{z}) &= \inf_{z\in B_{r^{(1)}}(X_0\oplus X_1)}(-I)(z) = -\sup_{z\in B_{r^{(1)}}(X_0\oplus X_1)} I(z) > -\infty.
\end{aligned} \tag{3.26}$$

By Lemma 3.3, $-\widetilde{I}$ satisfies the $(P.S.)^*_{\widetilde{c}}$ condition with respect to $(\widetilde{C}_n)_n$ for every real number $\widetilde{c}$ such that

$$0 < \inf_{\widetilde{z}\in\widetilde{S_{r^{(1)}}}}(-\widetilde{I})(\widetilde{z}) \le \widetilde{c} \le \sup_{\widetilde{z}\in\widetilde{\Delta_{R^{(1)}}}}(-\widetilde{I})(\widetilde{z}). \tag{3.27}$$

Thus by Theorem 2.5, there exist two critical points $\widetilde{z_1}$, $\widetilde{z_2}$ for the functional $-\widetilde{I}$ such that

$$\inf_{\widetilde{z}\in\widetilde{S_{r^{(1)}}}}(-\widetilde{I})(\widetilde{z}) \le (-\widetilde{I})(\widetilde{z}_i) \le \sup_{\widetilde{z}\in\widetilde{\Delta_{R^{(1)}}}}(-\widetilde{I})(\widetilde{z}), \quad i = 1, 2. \tag{3.28}$$

Setting $z_i = \widetilde{\Psi}(\widetilde{z}_i)$, $i = 1, 2$, we have

$$0 < \inf_{z\in S_{r^{(1)}}}(-I)(z) = \inf_{\widetilde{z}\in\widetilde{S_{r^{(1)}}}}(-\widetilde{I})(\widetilde{z}) \le (-I)(z_1) \le (-I)(z_2) \le \sup_{\widetilde{z}\in\widetilde{\Delta_{R^{(1)}}}}(-\widetilde{I})(\widetilde{z}) = \sup_{z\in\Delta_{R^{(1)}}}(-I)(z). \tag{3.29}$$

We claim that $\tilde{z}_i \notin \partial\tilde{C}$, that is $z_i \notin X_0 \oplus (X_2 \oplus X_3)$, which implies that z_i are the critical points for $-I$ in X_1, so z_i are the critical points for I in X_1. For this we assume by contradiction that $z_i \in X_0 \oplus (X_2 \oplus X_3)$. From (3.12), $P_{X_0 \oplus (X_2 \oplus X_3)} \nabla(-I)(z_i) = 0$, namely, z_i, $i = 1, 2$, are the critical points for $(-I)|_{X_0 \oplus (X_2 \oplus X_3)}$. By Lemma 3.2, $-I(z_i) = 0$, which is a contradiction for the fact that

$$0 < \inf_{z \in S_{r^{(1)}}(X_0 \oplus X_1)} (-I)(z) \le (-I)(z_i) \le \sup_{z \in \Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} (-I)(z). \tag{3.30}$$

Lemma 3.2 implies that there is no critical point $z \in X_0 \oplus (X_2 \oplus X_3)$ such that

$$0 < \inf_{z \in S_{r^{(1)}}(X_0 \oplus X_1)} (-I)(z) \le (-I)(z) \le \sup_{z \in \Delta_{R^{(1)}}(S_1(\rho), X_2 \oplus X_3)} (-I)(z). \tag{3.31}$$

Hence, $z_i \notin X_0 \oplus (X_2 \oplus X_3)$, $i = 1, 2$. This proves Proposition 3.4. □

Lemma 3.5 (second sphere-torus variational linking). *Assume that H satisfies the conditions (H1), (H3), (H4), and the condition*

(H2)″ *suppose that there exist $\gamma > 0$ and $\tau > 0$ such that $j_2 < \gamma < \beta$ and*

$$H(t, z) \ge \frac{1}{2}\gamma \|z\|^2 - \tau \quad \forall (t, z) \in R^1 \times R^{2n}. \tag{3.32}$$

Then there exist $\delta_2 > 0$, $\rho > 0$, $r^{(2)} > 0$, and $R^{(2)} > 0$ such that $r^{(2)} < R^{(2)}$, and for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta_2 < j_2 + 1 = j_3$ and $\alpha > 0$,

$$\begin{gathered} \sup_{z \in S_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)} I(z) < 0 < \inf_{\Sigma_{R^{(2)}}(S_2(\rho), X_3)} I(z), \\ \inf_{z \in \Delta_{R^{(2)}}(S_2(\rho), X_3)} I(z) > -\infty, \qquad \sup_{z \in B_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)} I(z) < \infty. \end{gathered} \tag{3.33}$$

Proof. Let $z = (z_0 + z_1) + z_2 \in (X_0 \oplus X_1) \oplus X_2$. By (H2)″, we have

$$I(z) = \frac{1}{2}\int_0^{2\pi} \dot{z} \cdot Jz\, dt - \int_0^{2\pi} H(t, z(t))dt \le \frac{1}{2}\|z\|^2 - \frac{\gamma}{2}\|z\|_{L^2}^2 + \tau \le \frac{1}{2}(j_2 - \gamma)\|z\|_{L^2}^2 + \tau \tag{3.34}$$

for some τ. Since $j_2 - \gamma < 0$, there exists $r^{(2)} > 0$ such that if $z \in S_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)$, then $I(z) < 0$. Thus we have $\sup_{z \in S_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)} I(z) < 0$. Moreover, if $z \in B_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)$, then $I(z) < \tau < \infty$, so we have $\sup_{z \in B_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)} I(z) < \infty$. Next, let $z = z_2 + z_3 \in X_2 \oplus X_3$ with $z_2 \in S_2(\rho)$, where ρ is a small number. We also let $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta < j_2 + 1 = j_3$ and $\alpha > 0$. Then $X_2 \oplus X_3 \subset E^+$ and $P^-(z_2 + z_3) = 0$. By (H1), there exists $\tau' > 0$ such that

$$\begin{aligned} I(z) &= \frac{1}{2}\int_0^{2\pi} \dot{z} \cdot Jz\, dt - \int_0^{2\pi} H(t, z(t))dt \\ &\ge \frac{1}{2}\left\|P^+(z_2 + z_3)\right\|^2 - \frac{\beta}{2}\left\|P^+(z_2 + z_3)\right\|_{L^2}^2 - \tau' \\ &= \frac{1}{2}\left\|P^+ z_2\right\|^2 + \frac{1}{2}\left\|P^+ z_3\right\|^2 - \frac{\beta}{2}\left\|P^+ z_2\right\|_{L^2}^2 - \frac{\beta}{2}\left\|P^+ z_3\right\|_{L^2}^2 - \tau' \\ &\ge \frac{1}{2}\left(1 - \frac{\beta}{j_2}\right)\rho^2 + \frac{1}{2}(j_3 - \beta)\left\|P^+ z_3\right\|_{L^2}^2 - \tau'. \end{aligned} \tag{3.35}$$

Since $1-\beta/j_2<0$ and $j_3-\beta>0$, there exist a small number $\delta_2>0$ and $R^{(2)}>0$ with $\delta_2<\delta$ and $R^{(2)}>r^{(2)}$ such that if $j_1-1<\alpha<j_1<j_2<\beta<j_2+\delta_2<j_2+1=j_3$ and $z=z_2+z_3\in\Sigma_{R^{(2)}}(S_2(\rho),X_3)$, then $I(z)>0$. Thus we have $\inf_{z\in\Sigma_{R^{(2)}}(S_2(\rho),X_3)}I(z)>0$.

Moreover, if $z\in\Delta_{R^{(2)}}(S_2(\rho),X_3)$, then $I(z)\geq(1/2)(1-\beta/j_2)\rho^2-\tau'>-\infty$. Thus we have $\inf_{\Delta_{R^{(2)}}(S_2(\rho),X_3)}I(z)>-\infty$. Thus we prove the lemma. □

Lemma 3.6. *For any $\Lambda\in]j_2,j_3[$ there exists a constant $\tau>0$ such that for any α and β with $j_1-1<\alpha<j_1<j_2\leq\beta\leq\Lambda<j_2+1=j_3$ and $\alpha>0$, if z is a critical point for $I|_{(X_0\oplus X_1)\oplus X_3}$ with $0\leq I(z)\leq\tau$, then $z=0$.*

Proof. By contradiction, we can suppose that there exist $\Lambda>0$, a sequence $(\alpha_n)_n$, $(\beta_n)_n$ such that $\alpha_n\to\alpha$, $\beta_n\to\beta$ with $\alpha\in]j_1-1,j_1[$, $\beta\in[j_2\cdot\Lambda]$, and a sequence $(z_n)_n$ in $(X_0\oplus X_1)\oplus X_3$ such that $I(z_n)\to 0$ and $P_{(X_0\oplus X_1)\oplus X_3}\nabla I(z_n)=0$. We claim that $(z_n)_n$ is bounded. If we do not suppose that $\|z_n\|\to+\infty$, let us set $w_n=z_n/\|z_n\|$. We have up to a subsequence, that $w_n\rightharpoonup w_0$ weakly for some $w_0\in(X_0\oplus X_1)\oplus X_3$. Furthermore,

$$0=\left\langle\nabla I(z_n),P_{X_0\oplus X_1}z_n\right\rangle=\left\|P^+P_{X_0\oplus X_1}z_n\right\|^2-\left\|P^-P_{X_0\oplus X_1}z_n\right\|^2-\left\langle H_z(t,z_n),P_{X_0\oplus X_1}z_n\right\rangle,\tag{3.36}$$

so we have

$$\left\|P_{X_0\oplus X_1}z_n\right\|^2=\left\langle H_z(t,z_n),P_{X_0\oplus X_1}z_n\right\rangle.\tag{3.37}$$

Moreover,

$$0=\left\langle\nabla I(z_n),P_{X_3}z_n\right\rangle=\|P_{X_3}z_n\|^2-\left\langle H_z(t,z_n),P_{X_3}z_n\right\rangle,\tag{3.38}$$

so we have

$$\left\|P_{X_3}z_n\right\|^2=\left\langle H_z(t,z_n),P_{X_{3}z_n}\right\rangle.\tag{3.39}$$

Adding (3.37) and (3.39), we have

$$\left\|z_n\right\|^2=\left\langle H_z(t,z_n),z_n\right\rangle.\tag{3.40}$$

From (3.40) we have

$$\|w_0\|^2=\lim_{n\to\infty}\left\langle H_z(t,z_n),w_n\right\rangle.\tag{3.41}$$

We also have

$$0=\left\langle P_{(X_0\oplus X_1)\oplus X_3}\nabla I(z_n),z_n\right\rangle=2I(z_n)+\int_0^{2\pi}\left[-2H(t,z_n)+H_z(t,z_n)\cdot z_n\right]dt.\tag{3.42}$$

Dividing by $\|z_n\|$ and going to the limit, we have

$$\lim_{n\to\infty}\int_0^{2\pi}H_z(t,z_n)\cdot w_n=0.\tag{3.43}$$

Thus

$$\left\|w_0\right\|^2=0,\tag{3.44}$$

which is a contradiction since $\|w_0\|=1$. So $(z_n)_n$ is bounded and we can suppose that $z_n\rightharpoonup z$ for $z\in(X_0\oplus X_1)\oplus X_3$. From (3.42), we have

$$\left\langle H_z(t,z_n),z_n\right\rangle=\int_0^{2\pi}2H(t,z_n)dt.\tag{3.45}$$

From (3.40),

$$\lim_{n\to\infty}\|z_n\|^2=\lim_{n\to\infty}\langle H_z(t,z_n),z_n\rangle=\lim_{n\to\infty}\int_0^{2\pi}2H(t,z_n)dt=\int_0^{2\pi}2H(t,z)dt. \tag{3.46}$$

Thus, z_n converges to z strongly. We claim that $z=0$. Assume that $z\neq 0$. By (H1) $\alpha\|z\|_{L^2}^2+c_1<2\int_0^{2\pi}H(t,z)dt<\beta\|z\|_{L^2}^2+c_2$, for some c_1 and c_2. If $z\in X_0\oplus X_1$ with $\|P_{X_0\oplus X_1}z\|^2\geq |j|\|z\|_{L^2}^2$ for $j<0$ and $|j|>\beta$,

$$|j|\left\|P_{X_0\oplus X_1}z\right\|_{L^2}^2\leq \left\|P_{X_0\oplus X_1}z\right\|^2\leq \beta\left\|P_{X_0\oplus X_1}z\right\|_{L^2}^2+c_2. \tag{3.47}$$

If $z\in X_3$, $\|P_{X_3}z\|^2\geq j_3\|P_{X_3}z\|_{L^2}^2$, and

$$j_3\left\|P_{X_3}z\right\|_{L^2}^2\leq \left\|P_{X_3}z\right\|^2\leq \beta\left\|P_{X_3}z\right\|_{L^2}^2+c_2. \tag{3.48}$$

Thus, we have

$$(|j|-\beta)\left\|P_{X_0\oplus X_1}z\right\|_{L^2}^2+(j_3-\beta)\left\|P_{X_3}z\right\|_{L^2}^2-2c_2\leq 0, \tag{3.49}$$

which is absurd because of $|j|>\beta$ and $j_3>\beta$. Thus $z=0$. We proved the lemma. □

Let P_{X_2} be the orthogonal projection from E onto X_2 and

$$\check{C}=\left\{z\in E \mid \left\|P_{X_2}z\right\|\geq 1\right\}. \tag{3.50}$$

Then $\check{C}$ is the smooth manifold with boundary. Let $\check{C}_n=\check{C}\cap E_n$. Let us define a functional $\check{\Psi}:E\setminus\{(X_0\oplus X_1)\oplus X_3\}\to E$ by

$$\check{\Psi}(z)=z-\frac{P_{X_2}z}{\left\|P_{X_2}z\right\|}=P_{(X_0\oplus X_1)\oplus X_3}z+\left(1-\frac{1}{\left\|P_{X_2}z\right\|}\right)P_{X_2}z. \tag{3.51}$$

We have

$$\nabla\check{\Psi}(z)(w)=w-\frac{1}{\left\|P_{X_2}z\right\|}\left(P_{X_2}w-\left\langle\frac{P_{X_2}z}{\left\|P_{X_2}z\right\|},w\right\rangle\frac{P_{X_2}z}{\left\|P_{X_2}z\right\|}\right). \tag{3.52}$$

Let us define the functional $\check{I}:\check{C}\to R$ by

$$\check{I}=I\circ\check{\Psi}. \tag{3.53}$$

Then $\check{I}\in C_{\text{loc}}^{1,1}$. We note that if $\check{z}$ is the critical point of $\check{I}$ and lies in the interior of $\check{C}$, then $z=\check{\Psi}(\check{z})$ is the critical point of I. We also note that

$$\left\|\operatorname{grad}_{\check{C}}^{-}\check{I}(\check{z})\right\|\geq \left\|P_{(X_0\oplus X_1)\oplus X_3}\nabla I\left(\check{\Psi}(\check{z})\right)\right\| \quad \forall\check{z}\in\partial\check{C}. \tag{3.54}$$

Let us set

$$\begin{aligned}\check{S}_{r^{(2)}} &= \check{\Psi}^{-1}\left(S_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)\right),\\ \check{B}_{r^{(2)}} &= \check{\Psi}^{-1}\left(B_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)\right),\\ \check{\Sigma}_{R^{(2)}} &= \check{\Psi}^{-1}\left(\Sigma_{R^{(2)}}(S_2(\rho), X_3)\right),\\ \check{\Delta}_{R^{(2)}} &= \check{\Psi}^{-1}\left(\Delta_{R^{(2)}}(S_2(\rho), X_3)\right).\end{aligned} \tag{3.55}$$

We note that $\check{S}_{r^{(2)}}, \check{B}_{r^{(2)}}, \check{\Sigma}_{R^{(2)}}$, and $\check{\Delta}_{R^{(2)}}$ have the same topological structure as $S_{r^{(2)}}$, $B_{r^{(2)}}$, $\Sigma_{R^{(2)}}$, and $\Delta_{R^{(2)}}$, respectively.

We have the following lemma whose proof has the same arguments as that of Lemma 3.5 except the space $(X_0 \oplus X_1) \oplus X_3$, $X_0 \oplus X_1$, X_3 instead of the space $X_0 \oplus (X_2 \oplus X_3)$, X_0, $X_2 \oplus X_3$.

Lemma 3.7. *$-\check{I}$ satisfies the $(P.S.)^*_{\check{c}}$ condition with respect to $(\check{C}_n)_n$ for every real number $\check{c}$ such that*

$$0 < \inf_{\check{z}\in\check{\Psi}^{-1}(S_{r^{(2)}}(X_0\oplus X_1\oplus X_2))} (-\widetilde{I})(\check{z}) \le \check{c} \le \sup_{\check{z}\in\check{\Psi}^{-1}(\Delta_{R^{(2)}}(S_2(\rho),X_3))} (-\widetilde{I})(\check{z}), \tag{3.56}$$

where ρ, $r^{(2)}$, and $R^{(2)}$ are introduced in Lemma 3.5.

Proposition 3.8. *Assume that H satisfies the conditions (H1), (H2)″, (H3), and (H4). Then there exists a small number $\delta_2 > 0$ such that for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta < j_2 + 1 = j_3$ and $\alpha > 0$, there exist at least two nontrivial critical points w_i, $i = 1, 2$, in X_2 for the functional I such that*

$$\inf_{z\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} I(z) \le I(w_i) \le \sup_{z\in S_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} I(z) < 0 < \inf_{z\in\Sigma_{R^{(2)}}(S_2(\rho),X_3)} I(z), \tag{3.57}$$

where ρ, $r^{(2)}$, and $R^{(2)}$ are introduced in Lemma 3.5.

Proof. It suffices to find the critical points for $-\check{I}$. By Lemma 3.5, $-\check{I}$ satisfies the torus-sphere variational linking inequality, that is, there exist $\delta_2 > 0$, $\rho > 0$, $r^{(2)} > 0$, and $R^{(2)} > 0$ such that $r^{(2)} < R^{(2)}$, and for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta_2 < j_2 + 1 = j_3$,

$$\begin{aligned}\sup_{\check{z}\in\check{\Sigma}_{R^{(2)}}} (-\widetilde{I})(\check{z}) &= \sup_{z\in\Sigma_{R^{(2)}}(S_2(\rho),X_3)} (-I)(z) < 0 < \inf_{z\in S_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} (-I)(z) = \inf_{\check{z}\in\check{S}_{r^{(2)}}} (-\widetilde{I})(\check{z}),\\ \sup_{\check{z}\in\check{\Delta}_{R^{(2)}}} (-\widetilde{I})(\check{z}) &= \sup_{z\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} (-I)(z) = -\inf_{z\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} I(z) < \infty,\\ \inf_{\check{z}\in\check{B}_{r^{(2)}}} (-\widetilde{I})(\check{z}) &= \inf_{z\in B_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} (-I)(z) = -\sup_{z\in B_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} I(z) > -\infty.\end{aligned} \tag{3.58}$$

By Lemma 3.7, $-\check{I}$ satisfies the $(P.S.)^*_{\check{c}}$ condition with respect to $(\check{C}_n)_n$ for every real number $\check{c}$ such that

$$0 < \inf_{\check{z}\in\check{S}_{r^{(2)}}} (-\widetilde{I})(\check{z}) \le \check{c} \le \sup_{\check{z}\in\check{\Delta}_{R^{(2)}}} (-\widetilde{I})(\check{z}). \tag{3.59}$$

Then by Theorem 2.5, there exist two critical points $\breve{w}_1$, $\breve{w}_2$ for the functional $-\check{I}$ such that

$$\inf_{\breve{w}\in \check{S}_{r^{(2)}}} (-\widetilde{I})(\breve{w}) \le (-\widetilde{I})(\breve{w}_i) \le \sup_{\breve{w}\in\check{\Delta}_{R^{(2)}}} (-\widetilde{I})(\breve{w}), \quad i = 1,2. \tag{3.60}$$

Setting $w_i = \check{\Psi}(\breve{w}_i)$, $i = 1,2$, we have

$$0 < \inf_{w\in S_{r^{(2)}}} (-I)(w) = \inf_{\breve{w}\in\check{S}_{r^{(2)}}} (-\widetilde{I})(\breve{w}) \le (-I)(w_1) \le -I(w_2) \le \sup_{\breve{w}\in\check{\Delta}_{R^{(2)}}} (-\widetilde{I})(\breve{w}) = \sup_{w\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} (-I)(w). \tag{3.61}$$

We claim that $\breve{w}_i \notin \partial\check{C}$, that is $w_i \notin (X_0 \oplus X_1) \oplus X_3$, which implies that w_i are the critical points for $-I$, so w_i are the critical points for I. For this we assume by contradiction that $w_i \in (X_0 \oplus X_1) \oplus X_3$. From (3.54), $P_{(X_0\oplus X_1)\oplus X_3}\nabla(-I)(w_i) = 0$, namely, w_i, $i = 1,2$, are the critical points for $(-I)|_{(X_0\oplus X_1)\oplus X_3}$. By Lemma 3.6, $-I(w_i) = 0$, which is a contradiction for the fact that

$$0 < \inf_{w\in S_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} (-I)(w) \le (-I)(w_i) \le \sup_{w\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} (-I)(w). \tag{3.62}$$

It follows from Lemma 3.6 that there is no critical point $w \in (X_0 \oplus X_1) \oplus X_3$ such that

$$0 < \inf_{w\in S_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} (-I)(w) \le (-I)(w) \le \sup_{w\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} (-I)(w). \tag{3.63}$$

Hence, $w_i \notin (X_0 \oplus X_1) \oplus X_3$, $i = 1,2$. This proves Proposition 3.8. □

Proof of Theorem 1.1. Assume that H satisfies conditions (H1)–(H4). By Proposition 3.4, there exist $\delta_1 > 0$, $\rho > 0$, $r^{(1)} > 0$, and $R^{(1)} > 0$ such that for any α and β with $j_1 - 1 < \alpha < j_1 < \beta < j_2 + \delta_1 < j_2 + 1 = j_3$, (1.1) has at least two nontrivial solutions z_i, $i = 1,2$, in X_1 for the functional I such that

$$\inf_{z\in\Delta_{R^{(1)}}(S_1(\rho),X_2\oplus X_3)} I(z) \le I(z_i) \le \sup_{z\in S_{r^{(1)}}(X_0\oplus X_1)} I(z) < 0 < \inf_{z\in\Sigma_{R^{(1)}}(S_1(\rho),X_2\oplus X_3)} I(z). \tag{3.64}$$

By Proposition 3.8, there exist $\delta_2 > 0$, $\rho > 0$, $r^{(2)} > 0$, and $R^{(2)} > 0$ such that for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta_2 < j_2 + 1 = j_3$ and $\alpha > 0$, (1.1) has at least two nontrivial solutions w_i, $i = 1,2$, in X_2 for the functional I such that

$$\inf_{z\in\Delta_{R^{(2)}}(S_2(\rho),X_3)} I(z) \le I(w_i) \le \sup_{z\in S_{r^{(2)}}(X_0\oplus X_1\oplus X_2)} I(z) < 0 < \inf_{z\in\Sigma_{R^{(2)}}(S_2(\rho),X_3)} I(z). \tag{3.65}$$

Let

$$\delta = \min\{\delta_1, \delta_2\}. \tag{3.66}$$

Then for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta < j_2 + 1 = j_3$ and $\alpha > 0$, (1.1) has at least four nontrivial solutions, two of which are in X_1 and two of which are in X_2. □

4. Proof of Theorem 1.2

Assume that H satisfies conditions (H1)–(H4) with $\alpha < \beta < 0$. Let us set

$$\begin{aligned}
X_0 &\equiv \mathrm{span}\{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid j \geq -j_1 + 1,\ j \in N,\ 1 \leq k \leq n\},\\
X_1 &\equiv \mathrm{span}\{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid j \geq -j_1,\ j \in N,\ 1 \leq k \leq n\},\\
X_2 &\equiv \mathrm{span}\{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid j = -j_2,\ j \in N,\ 1 \leq k \leq n\},\\
X_3 &\equiv \mathrm{span}\left\{\{e_1, e_2, \ldots, e_{2n}, (\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid j > 0,\ j \in N,\ 1 \leq k \leq n\}\right.\\
&\quad \left.\cup \{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid j \leq -j_2 - 1 = -j_3,\ j \in N,\ 1 \leq k \leq n\}\right\}.
\end{aligned} \tag{4.1}$$

Then the space E is the topological direct sum of the subspaces X_0, X_1, X_2, and X_3, where X_1 and X_2 are finite dimensional subspaces.

Proof of Theorem 1.2. By the same arguments as that of the proof of Theorem 1.1, there exist $\delta > 0$, $\rho > 0$, $r^{(1)} > 0$, $R^{(1)}$, $r^{(2)} > 0$, and $R^{(2)} > 0$ such that for any α and β with $j_1 - 1 < \alpha < j_1 < j_2 < \beta < j_2 + \delta$, (1.1) has at least four nontrivial solutions, two of which are nontrivial solutions z_i, $i = 1, 2$, in X_1 with

$$\inf_{z \in \Delta_{R^{(1)}}(S_{12}(\rho), X_3)} I(z) \leq I(z_i) \leq \sup_{z \in S_{r^{(1)}}(X_0 \oplus X_1)} I(z) < 0 < \inf_{z \in \Sigma_{R^{(1)}}(S_{12}(\rho), X_3)} I(z), \tag{4.2}$$

and two of which are nontrivial solutions w_i, $i = 1, 2$,in X_2 with

$$\inf_{z \in \Delta_{R^{(2)}}(S_2(\rho), X_3)} I(z) \leq I(w_i) \leq \sup_{z \in S_{r^{(2)}}(X_0 \oplus X_1 \oplus X_2)} I(z) < 0 < \inf_{z \in \Sigma_{R^{(2)}}(S_2(\rho), X_3)} I(z). \tag{4.3}$$

□

Acknowledgment

This research is supported in part by Inha University research grant.

References

[1] K.-C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, vol. 6 of *Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser, Boston, Mass, USA, 1993.

[2] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[3] G. Fournier, D. Lupo, M. Ramos, and M. Willem, "Limit relative category and critical point theory," in *Dynamics Reported*, vol. 3, pp. 1–24, Springer, Berlin, Germany, 1994.

[4] A. M. Micheletti and C. Saccon, "Multiple nontrivial solutions for a floating beam equation via critical point theory," *Journal of Differential Equations*, vol. 170, no. 1, pp. 157–179, 2001.