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*Research Article*

# Existence of Three Monotone Solutions of Nonhomogeneous Multipoint BVPs for Second-Order Differential Equations

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This paper is concerned with nonhomogeneous multipoint boundary value problems of second-order differential equations with one-dimensional $p$-Laplacian. Sufficient conditions to guarantee the existence of at least three solutions (may be not positive) of these BVPs are established.



## 1. Introduction

In recent years, there are several papers concerned with the existence of positive solutions of BVPs for differential equations with nonhomogeneous BCs. Kwong and Wong in [1] studied the following BVP:

$$\begin{gathered} y''(t) = -f\big(t, y(t)\big), \quad 0 < t < 1, \\ \sin\theta y(0) - \cos\theta y'(0) = 0, \\ y(1) - \sum_{i=1}^{m-2} \alpha_i y\big(\xi_i\big) = b \geq 0, \end{gathered} \tag{1.1}$$

where $\xi_i \in (0,1)$, $\alpha_i \geq 0$, $\theta \in [0, 3\pi/4]$, $f$ is a nonnegative and continuous function. Under some assumptions, it was proved that there exists a constant $b^* > 0$ such that

(i) BVP(1.1) has at least two positive solutions if $b \in (0, b^*)$;

(ii) BVP(1.1) has at least one solution if $b = 0$ or $b = b^*$;

(iii) BVP(1.1) has no positive solution if $b > b^*$.

Sun et al. in [2] studied the existence of positive solutions for the following three-point boundary value problem:

$$u''(t) + a(t)f(u(t)) = 0, \quad 0 \le t \le 1,$$
$$u'(0) = 0, \tag{1.2}$$
$$u(1) - \sum_{i=1}^{m-2} \alpha_i u(\xi_i) = b \ge 0,$$

where $\xi_i \in (0,1)$, $\alpha_i \ge 0$ are given. It was proved that there exists $b^* > 0$ such that BVP(1.2) has at least one positive solution if $b \in (0, b^*)$ and no positive solution if $b > b^*$. To study the existence of positive solutions of above BVPs, the Green's functions of the corresponding problems are established and play an important role in the proofs of the main results.

For the following multipoint boundary value problems

$$\begin{aligned} &x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ &x'(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = \lambda_1, \\ &x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \\ &x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ &x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = \lambda_1, \\ &x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{aligned} \tag{1.3}$$

in papers [3–5], sufficient conditions are found for the existence of solutions of BVP(1.3) based on the existence of lower and upper solutions with certain relations. Using the obtained results, under some other assumptions, the explicit ranges of values of $\lambda_1$ and $\lambda_2$ are presented with which BVP has a solution, has a positive solution, and has no solution, respectively. Furthermore, it is proved that the whole plane for $\lambda_1$ and $\lambda_2$ can be divided into two disjoint connected regions $\wedge E$ and $\wedge N$ such that BVP has a solution for $(\lambda_1, \lambda_2) \in \wedge E$ and has no solution for $(\lambda_1, \lambda_2) \in \wedge N$.

In a recent paper [6], Liu, by using the Schauder fixed point theorem and imposing growth conditions on $f$, obtained at least one positive solution of the following BVPs:

$$\begin{aligned} &[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ &x'(0) = \sum_{i=1}^{m} \alpha_i x'(\xi_i) + A, \\ &x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i) + B, \\ &[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ &x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i) + A, \\ &x'(1) = \sum_{i=1}^{m} \beta_i x'(\xi_i) + B. \end{aligned} \tag{1.4}$$

Motivated by the results obtained in the papers, this paper is concerned with the following BVPs for differential equation with $p$-Laplacian coupled with nonhomogeneous multipoint BCs, that is, the BVPs

$$
\begin{gathered}
[\phi(x'(t))]' + q(t)f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\
x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i) + A, \\
x'(1) = \sum_{i=1}^{m} \beta_i x'(\xi_i),
\end{gathered}
\tag{1.5}
$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $A \in R$, $\alpha_i \geq 0$, $\beta_i \geq 0$ for all $i = 1,\ldots,m$, $f : [0,1] \times R^2 \to R$ is continuous and nonnegative, $q : (0,1) \to [0,+\infty)$ is continuous with $\int_0^1 q(u)du < +\infty$, $\phi$ is called $p$-Laplacian, $\phi(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x)$.

Suppose

$(H_1)$ $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous with $f(t,c+h,0) \not\equiv 0$ on each subinterval of $[0,1]$ for all $c \geq 0$, where $h = A/1 - \sum_{i=1}^{m}\alpha_i$;

$(H_2)$ $A \geq 0$;

$(H_3)$ $\alpha_i \geq 0$, $\beta_i \geq 0$ satisfy $\sum_{i=1}^{m}\alpha_i < 1$, $\sum_{i=1}^{m}\beta_i < 1$ and there exists a constant $\sigma > 0$ such that $\phi^{-1}(1 + (1/\sigma))\sum_{i=1}^{m}\beta_i < 1$.

The purpose is to establish sufficient conditions for the existence of at least three solutions of BVP(1.5). It is proved that BVP(1.5) has three monotone solutions under the growth conditions imposed on $f$ for all $A \in R$. These solutions may not be positive. The proofs of the main results are proved by using fixed point theorem in cones in Banach spaces, Green's functions and the existence of upper and lower solutions are not used in this paper.

The remainder of this paper is organized as follows. The main results are given in Section 2 and an example to show the main results is given in Section 3.

## 2. Main Results

In this section, we first present some background definitions in Banach spaces and state an important three fixed point theorem. Then the main results are given and proved.

*Definition 2.1.* Let $X$ be a semi-ordered real Banach space. The nonempty convex closed subset $P$ of $X$ is called a cone in $X$ if $ax \in P$ for all $x \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

*Definition 2.2.* A map $\psi : P \to [0,+\infty)$ is a nonnegative continuous concave or convex functional map provided $\psi$ is nonnegative and continuous and satisfies

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y), \tag{2.1}$$

or

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y), \tag{2.2}$$

for all $x, y \in P$ and $t \in [0,1]$.

*Definition 2.3.* An operator $T;\ X \to X$ is completely continuous if it is continuous and maps bounded sets into relative compact sets.

*Definition 2.4.* Let $a_1, a_2, a_3, a_4, a_5 > 0$ be positive constants, $\alpha_1,\ \alpha_2$ be two nonnegative continuous concave functionals on cone $P$, $\beta_1,\ \beta_2,\ \beta_3$ be three nonnegative continuous convex functionals on cone $P$. Define the convex sets as follows:

$$\begin{aligned}
P_c &= \{x \in P : \|x\| < a_5\},\\
P(\beta_1, \alpha_1; a_2, a_5) &= \{x \in P : \alpha_1(x) \ge a_2,\ \beta_1(x) \le a_5\},\\
P(\beta_1, \beta_3, \alpha_1; a_2, a_3, a_5) &= \{x \in P : \alpha_1(x) \ge a_2,\ \beta_3(x) \le a_3,\ \beta_1(x) \le a_5\},\\
Q(\beta_1, \beta_2; a_1, a_5) &= \{x \in P : \beta_2(x) \le a_1,\ \beta_1(x) \le a_5\},\\
Q(\beta_1, \beta_2, \alpha_2; a_4, a_1, a_5) &= \{x \in P : \alpha_2(x) \ge a_4,\ \beta_2(x) \le a_1,\ \beta_1(x) \le a_5\}.
\end{aligned} \tag{2.3}$$

**Lemma 2.5** (see [7]). *Let $X$ be a semi-ordered real Banach space with the norm $\|\cdot\|$, let $P$ be a cone in $X$, let $\alpha_1,\ \alpha_2$ be two nonnegative continuous concave functionals on cone $P$, let $\beta_1,\ \beta_2,\ \beta_3$ be three nonnegative continuous convex functionals on cone $P$. There exists constant $M > 0$ such that*

$$\alpha_1(x) \le \beta_2(x), \quad \|x\| \le M\beta_1(x) \quad \forall x \in P. \tag{2.4}$$

*Furthermore, suppose that $a_1, a_2, a_3, a_4, a_5 > 0$ are constants with $a_1 < a_2$. Let $T : \overline{P_{a_5}} \to \overline{P_{a_5}}$ be a completely continuous operator. If*

*($C_1$) $\{y \in P(\beta_1, \beta_3, \alpha_1; a_2, a_3, a_5) \mid \alpha_1(x) > a_2\} \ne \varnothing$ and*

$$\alpha_1(Tx) > a_2 \quad \text{for every } x \in P(\beta_1, \beta_3, \alpha_1; a_2, a_3, a_5); \tag{2.5}$$

*($C_2$) $\{y \in Q(\beta_1, \beta_3, \alpha_2; a_4, a_1, a_5) \mid \beta_2(x) < a_1\} \ne \varnothing$ and*

$$\beta_2(Tx) < a_1 \quad \text{for every } x \in Q(\beta_1, \beta_3, \alpha_2; a_4, a_1, a_5); \tag{2.6}$$

*($C_3$) $\alpha_1(Ty) > a_2$ for $y \in P(\beta_1, \alpha_1; a_2, a_5)$ with $\beta_3(Ty) > a_3$;*

*($C_4$) $\beta_2(Tx) < a_1$ for each $x \in Q(\beta_1, \beta_2; a_1, a_5)$ with $\alpha_2(Tx) < a_4$, then $T$ has at least three fixed points $y_1$, $y_2$, and $y_3$ such that*

$$\beta_2(y_1) < a_1, \qquad \alpha_1(y_2) > a_2, \qquad \beta_2(y_3) > a_1, \qquad \alpha_1(y_3) < a_2. \tag{2.7}$$

*Choose $X = C^1[0,1]$. We call $x \le y$ for $x, y \in X$ if $x(t) \le y(t)$ for all $t \in [0,1]$, define the norm $\|x\| = \max\{\max_{t\in[0,1]}|x(t)|,\ \max_{t\in[0,1]}|x'(t)|\}$ for $x \in X$. It is easy to see that $X$ is a semi-ordered real Banach space.*

*Choose $k \in (0, 1/2)$. For a cone $P \subseteq X$ of the Banach space $X = C^1[0,1]$, define the functionals on $P : P \to R$ by*

$$\begin{aligned}
\beta_1(y) &= \max_{t\in[0,1]}|y'(t)|, \quad y \in P,\\
\beta_2(y) &= \max_{t\in[0,1]}|y(t)|, \quad y \in P,\\
\beta_3(y) &= \max_{t\in[k,1-k]}|y(t)|, \quad y \in P,\\
\alpha_1(y) &= \min_{t\in[k,1-k]}|y(t)|, \quad y \in P,\\
\alpha_2(y) &= \min_{t\in[k,1-k]}|y(t)|, \quad y \in P.
\end{aligned} \tag{2.8}$$

*It is easy to see that $\alpha_1$, $\alpha_2$ are two nonnegative continuous concave functionals on the cone $P$, $\beta_1$, $\beta_2$, $\beta_3$ are three nonnegative continuous convex functionals on cone $P$ and $\alpha_1(y) \leq \beta_2(y)$ for all $y \in P$.*

**Lemma 2.6.** *Suppose that $x \in X$, $x(t) \geq 0$ for all $t \in [0,1]$ and $x'(t)$ is decreasing on $[0,1]$. Then*

$$x(t) \geq \min\{t,\ 1-t\} \max_{t\in[0,1]} x(t), \quad t \in [0,1]. \tag{2.9}$$

*Proof.* Suppose that $x(t_0) = \max_{t\in[0,1]} x(t)$. If $t \in (0,t_0)$, we get that there exists $0 \leq \eta \leq t \leq \xi \leq t_0$ such that

$$\begin{aligned}\frac{x(t)-x(0)}{t-0} - \frac{x(t_0)-x(0)}{t_0-0} &= -\frac{t\left[x(t_0)-x(t)\right]-(t_0-t)\left[x(t)-x(0)\right]}{tt_0} \\ &= -\frac{t(t_0-t)x'(\xi)-(t_0-t)tx'(\eta)}{tt_0} \\ &\geq -\frac{t(t_0-t)x'(\eta)-(t_0-t)tx'(\eta)}{tt_0} = 0.\end{aligned} \tag{2.10}$$

Then

$$x(t) \geq \frac{t}{t_0}x(t_0) + \left(1-\frac{t}{t_0}x(0)\right) \geq \frac{t}{t_0}x(t_0) \geq tx(t_0), \quad t \in (0,t_0). \tag{2.11}$$

Similarly we can get that

$$x(t) \geq (1-t)x(t_0), \quad t \in (t_0,1). \tag{2.12}$$

It follows that $x(t) \geq \min\{t,1-t\}\max_{t\in[0,1]} x(t)$ for all $t \in [0,1]$. The proof is complete. □

Consider the following BVP:

$$\begin{gathered}\left[\phi(y'(t))\right]' + h(t) = 0, \quad t \in (0,1), \\ y(0) - \sum_{i=1}^{m} \alpha_i y(\xi_i) = 0, \\ y'(1) - \sum_{i=1}^{m} \beta_i y'(\xi_i) = 0,\end{gathered} \tag{2.13}$$

**Lemma 2.7.** *Suppose that $h$ is a nonnegative continuous function, $(H_2)$ and $(H_3)$ hold. If $y$ is a solution of BVP(2.13), then $y$ is increasing and positive on $(0,1)$.*

*Proof.* Suppose that $y$ satisfies (2.13). It follows from the assumptions that $y'$ is decreasing on $[0,1]$. Then the BCs in (2.13) and $(H_2)$ imply that

$$y'(1) = \sum_{i=1}^{m} \beta_i y'(\xi_i) \geq \sum_{i=1}^{m} \beta_i y'(1). \tag{2.14}$$

It follows that $y'(1) \geq 0$. We get that $y'(t) \geq 0$ for $t \in [0,1]$. Then

$$y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i) \geq \sum_{i=1}^{m} \alpha_i y(0). \tag{2.15}$$

It follows that $y(0) \geq 0$. Then $y(t) > y(0) \geq 0$ for $t \in (0,1)$ since $y'(t) \geq 0$ for all $t \in [0,1]$. The proof is complete. □

**Lemma 2.8.** *Suppose that $h$ is a nonnegative continuous function, $(H_2)$ and $(H_3)$ hold. If $y$ is a solution of BVP*(2.13), *then*

$$y(t) = B_h + \int_0^t \phi^{-1}\left(A_h + \int_s^1 h(u)du\right)ds, \tag{2.16}$$

*and $A_h \in [0, \sigma\int_0^1 h(u)du]$ satisfies*

$$\phi^{-1}(A_h) = \sum_{i=1}^{m} \beta_i \phi^{-1}\left(A_h + \int_{\xi_i}^1 h(s)ds\right), \tag{2.17}$$

*and $B_h$ satisfies*

$$B_h = \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi^{-1}\left(A_h + \int_s^1 h(u)du\right)ds. \tag{2.18}$$

*Proof.* It follows from (2.13) that

$$y(t) = y(0) + \int_0^t \phi^{-1}\left(\phi(y'(1)) + \int_s^1 h(u)du\right)ds, \tag{2.19}$$

and the BCs in (2.13) imply that

$$\begin{aligned} y'(1) &= \sum_{i=1}^{m} \beta_i \phi^{-1}\left(\phi(y'(1)) + \int_{\xi_i}^1 h(s)ds\right), \\ y(0) &= \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi^{-1}\left(\phi(y'(1)) + \int_s^1 h(u)du\right)ds. \end{aligned} \tag{2.20}$$

Let

$$G(c) = \phi^{-1}(c) - \sum_{i=1}^{m} \beta_i \phi^{-1}\left(c + \int_{\xi_i}^1 h(s)ds\right). \tag{2.21}$$

It is easy to see that $G(0) \le 0$. On the other hand, it follows from $(H_3)$ that $\phi^{-1}(1 + (1/\sigma))\sum_{i=1}^m \beta_i < 1$, one sees that

$$\begin{aligned} \frac{G(\sigma\int_0^1 h(u)du)}{\phi^{-1}(\sigma\int_0^1 h(u)du)} &= 1 - \sum_{i=1}^{m} \beta_i \phi^{-1}\left(1 + \frac{\int_{\xi_i}^1 h(s)ds}{\sigma\int_0^1 h(u)du}\right) \\ &\ge 1 - \sum_{i=1}^{m} \beta_i \phi^{-1}\left(1 + \frac{1}{\sigma}\right) \\ &\ge 0. \end{aligned} \tag{2.22}$$

Hence $G(\sigma\int_0^1 h(u)du) \ge 0$. Since $G(x)$ is increasing for $x \in R$, we get that there exists unique constant $A_h = \phi(y(1)) \in [0, \sigma\int_0^1 h(u)du]$ such that (2.17) holds. The proof is completed. □

Note $h = A/1 - \sum_{i=1}^{m}\alpha_i$, and let $x(t) = y(t) + h$. Then BVP(1.5) is transformed into the following BVP:

$$\begin{gathered} \left[\phi(y'(t))\right]' + f(t, y(t) + h, y'(t)) = 0, \quad t \in (0,1), \\ y(0) - \sum_{i=1}^{m}\alpha_i y(\xi_i) = 0, \\ y'(1) - \sum_{i=1}^{m}\beta_i y'(\xi_i) = 0. \end{gathered} \tag{2.23}$$

Let

$$P = \left\{ y \in X : \begin{array}{l} y(t) \geq 0, \quad \forall t \in [0,1], \\ y'(t) \geq 0 \text{ is decreasing on } [0,1], \\ y(t) \geq \min\{t, (1-t)\} \max_{t\in[0,1]} y(t), \quad \forall t \in [0,1], \\ y(0) - \sum_{i=1}^{m}\alpha_i y(\xi_i) = 0, \\ y'(1) - \sum_{i=1}^{m}\beta_i y'(\xi_i) = 0 \end{array} \right\}. \tag{2.24}$$

Then $P$ is a cone in $X$.

Since

$$|y(0)| = \left|\frac{\sum_{i=1}^{m}\alpha_i y(\xi_i) - \sum_{i=1}^{m}\alpha_i y(0)}{1 - \sum_{i=1}^{m}\alpha_i}\right| \leq \frac{\sum_{i=1}^{m}\alpha_i\xi_i}{1 - \sum_{i=1}^{m}\alpha_i} \max_{t\in[0,1]}|y'(t)| = \frac{\sum_{i=1}^{m}\alpha_i\xi_i}{1 - \sum_{i=1}^{m}\alpha_i}\gamma(y), \tag{2.25}$$

we get that

$$\max_{t\in[0,1]}|y(t)| = y(1) = \int_0^1 y'(s)ds + y(0) \leq \left(1 + \frac{\sum_{i=1}^{m}\alpha_i\xi_i}{1 - \sum_{i=1}^{m}\alpha_i}\right)\gamma(y). \tag{2.26}$$

It is easy to see that there exists a constant $M > 0$ such that $\|y\| \leq M\gamma(y)$ for all $y \in P$.

Define the nonlinear operator $T : P \to X$ by

$$(Ty)(t) = B_y + \int_0^t \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\right)ds, \quad y \in P, \tag{2.27}$$

where $A_y$ satisfies

$$\phi^{-1}(A_y) = \sum_{i=1}^{m}\beta_i\phi^{-1}\left(A_y + \int_{\xi_i}^1 f(s, y(s) + h, y'(s))ds\right), \tag{2.28}$$

and $B_y$ satisfies

$$B_y = \frac{1}{1 - \sum_{i=1}^{m}\alpha_i}\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))ds\right)ds. \tag{2.29}$$

Then

$$\begin{aligned} (Ty)(t) = &\int_0^t \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\right)ds \\ &+ \frac{1}{1 - \sum_{i=1}^{m}\alpha_i}\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))ds\right)ds, \quad y \in P. \end{aligned} \tag{2.30}$$

**Lemma 2.9.** *Suppose that $(H_1)$, $(H_2)$, and $(H_3)$ hold. It is easy to show that*

(i) *$y$ is a solution of the BVP*

$$\begin{aligned}
&[\phi((Ty)'(t))]' + f(t, y(t) + h, y'(t)) = 0, \quad t \in (0,1), \\
&(Ty)(0) - \sum_{i=1}^{m} \alpha_i (Ty)(\xi_i) = 0, \\
&(Ty)'(1) - \sum_{i=1}^{m} \beta_i (Ty)'(\xi_i) = 0;
\end{aligned} \tag{2.31}$$

(ii) *$Ty \in P$ for each $y \in P$;*

(iii) *$x$ is a solution of BVP(1.5) if and only if $x = y + h$ and $y$ is a solution of the operator equation $y = Ty$ in cone $P$;*

(iv) *$T : P \to P$ is completely continuous.*

*Proof.* The proofs are simple and are omitted. □

**Theorem 2.10.** *Suppose that $(H_1)$, $(H_2)$, and $(H_3)$ hold and there exist positive constants $e_1$, $e_2$, $c$ and $Q$, $W$, and $E$ given by*

$$\begin{aligned}
L &= \int_0^1 \phi^{-1}(\sigma + 1 - s)ds + \frac{1}{1 - \sum_{i=1}^{m} \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi^{-1}(\sigma + 1 - s)ds; \\
Q &= \min\left\{ \phi\left(\frac{c}{L}\right), \frac{\phi(c)}{\sigma + 1} \right\}; \\
W &= \phi\left( \frac{e_2}{\sigma_0 \int_k^{1-k} \phi^{-1}(1 - k - s)ds} \right); \\
E &= \phi\left(\frac{e_1}{L}\right).
\end{aligned} \tag{2.32}$$

*such that*

$$c \geq \frac{e_2}{\sigma_0} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0, \qquad Q > W. \tag{2.33}$$

*If*

$(A_1)$ *$f(t, u, v) < Q$ for all $t \in [0,1]$, $u \in [h, c + h]$, $v \in [-c, c]$;*
$(A_2)$ *$f(t, u, v) > W$ for all $t \in [k, 1 - k]$, $u \in [e_2 + h, e_2/\sigma_0 + h]$, $v \in [-c, c]$;*
$(A_3)$ *$f(t, u, v) \leq E$ for all $t \in [0,1]$, $u \in [h, e_1/\sigma_0 + h]$, $v \in [-c, c]$;*
*then BVP(1.5) has at least three increasing positive solutions $x_1$, $x_2$, $x_3$ such that*

$$x_1(1) < e_1 + h, \qquad x_2(k) > e_2 + h, \qquad x_3(1) > e_1 + h, \qquad x_3(k) < e_2 + h. \tag{2.34}$$

*Proof.* To apply Lemma 2.5, we prove that all conditions in Lemma 2.5 are satisfied. By the definitions, it is easy to see that $\alpha_1$, $\alpha_2$ are two nonnegative continuous concave functionals on cone $P$, $\beta_1$, $\beta_2$, $\beta_3$ are three nonnegative continuous convex functionals on cone $P$ and

$\alpha_1(y) \le \beta_2(y)$ for all $y \in P$, there exist constants $M > 0$ such that $\|y\| \le M\beta_1(y)$ for all $y \in P$. Lemma 2.9 implies that $x = x(t)$ is a positive solution of BVP(1.5) if and only if $x(t) = y(t) + h$ and $y(t)$ is a solution of the operator equation $y = Ty$ and $T : P \to P$ is completely continuous.

Corresponding to Lemma 2.5,

$$a_1 = e_1, \qquad a_2 = e_2, \qquad a_3 = \frac{e_2}{\sigma_0}, \qquad a_4 = \sigma_0 e_1, \qquad a_5 = c. \tag{2.35}$$

Now, we prove that all conditions of Lemma 2.5 hold. One sees that $0 < a_1 < a_2$. The remainder is divided into four steps.

*Step 1.* Prove that $T : \overline{P_{a_5}} \to \overline{P_{a_5}}$.

For $y \in \overline{P_{a_5}}$, we have $\|y\| \le a_5$. Then $0 \le y(t) \le a_5$ for $t \in [0,1]$ and $-a_5 \le y'(t) \le a_5$ for all $n \in [0,1]$. So $(A_1)$ implies that

$$f\left(t, y(t) + h, y'(t)\right) \le Q, \quad t \in [0,1]. \tag{2.36}$$

It follows from Lemma 2.9 that $Ty \in P$. Then Lemma 2.9 implies that

$$\begin{aligned}
0 \le (Ty)(t) &\le \int_0^1 \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\right)ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\right)ds \\
&\le \int_0^1 \phi^{-1}\left(\sigma \int_0^1 f(u, y(u) + h, y'(u))du + \int_s^1 f(u, y(u) + h, y'(u))du\right)ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(\sigma \int_0^1 f(u, y(u) + h, y'(u))du \right. \\
&\qquad\qquad \left. + \int_s^1 f(u, y(u) + h, y'(u))du\right)ds \\
&\le \int_0^1 \phi^{-1}(\sigma Q + Q(1-s))ds + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}(\sigma Q + Q(1-s))ds \\
&= \phi^{-1}(Q)\left[\int_0^1 \phi^{-1}(\sigma + 1 - s)ds + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}(\sigma + 1 - s)ds\right] \\
&\le a_5.
\end{aligned} \tag{2.37}$$

On the other hand, similarly to above discussion, we have from Lemma 2.9 that

$$\begin{aligned}
\left|(Ty)'(t)\right| \le (Ty)'(0) &= \phi^{-1}\left(A_y + \int_0^1 f(u, y(u) + h, y'(u))du\right) \\
&\le \phi^{-1}\left(\sigma \int_0^1 f(u, y(u) + h, y'(u))du + \int_0^1 f(u, y(u) + h, y'(u))du\right) \\
&\le \phi^{-1}((\sigma + 1)Q) \\
&\le a_5.
\end{aligned} \tag{2.38}$$

It follows that $\|Ty\| = \max\{\max_{t\in[0,1]}|(Ty)(t)|, \max_{t\in[0,1]}|(Ty)'(t)|\} \le a_5$. Then $T : \overline{P_{a_5}} \to \overline{P_{a_5}}$.

*Step 2.* Prove that

$$\{y \in P(\beta_1, \beta_3, \alpha_1; a_2, a_3, a_5) \mid \alpha_1(y) > a_2\} = \left\{y \in P\left(\beta_1, \beta_3, \alpha_1; e_2, \frac{e_2}{\sigma_0}, c\right) \mid \alpha_1(y) > e_2\right\} \neq \varnothing \tag{2.39}$$

and $\alpha_1(Ty) > e_2$ for every $y \in P(\beta_1, \beta_3, \alpha_1; e_2, e_2/\sigma_0, a_5)$.

Choose $y(t) = e_2/2\sigma_0$ for all $t \in [0,1]$. Then $y \in P$ and

$$\alpha_1(y) = \frac{e_2}{2\sigma_0} > e_2, \qquad \beta_3(y) = \frac{e_2}{2\sigma_0} \leq \frac{e_2}{\sigma_0}, \qquad \beta_1(y) = 0 < a_5. \tag{2.40}$$

It follows that $\{y \in P(\beta_1, \beta_3, \alpha_1; a_2, a_3, a_5) \mid \alpha_1(y) > a_2\} \neq \varnothing$.

For $y \in P(\beta_1, \beta_3, \alpha_1; a_2, a_3, a_5)$, one has that

$$\alpha_1(y) = \min_{t\in[k,1-k]} y(t) \geq e_2, \qquad \beta_3(y) = \max_{t\in[k,1-k]} y(t) \leq \frac{e_2}{\sigma_0}, \qquad \beta_1(y) = \max_{t\in[0,1]} \left|y'(t)\right| \leq a_5. \tag{2.41}$$

Then

$$e_2 \leq y(t) \leq \frac{e_2}{\sigma_0}, \quad t \in [k, 1-k], \quad \left|y'(t)\right| \leq a_5. \tag{2.42}$$

Thus $(A_2)$ implies that

$$f\left(t, y(t) + h, y'(t)\right) \geq W, \quad n \in [k, 1-k]. \tag{2.43}$$

Since

$$\alpha_1(Ty) = \min_{t\in[k,1-k]} (Ty)(t) \geq \sigma_0 \max_{t\in[0,1]} (Ty)(t), \tag{2.44}$$

we get from Lemma 2.9 that

$$\begin{aligned}
\alpha_1(Ty) &\geq \sigma_0 \max_{t\in[0,1]} (Ty)(t) \\
&= \sigma_0 \left[\int_0^1 \phi^{-1}\left(A_y + \int_s^1 f\left(u, y(u) + h, y'(u)\right) du\right) ds \right.\\
&\qquad \left. + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(A_h + \int_s^1 f\left(u, y(u) + h, y'(u)\right) ds\right) ds\right] \\
&\geq \sigma_0 \left[\int_0^1 \phi^{-1}\left(\int_s^1 f\left(u, y(u) + h, y'(u)\right) du\right) ds\right] \\
&\geq \sigma_0 \int_k^{1-k} \phi^{-1}\left(\int_s^{1-k} f\left(u, y(u) + h, y'(u)\right) du\right) ds \\
&\geq \sigma_0 \int_k^{1-k} \phi^{-1}\left(W(1-k-s)\right) ds \\
&= e_2.
\end{aligned} \tag{2.45}$$

This completes Step 2.

*Step 3.* Prove that $\{y \in Q(\beta_1, \beta_3, \alpha_2; a_4, a_1, a_5) \mid \beta_2(y) < a_1\} = \{y \in Q(\beta_1, \beta_3, \alpha_2; \sigma_0 e_1, e_1, c) \mid \beta_2(y) < e_1\} \neq \emptyset$ and

$$\beta_2(Ty) < e_1 \quad \text{for every } y \in Q(\beta_1, \beta_3, \alpha_2; a_4, a_1, a_5) = Q(\beta_1, \beta_3, \alpha_2; \sigma_0 e_1, e_1, a_5). \tag{2.46}$$

Choose $y(t) = \sigma_0 e_1$. Then $y \in P$, and

$$\alpha_2(y) = \sigma_0 e_1 \geq h, \qquad \beta_2(y) = \beta_3(y) = \sigma_0 e_1 < e_1 = a_1, \qquad \beta_1(y) = 0 \leq a_5. \tag{2.47}$$

It follows that $\{y \in Q(\beta_1, \beta_3, \alpha_2; a_4, a_1, a_5) \mid \beta_2(y) < a_1\} \neq \emptyset$.

For $y \in Q(\beta_1, \beta_3, \alpha_2; a_4, a_1, a_5)$, one has that

$$\alpha_2(y) = \min_{t\in[k,1-k]} y(t) \geq h = e_1\sigma_0, \qquad \beta_3(y) = \max_{t\in[k,1-k]} y(t) \leq a_1 = e_1, \qquad \beta_1(y) = \max_{t\in[0,1]} |y'(t)| \leq a_5. \tag{2.48}$$

Hence we get that

$$0 \leq y(t) \leq \frac{e_1}{\sigma_0}, \quad t \in [0,1]; \quad -a_5 \leq y'(t) \leq a_5, \quad t \in [0,1]. \tag{2.49}$$

Then $(A_3)$ implies that

$$f(t, y(t) + h, y'(t)) \leq E, \quad t \in [0,1]. \tag{2.50}$$

So

$$\begin{aligned}
\beta_2(Ty) &= \max_{t\in[0,1]} (Ty)(t) \\
&= \int_0^1 \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))\,du\right) ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))\,du\right) ds \\
&\leq \int_0^1 \phi^{-1}\left(\sigma \int_0^1 f(u, y(u) + h, y'(u))\,du + \int_s^1 f(u, y(u) + h, y'(u))\,du\right) ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(\sigma \int_0^1 f(u, y(u) + h, y'(u))\,du + \int_s^1 f(u, y(u) + h, y'(u))\,du\right) ds \\
&\leq \int_0^1 \phi^{-1}(\sigma E + E(1-s))\,ds + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}(\sigma E + E(1-s))\,ds \\
&= \phi^{-1}(E)\left[\int_0^1 \phi^{-1}(\sigma + 1 - s)\,ds + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}(\sigma + 1 - s)\,ds\right] \\
&= e_1 = a_1.
\end{aligned} \tag{2.51}$$

This completes Step 3.

*Step 4.* Prove that $\alpha_1(Ty) > a_2$ for $y \in P(\beta_1, \alpha_1; a_2, a_5)$ with $\beta_3(Ty) > a_3$.

For $y \in P(\beta_1, \alpha_1; a_2, a_5) = P(\beta_1, \alpha_1; e_2, a_5)$ with $\beta_3(Ty) > a_3 = e_2/\sigma_0$, we have that $\alpha_1(y) = \min_{t\in[k,1-k]} y(t) \geq e_2$ and $\beta_1(y) = \max_{t\in[0,1]}|y(t)| \leq a_5$ and $\max_{t\in[k,1-k]}(Ty)(t) > e_2/\sigma_0$. Then

$$\alpha_1(Ty) = \min_{t\in[k,1-k]}(Ty)(t) \geq \sigma_0\beta_2(Ty) > \sigma_0\frac{e_2}{\sigma_0} = e_2 = a_2. \tag{2.52}$$

This completes Step 4.

*Step 5.* Prove that $\beta_2(Ty) < a_1$ for each $y \in Q(\beta_1, \beta_2; a_1, a_5)$ with $\alpha_2(Ty) < a_4$.

For $y \in Q(\beta_1, \beta_2; a_1, a_5)$ with $\alpha_2(Ty) < a_1$, we have $\beta_1(y) = \max_{t\in[0,1]}|y(t)| \leq a_5$ and $\beta_2(y) = \max_{t\in[0,1]} y(t) \leq a_1 = e_1$ and $\alpha_2(Ty) = \min_{t\in[k,1-k]}(Ty)(t) < a_4 = e_1\sigma_0$. Then

$$\beta_2(Ty) = \max_{t\in[0,1]}(Ty)(t) \leq \frac{1}{\sigma_0}\min_{t\in[k,1-k]}(Ty)(t) < \frac{1}{\sigma_0}e_1\sigma_0 = e_1 = a_1. \tag{2.53}$$

This completes Step 5.

Then Lemma 2.5 implies that $T$ has at least three fixed points $y_1$, $y_2$, and $y_3$ in $P$ such that

$$\beta_2(y_1) < e_1, \qquad \alpha_1(y_2) > e_2, \qquad \beta_2(y_3) > e_1, \qquad \alpha_1(y_3) < e_2. \tag{2.54}$$

Hence BVP(1.5) has three increasing positive solutions $x_1, x_2$, and $x_3$ such that

$$\begin{aligned} &\max_{t\in[0,1]} x_1(t) < e_1 + h, \qquad \min_{t\in[k,1-k]} x_2(t) > e_2 + h, \\ &\max_{t\in[0,1]} x_3(t) > e_1 + h, \qquad \min_{t\in[k,1-k]} x_3(t) < e_2 + h. \end{aligned} \tag{2.55}$$

Hence

$$x_1(1) < e_1 + h, \qquad x_2(k) > e_2 + h, \qquad x_3(1) > e_1 + h, \qquad x_3(k) < e_2 + h. \tag{2.56}$$

The proof is complete. □

## 3. Examples

Now, we present one example, whose three solutions cannot be obtained by theorems in known papers, to illustrate the main results.

*Example 3.1.* Consider the following BVP:

$$\begin{aligned} &x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ &x(0) = \frac{1}{4}x\left(\frac{1}{4}\right) + 6, \\ &x'(1) = \frac{1}{4}x'\left(\frac{1}{2}\right). \end{aligned} \tag{3.1}$$

Corresponding to BVP(1.5), one sees that $\phi(x) = x = \phi^{-1}(x)$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $\alpha_1 = 1/4$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 1/4$, $A = 6$. It is easy to see that $h = A/1 - \alpha_i = 8$, choose $\sigma = 1/2$, then $\phi^{-1}(1 + 1/\sigma)\sum_{i=1}^{m}\beta_i < 1$.

Choose $k = 1/4$, then $\sigma_0 = 1/4$, choose $e_1 = 10$, $e_2 = 50$, $c = 20000$ and $Q$, $W$ and $E$ are given by

$$\begin{aligned} L &= \int_0^1 \phi^{-1}(\sigma + 1 - s)ds + \frac{1}{1 - \sum_{i=1}^{m}\alpha_i}\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\phi^{-1}(\sigma + 1 - s)ds = \frac{107}{96}; \\ Q &= \min\left\{\phi\left(\frac{c}{L}\right), \frac{\phi(c)}{\sigma + 1}\right\} = \frac{40000}{3}; \\ W &= \phi\left(\frac{e_2}{\sigma_0\int_k^{1-k}\phi^{-1}(1 - k - s)ds}\right) = 1600; \\ E &= \phi\left(\frac{e_1}{L}\right) = \frac{960}{107}, \end{aligned} \tag{3.2}$$

such that

$$c \geq \frac{e_2}{\sigma_0} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0, \quad Q > W. \tag{3.3}$$

If

$$f_0(u) = \begin{cases} \dfrac{480}{107}, & x \in [8, 48], \\ \dfrac{480}{107} + \dfrac{8000 - (480/107)}{58 - 48} \times (x - 48), & x \in \left[\dfrac{140}{3}, \dfrac{146}{3}\right], \\ 8000, & x \in [58, 20008], \\ (x - 20008)^3 + 8000, & x \geq 20008, \end{cases} \tag{3.4}$$

let

$$f(t, u, v) = f_0(u) + \frac{1 + \sin t}{10000} + \frac{u^2 + v^2}{2 \times 10^1 2}, \tag{3.5}$$

then

$(A_1)$ $f(t, u, v) < 40000/3$ for all $t \in [0, 1]$, $u \in [8, 20008]$, $v \in [-20000, 20000]$;

$(A_2)$ $f(t, u, v) > 1600$ for all $t \in [1/4, 3/4]$, $u \in [58, 808]$, $v \in [-20000, 20000]$;

$(A_3)$ $f(t, u, v) \leq 960/107$ for all $t \in [0, 1]$, $u \in [8, 48]$, $v \in [-20000, 20000]$;

then Theorem 2.10 implies that BVP(3.1) has at least three decreasing and positive solutions $x_1$, $x_2$, $x_3$ such that

$$x_1(1) < \frac{50}{3}, \qquad x_2\left(\frac{1}{4}\right) > \frac{146}{3}, \qquad x_3(1) > \frac{50}{3}, \qquad x_3\left(\frac{1}{4}\right) < \frac{146}{3}. \tag{3.6}$$

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