Hindawi Publishing Corporation
Boundary Value Problems
Volume 2008, Article ID 752827, 14 pages
doi:10.1155/2008/752827

Research Article

Existence and Uniqueness of Solutions for a Second-Order Delay Differential Equation Boundary Value Problem on the Half-Line

Yuming Wei[1, 2]

[1] *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China*
[2] *School of Mathematical Science, Guangxi Normal University, Guilin 541004, China*

Correspondence should be addressed to Yuming Wei, ymwei@gxnu.edu.cn

Received 5 May 2008; Accepted 8 September 2008

Recommended by Ivan T. Kiguradze

This paper is concerned with the existence and uniqueness of solutions for the second-order nonlinear delay differential equations. By the use of the Schauder fixed point theorem, the existence of the solutions on the half-line is derived. Via the Banach contraction principle, another result concerning the existence and uniqueness of solutions on the half-line is established. The main results in this paper extend some of the existing literatures.

1. Introduction

Boundary value problems on unbounded interval have many applications in physical problems. Such problems arise, for example, in the study of linear elasticity, fluid flows, and foundation engineering (see [1] and the references therein). An interesting overview on unbounded interval problems, including real-world examples, history, and various methods of proving solvability, can be found in the recent book by Agarwal and O'Regan [2]. Boundary value problems on unbounded interval concerning second-order delay differential equations are of specific interest in these applications.

For second-order delay differential equations, boundary value problems on the half-line are closely related to the problems of existence of global solutions on the half-line with prescribed asymptotic behavior. Recently, there is, in particular, a growing interest in solutions of such boundary value problems (see, e.g., [3]). For the basic theory of delay differential equations, the reader is referred to the books by Diekmann et al. [4] and Hale and Verduyn Lunel [5]. In particular, concerning initial value problems, we refer to the monograph by Lakshmikantham and Leela [6], while, regarding boundary value problems, we mention the monographs by Azbelev et al. [7] and Azbelev and Rakhmatullina [8].

However, to the best of our knowledge, the literature on the existence and uniqueness of solutions on the half-line for delay differential equations seems to be rather limited. Motivated by the papers by Agarwal et al. [9] and Mavridis et al. [10], this paper aims to fill this gap by improving and generalizing results mentioned in the references.

Throughout the paper, for any interval J of the real line $\mathbb{R}$ and any subset X of $\mathbb{R}$, by $C(J,X)$ we will denote the set of all continuous functions defined on J and having values in X. Moreover, r will be a nonnegative real number. Furthermore, if t is a point in the interval $[0,\infty)$ and x is a continuous real-valued function defined at least on $[t-r,t]$, the notation x_t will be used for the function in $C([-r,0],\mathbb{R})$ defined by the formula

$$x_t(\tau) = x(t+\tau) \quad \text{for } -r \le \tau \le 0. \tag{1.1}$$

We notice that the set $C([-r,0],\mathbb{R})$ is a Banach space endowed by the usual sup-norm $|\cdot|$:

$$|\psi| = \max_{-r\le\tau\le 0} \left|\psi(\tau)\right| \quad \text{for } \psi \in C\left([-r,0],\mathbb{R}\right). \tag{1.2}$$

Consider the second-order nonlinear delay differential equation

$$\left[p(t)x'(t)\right]' + f\left(t, x_t, p(t)x'(t)\right) = 0, \tag{1.3}$$

where p is a positive continuous real-valued function on the interval $(0,\infty)$ such that

$$\int_{0+} \frac{dt}{p(t)} < \infty, \tag{1.4}$$

and f is a real-valued function defined on the set $(0,\infty)\times C([-r,0],\mathbb{R})\times\mathbb{R}$, which satisfies the continuity condition: $f(t,x_t,p(t)x'(t))$ is continuous with respect to t in $(0,\infty)$ for each given function x in $C([-r,\infty),\mathbb{R})$ that is continuously differentiable on the interval $(0,\infty)$.

Our interest will be concentrated on global solutions of the delay differential equation (1.3), for example, on solution of (1.3) on the whole interval $(0,\infty)$. By a solution on $(0,\infty)$ of (1.3), we mean a function x in $C([-r,\infty),\mathbb{R})$, which is continuously differentiable on the interval $(0,\infty)$ and such that px' is continuously differentiable on $(0,\infty)$ and (1.3) is satisfied for all $t>0$.

With the delay differential equation (1.3), one associates a condition of the form

$$x_0 = \phi, \tag{1.5}$$

or, equivalently,

$$x(t) = \phi(t) \quad \text{for } -r \le t \le 0, \tag{1.5'}$$

where ϕ in $C([-r,0],\mathbb{R})$ is given with

$$\phi(0) = 0. \tag{1.6}$$

Also, together with (1.3), we specify another condition of the form

$$\lim_{t\to\infty} p(t)x'(t) = L, \tag{1.7}$$

where L is a given real number.

Equations (1.3)–(1.7) constitute a boundary value problem (BVP) on the half-line. A solution on $(0,\infty)$ of the delay differential equation (1.3) satisfying the boundary value conditions (1.5) and (1.7) is said to be a solution on $(0,\infty)$ of the boundary value problem (1.3)–(1.7) or, more briefly, a solution on $(0,\infty)$ of the BVP (1.3)–(1.7).

In the sequel, by $P(t)$ we will denote the positive continuous real-valued function on the interval $(0,\infty)$ defined by the formula

$$P(t) = \int_{0+}^{t} \frac{ds}{p(s)} \quad \text{for } t > 0. \tag{1.8}$$

A useful integral representation of the BVP (1.3)–(1.7) is given by the following lemma, which will be used in proving the main result of the paper.

Lemma 1.1. *Let x be a function in $C([-r,\infty),\mathbb{R})$ that is continuously differentiable on the interval $(0,\infty)$. Then x is a solution on $(0,\infty)$ of the BVP (1.3)–(1.7) if and only if it satisfies*

$$x(t) = \begin{cases} \phi(t) & \text{for } -r \le t \le 0, \\ LP(t) + \displaystyle\int_{0+}^{t} \frac{1}{p(\sigma)} \int_{\sigma}^{\infty} f\left(s, x_s, p(s)x'(s)\right) ds\, d\sigma & \text{for } t > 0. \end{cases} \tag{1.9}$$

Proof. Assume that x satisfies (1.9). Then (1.5′) or (1.5) is fulfilled. Moreover, we immediately obtain

$$p(t)x'(t) = L + \int_{t}^{\infty} f\left(s, x_s, p(s)x'(s)\right) ds, \tag{1.10}$$

which implies that $\lim_{t\to\infty} p(t)x'(t) = L$, for example, (1.7) holds true. Furthermore, from (1.10) we get

$$\left(p(t)x'(t)\right)' = -f\left(t, x_t, p(t)x'(t)\right) \quad \text{for } t > 0, \tag{1.11}$$

which means that x is a solution on $(0,\infty)$ of (1.3). Thus, x is a solution on $(0,\infty)$ of the BVP (1.3)–(1.7).

Conversely, let us suppose that x is a solution on $(0,\infty)$ of the BVP (1.3)–(1.7). In view of (1.5′) we have $x(t) = \phi(t)$ for $-r \le t \le 0$. Furthermore, from (1.3) it follows that x satisfies (1.11) and consequently

$$p(t)x'(t) - \lim_{T\to\infty} p(T)x'(T) = -\int_{\infty}^{t} f\left(s, x_s, p(s)x'(s)\right) ds \quad \text{for } t > 0. \tag{1.12}$$

So, because of (1.7), we conclude that (1.10) is satisfied. By using (1.10) and taking into account the fact that $x(0) = \phi(0) = 0$, we obtain for every $t > 0$,

$$\begin{aligned} x(t) &= x(0) + L\int_{0+}^{t}\frac{d\sigma}{p(\sigma)} + \int_{0+}^{t}\frac{1}{p(\sigma)}\int_{\sigma}^{\infty} f\big(s, x_s, p(s)x'(s)\big)ds\,d\sigma \\ &= LP(t) + \int_{0+}^{t}\frac{1}{p(\sigma)}\int_{\sigma}^{\infty} f\big(s, x_s, p(s)x'(s)\big)ds\,d\sigma. \end{aligned} \tag{1.13}$$

We have thus proved that x satisfies (1.9). The proof of the lemma is complete. □

Our results in this paper are presented in the form of two theorems (Theorems 2.1 and 2.2). In Theorem 2.1, sufficient conditions are established in order that the BVP (1.3)–(1.7) has at least one solution in the interval $(0,\infty)$. Theorem 2.2 provides sufficient conditions for the BVP (1.3)–(1.7) to have exactly one solution on the interval $(0,\infty)$. The results of this paper are stated in Section 2. The proofs of Theorems 2.1 and 2.2 are given in Section 3. The proof of Theorem 2.1 is based on the use of the classical Schauder fixed point theorem, while the well-known Banach contraction principle is used in the proof of Theorem 2.2.

2. Main results

The first main result of this paper is the following theorem, which provides sufficient conditions for BVP (1.3)–(1.7) to have at least one solution on the interval $(0,\infty)$.

Theorem 2.1. *Suppose that*

$$\big|f(t,\psi,z)\big| \le F\big(t,|\psi|,|z|\big)\ \forall(t,\psi,z) \in (0,\infty)\times C([-r,0],\mathbb{R})\times\mathbb{R}, \tag{2.1}$$

where F is a nonnegative real-valued function defined on $(0,\infty)\times C([-r,0],[0,\infty))\times[0,\infty)$, which satisfies the following continuity condition:

(C) $F(t,|x_t|,p(t)|x'(t)|)$ is continuous with respect to t in $(0,\infty)$ for each given function x in $C([-r,\infty),\mathbb{R})$ which is continuously differentiable on the interval $(0,\infty)$.

Assume that

(A) for each $t > 0$, the function $F(t,\cdot,\cdot)$ is increasing on $C([-r,0],[0,\infty))\times[0,\infty)$ in the sense that $F(t,\psi,z) \le F(t,\omega,v)$ for any ψ, ω in $C([-r,0],[0,\infty))$ with $\psi \le \omega$ (i.e., $\psi(\tau) \le \omega(\tau)$ for $-r \le \tau \le 0$) and any z, v in $[0,\infty)$ with $z \le v$. Let a real number c exist with $c > |L|$ such that

$$\int_{0+}^{\infty} F\big(t,\eta_t,c\big)dt \le c - |L|, \tag{2.2}$$

where the function η in $C([-r,\infty),[0,\infty))$ depends on ϕ, c and is defined by

$$\eta(t) = \begin{cases} \big|\phi(t)\big| & \text{for } -r \le t \le 0, \\ cP(t) & \text{for } t > 0. \end{cases} \tag{2.3}$$

Then, the BVP (1.3)–(1.7) *has at least one solution x on the interval* $(0,\infty)$ *such that*

$$\big(-c+|L|+L\big)P(t) \le x(t) \le \big(c-|L|+L\big)P(t) \quad \text{for every } t>0, \tag{2.4}$$

$$-c+|L|+L \le p(t)x'(t) \le c-|L|+L \quad \text{for every } t>0. \tag{2.5}$$

In addition, for this solution x of the BVP (1.3)–(1.7),

$$\lim_{t\to 0+} p(t)x'(t) = \xi_x, \tag{2.6}$$

where ξ_x is some real number (depending on the solution x); the number ξ_x is given by

$$\xi_x = L + \int_{0+}^{\infty} f\big(s, x_s, p(s)x'(s)\big)ds. \tag{2.7}$$

It is remarkable that (2.6) *implies*

$$\lim_{t\to 0+} \frac{x(t)}{P(t)} = \xi_x, \tag{2.8}$$

provided that p is such that

$$\int_{0+}^{\infty} \frac{1}{p(t)} = \infty. \tag{2.9}$$

Our second main result is Theorem 2.2 below. This theorem establishes conditions under which the BVP (1.3)–(1.7) has exactly one solution on the interval $(0,\infty)$.

Theorem 2.2. *Let the following generalized Lipschitiz condition be satisfied:*

$$\begin{gathered}\big|f(t,\varphi,z) - f(t,\omega,v)\big| \le K(t)\max\big\{|\varphi-\omega|, |z-v|\big\},\\ \forall (t,\varphi,z), (t,\omega,v) \text{ in } (0,\infty)\times C([-r,0],\mathbb{R})\times\mathbb{R},\end{gathered} \tag{2.10}$$

where K is a nonnegative continuous real-valued function on the interval $(0,\infty)$ *such that*

$$\int_{0+}^{\infty} \max\big\{1, P(t)\big\}K(t)dt < 1. \tag{2.11}$$

Moreover, suppose that (2.1) *holds, where F is a nonnegative real-valued function on the set* $(0,\infty)\times C([-r,0],[0,\infty))\times[0,\infty)$, *which satisfies the continuity condition (C). Assume that (A) is satisfied.*

Let there exist a real number c with $c > |L|$ so that (2.2) *holds, where the function η in* $C([-r,\infty),[0,\infty))$ *depends on ϕ, c and is defined by* (2.3). *Then, the BVP* (1.3)–(1.7) *has exactly one solution x on the interval* $(0,\infty)$ *with*

$$p(t)\big|x'(t)\big| \le c \quad \text{for every } t>0, \tag{2.12}$$

this unique solution x is such that (2.4) *and* (2.5) *hold. In addition, for this unique solution x of the BVP* (1.3)–(1.7), *statement* (2.6) *is true, where ξ_x is some real number (depending on the solution x); the number ξ_x is given by* (2.7).

3. Proofs of Theorems 2.1 and 2.2

To prove Theorem 2.1, we will use the fixed point technique, by applying the following Schauder theorem (see [11]).

The Schauder theorem

Let E be a Banach space and Ω any nonempty convex and closed subset of E. If M is a continuous mapping of Ω into itself and $M\Omega$ is relatively compact, then the mapping M has at least one fixed point (i.e., there exists an $x \in \Omega$ with $x = Mx$).

Let $\mathrm{BC}((0,\infty),\mathbb{R})$ be the Banach space of all bounded continuous real-valued functions on the interval $(0,\infty)$, endowed with the sup-norm $\|\cdot\|$ defined by

$$\|\lambda\| = \sup_{t>0}|\lambda(t)| \quad \text{for } \lambda \in \mathrm{BC}((0,\infty),\mathbb{R}). \tag{3.1}$$

We need the following compactness criterion for subset of $\mathrm{BC}((0,\infty),\mathbb{R})$, which is a consequence of the well-known Arzela-Ascoli theorem. This compactness criterion is an adaptation of a lemma due to Avramescu [12]. In order to formulate this criterion, we note that a set U of real-valued functions defined on the interval $(0,\infty)$ is called equiconvergent at $0+$ (resp., ∞) if all the functions in U are convergent in $\mathbb{R}$ at the point $0+$ (resp., ∞) and, in addition, for each $\epsilon > 0$, there exists $T \equiv T(\epsilon) > 0$ such that, for all the functions λ in U, it holds that $|\lambda(t) - \lim_{s\to 0+}\lambda(s)| < \epsilon$ for $0 < t < T$ (resp., $|\lambda(t) - \lim_{s\to\infty}\lambda(s)| < \epsilon$ for $t > T$).

Compactness criterion

Let U be an equicontinuous and uniformly bounded subset of the Banach space $\mathrm{BC}((0,\infty),\mathbb{R})$. If U is equiconvergent at $0+$ and at ∞, it is also relatively compact.

In order to prove Theorem 2.2, we will make use of the well-known Banach contraction principle (see [13]).

The Banach contraction principle

Let E be a Banach space and Ω any nonempty closed subset of E. If M is a contraction of Ω into itself, then the mapping M has exactly one fixed point (i.e., there exists an $x \in \Omega$ with $x = Mx$).

Throughout the remainder of this section, E stands for the set of all functions u in $C([-r,\infty),\mathbb{R})$, which are continuously differentiable on the interval $(0,\infty)$ and such that pu' is bounded on $(0,\infty)$. The set E is a Banach space endowed with the norm $\|\cdot\|_E$ defined as follows:

$$\|u\|_E = \max\left\{\max_{-r\le t\le 0}|u(t)|, \sup_{t>0} p(t)|u'(t)|\right\} \quad \text{for } u \in E. \tag{3.2}$$

In order to prove Theorems 2.1 and 2.2, we will first establish the following proposition.

Proposition 3.1. *Suppose that* (2.1) *holds, where F is a nonnegative real-valued function defined on the set* $(0,\infty)\times C([-r,0],[0,\infty))\times[0,\infty)$*, which satisfies the continuity condition (C). Assume that (A) is satisfied. Let c be a positive real number such that*

$$\int_{0+}^{\infty} F(t,\eta_t,c)<\infty, \tag{3.3}$$

where the function η *in* $C([-r,\infty),[0,\infty))$ *depends on* ϕ*, c and is defined by* (2.3)*. Also, let* Ω *be the subset of the Banach space Econstituted of all functions x in E, which satisfies* (1.5′) *and* (2.12)*. Then the formula*

$$(Mx)(t)=\begin{cases}\phi(t) & \text{for } -r\le t\le 0,\\ LP(t)+\displaystyle\int_{0+}^{t}\frac{1}{p(\sigma)}\int_{\sigma}^{\infty} f\left(s,x_s,p(s)x'(s)\right)ds\,d\sigma & \text{for } t>0\end{cases} \tag{3.4}$$

makes sense for any function x in Ω*, and this formula defines a mapping M of* Ω *into E. Moreover,* $M\Omega$ *is relatively compact and the mapping M is continuous.*

Proof. Let x be an arbitrary function in Ω. By the definition of Ω, the function x satisfies (1.5′) and (2.12). Since $\phi(0)=0$, it follows from (1.5′) that $x(0)=0$. By taking into account this fact and using (2.12), we can easily obtain

$$\left|x(t)\right|\le cP(t)\quad\text{for every } t\ge 0. \tag{3.5}$$

By virtue of (1.5′), (3.5), and (2.3), it holds that $|x(t)|\le\eta(t)\ \forall t\ge -r$, which ensures that

$$\left|x_t\right|\le\eta_t\quad\text{for every } t\ge 0. \tag{3.6}$$

In view of (3.6) and (2.12) and the assumption (A), we get

$$F\left(t,\left|x_t\right|,p(t)\left|x'(t)\right|\right)\le F\left(t,\eta_t,c\right)\quad\text{for } t>0. \tag{3.7}$$

On the other hand, the hypothesis (2.1) guarantees that

$$\left|f\left(t,x_t,p(t)x'(t)\right)\right|\le F\left(t,\left|x_t\right|,p(t)\left|x'(t)\right|\right)\quad\text{for } t>0. \tag{3.8}$$

Thus, we have

$$\left|f\left(t,x_t,p(t)x'(t)\right)\right|\le F\left(t,\eta_t,c\right)\quad\text{for } t>0. \tag{3.9}$$

From (3.3) and (3.9), it follows that

$$\int_{0+}^{\infty} \left| f\left(t, x_t, p(t)x'(t)\right)\right| dt < \infty, \tag{3.10}$$

and consequently

$$\int_{0+}^{\infty} f\left(t, x_t, p(t)x'(t)\right) dt \text{ exists in } \mathbb{R}. \tag{3.11}$$

Furthermore, by taking into account (3.10) and using the hypothesis that $\int_{0+}[1/p(t)dt] < \infty$, we can conclude that

$$\int_{0+} \frac{1}{p(\sigma)} \int_{\sigma}^{\infty} \left| f\left(t, x_t, p(t)x'(t)\right)\right| dt\, d\sigma < \infty. \tag{3.12}$$

It follows from (3.12) that

$$\int_{0+} \frac{1}{p(\sigma)} \int_{\sigma}^{\infty} f\left(t, x_t, p(t)x'(t)\right) dt\, d\sigma \quad \text{exists in } \mathbb{R}. \tag{3.13}$$

As (3.13) holds true for all functions x in Ω, we can immediately see that the formula (3.4) makes sense for any function x in Ω, and this formula defines a mapping M of Ω into $C([-r, \infty), \mathbb{R})$. We will show that M is a mapping of Ω into E, for example, that $M\Omega \subseteq E$. For this purpose, let us consider an arbitrary function x in Ω. Then, by taking into account (3.9), from (3.4) we obtain for $t > 0$,

$$\begin{aligned} p(t)\left|(Mx)'(t)\right| &= \left| L + \int_{t}^{\infty} f\left(s, x_s, p(s)x'(s)\right) ds \right| \\ &\leq |L| + \int_{t}^{\infty} \left| f\left(s, x_s, p(s)x'(s)\right)\right| ds \\ &\leq |L| + \int_{t}^{\infty} F\left(s, \eta_s, c\right) ds \\ &\leq |L| + \int_{0+}^{\infty} F\left(s, \eta_s, c\right) ds. \end{aligned} \tag{3.14}$$

Hence,

$$p(t)\left|(Mx)'(t)\right| \leq Q \quad \text{for every } t > 0, \tag{3.15}$$

here

$$Q = |L| + \int_{0+}^{\infty} F\left(s, \eta_s, c\right) ds. \tag{3.16}$$

Note that, because of (3.3), Q is a nonnegative real constant. Inequality (3.15) means that $p(Mx)'$ is bounded on the interval $(0,\infty)$ and so Mx belongs to E. We have thus proved that, for any $x \in \Omega$, $Mx \in E$, for example, that $M\Omega \subseteq E$.

Now we will prove that $M\Omega$ is relatively compact. We observe that, for any function x in Ω, we have $(Mx)(t) = \phi(t)$ for $-r \le t \le 0$. By taking into account this fact as well as the definition of the norm $\|\cdot\|_E$, we can easily conclude that it is enough to show that the set

$$U = \left\{p\left((Mx)|_{(0,\infty)}\right)' : x \in \Omega\right\} \tag{3.17}$$

is relatively compact in the Banach space $\mathrm{BC}((0,\infty),\mathbb{R})$. By using (3.9), for any function x in Ω and every $t_1,\ t_2$ with $0 < t_1 \le t_2$, we obtain

$$\begin{aligned} \left|p(t_1)(Mx)'(t_1) - p(t_2)(Mx)'(t_2)\right| &= \left|\int_{t_1}^{t_2} f\left(s, x_s, p(s)x'(s)\right)ds\right| \\ &\le \int_{t_1}^{t_2} \left|f\left(s, x_s, p(s)x'(s)\right)\right|ds \\ &\le \int_{t_1}^{t_2} F\left(s, \eta_s, c\right)ds. \end{aligned} \tag{3.18}$$

So, by taking into account (3.3), we can easily verify that U is equicontinuous. Moreover, each function x in Ω satisfies (3.15), where the nonnegative real number Q is defined by (3.16) (and it is independent of x). This guarantees that U is uniformly bounded. Furthermore, if x is an arbitrary function in Ω, then we have for $t > 0$,

$$\begin{aligned} \left|p(t)(Mx)'(t) - L\right| &= \left|\int_{t}^{\infty} f\left(s, x_s, p(s)x'(s)\right)ds\right| \\ &\le \int_{t}^{\infty} \left|f\left(s, x_s, p(s)x'(s)\right)\right|ds, \end{aligned} \tag{3.19}$$

and hence, because of (3.9), it holds that

$$\left|p(t)(Mx)'(t) - L\right| \le \int_{t}^{\infty} F\left(s, \eta_s, c\right)ds \quad \text{for every } t > 0. \tag{3.20}$$

For any function x in Ω, (3.20) together with (3.3) implies that

$$\lim_{t\to\infty} p(t)(Mx)'(t) = L. \tag{3.21}$$

By again using (3.3) and (3.20), we immediately see that U is equiconvergent at ∞. Now, for each function x in Ω, we define ξ_x by (2.7). For any function x in Ω, (3.11) guarantees that ξ_x is a real number. If x is an arbitrary function in Ω, then we obtain for $t > 0$,

$$\begin{aligned} \left|p(t)(Mx)'(t) - \xi_x\right| &= \left|\int_{0+}^{t} f\left(s, x_s, p(s)x'(s)\right) ds\right| \\ &\leq \int_{0+}^{t} \left|f\left(s, x_s, p(s)x'(s)\right)\right| ds, \end{aligned} \tag{3.22}$$

and so, by the virtue of (3.9), we get

$$\left|p(t)(Mx)'(t) - \xi_x\right| \leq \int_{0+}^{t} F(s, \eta_s, c) ds \quad \text{for every } t > 0. \tag{3.23}$$

It follows from (3.23) and (3.3) that

$$\lim_{t \to 0+} p(t)(Mx)'(t) = \xi_x, \tag{3.24}$$

for any function x in Ω. Also, by again using (3.3) and (3.23) and taking into account (3.24), we conclude that U is equiconvergent at $0+$. Furthermore, by the given compactness criterion, the set U is relatively compact in $\mathrm{BC}((0,\infty),\mathbb{R})$.

Next we will prove that the mapping M is continuous. Consider an arbitrary function x in Ω, and let $\{x^{(n)}\}$, $n \geq 1$, be any sequence of functions in Ω such that $\lim_{n\to\infty} x^{(n)} = x$ in the sense of $\|\cdot\|_E$. It is not difficult to verify that $\lim_{n\to\infty} x^{(n)}(t) = x(t)$ uniformly in $t \in [-r,\infty)$ and $\lim_{n\to\infty}(x^{(n)})'(t) = x'(t)$ uniformly in $t \in (0,\infty)$. On the other hand, by (3.9), it holds that

$$\left|f\left(t, x_t^{(n)}, p(t)\left(x^{(n)}\right)'(t)\right)\right| \leq F(t, \eta_t, c) \quad \text{for every } t > 0,\ \forall n \geq 1. \tag{3.25}$$

Thus by taking into account the fact that

$$\int_{0+} \frac{1}{p(\sigma)} \int_{\sigma}^{\infty} F(t, \eta_t, c) dt\, d\sigma < \infty \tag{3.26}$$

(which is a consequence of (3.3) and the hypothesis that $\int_{0+}[1/p(t)]dt < \infty$), we can apply the Lebesgue dominated convergence theorem to obtain, for every $t > 0$,

$$\lim_{n\to\infty} \int_{0+}^{t} \frac{1}{p(\sigma)} \int_{\sigma}^{\infty} f\left(s, x_s^{(n)}, p(s)\left(x^{(n)}\right)'(s)\right) ds\, d\sigma = \int_{0+}^{t} \frac{1}{p(\sigma)} \int_{\sigma}^{\infty} f\left(s, x_s, p(s)x'(s)\right) ds\, d\sigma. \tag{3.27}$$

This, together with the fact that

$$\left(Mx^{(n)}\right)(t) = (Mx)(t) = \phi(t) \quad \text{for } -r \leq t \leq 0 \ (n = 1, 2, \ldots), \tag{3.28}$$

guarantees the pointwise convergence

$$\lim_{n\to\infty}\left(Mx^{(n)}\right)(t) = (Mx)(t) \quad \text{for } t \ge -r. \tag{3.29}$$

It remains to show that this convergence is also convergent in the sense of $\|\cdot\|_E$, for example, that

$$\lim_{n\to\infty} Mx^{(n)} = Mx. \tag{3.30}$$

For this purpose, we consider an arbitrary subsequence $(Mx^{(k)})$ of $(Mx^{(n)})$. Since $M\Omega$ is relatively compact, there exist a subsequence $(Mx^{(j)})$ of the sequence $(Mx^{(k)})$ and a function u in E so that $\lim_{j\to\infty} Mx^{(j)} = u$ in the sense of $\|\cdot\|_E$. As the $\|\cdot\|_E$ convergence implies the pointwise one to the same limit function, we must have $u = Mx$. That is, (3.30) holds true. Consequently, M is continuous. The proof of the proposition has been finished. □

Proof of Theorem 2.1. First of all, we observe that the hypothesis (2.2) implies (3.3). Let Ω be the subset of the Banach space of E defined as in the proposition. Clearly, Ω is a nonempty convex and closed subset of E. By our proposition, the formula (3.4) makes sense for any function x in Ω, and this formula defines a continuous mapping M of Ω into E such that $M\Omega$ is relatively compact. We will show that M is a mapping of Ω into itself, for example, that $M\Omega \subseteq \Omega$. Let us consider an arbitrary function x in Ω. The function x satisfies (3.20). By using the hypothesis (2.2), from (3.20) we obtain

$$\left|p(t)(Mx)'(t) - L\right| \le c - |L| \quad \text{for every } t > 0, \tag{3.31}$$

which obviously gives

$$p(t)\left|(Mx)'(t)\right| \le c \quad \text{for every } t > 0. \tag{3.32}$$

By the definition of Ω, (3.32) together with the fact that $(Mx)(t) = \phi(t)$ for $-r \le t \le 0$ guarantees that Mx belongs to Ω. We have thus proved that, for each x in $\Omega, Mx \in \Omega$, for example, that $M\Omega \subseteq \Omega$.

Now the Schauder theorem guarantees the existence of an x in Ω such that (1.9) is satisfied. By our Lemma 1.1, x is a solution on the interval $(0,\infty)$ of the BVP (1.3)–(1.7). Also, as $x \in \Omega$ and $x = Mx$, (3.31) ensures that the solution x satisfies

$$\left|p(t)x'(t) - L\right| \le c - |L| \quad \text{for every } t > 0. \tag{3.33}$$

That is, x satisfies (2.5). Moreover, since $x(0) = \phi(0) = 0$, it follows from (2.5) that x is also such that (2.4) holds. Furthermore, since $x \in \Omega$, (3.11) and (3.24) are satisfied, where ξ_x is given by (2.7). Because of (3.11), ξ_x is a real number. As $x = Mx$, it follows from (3.24) that the solution satisfies (2.6). The proof of the theorem is complete. □

Proof of Theorem 2.2. Let Ω be the subset of the Banach space E defined as in the proposition. Clearly, Ω is a nonempty closed subset of E. Since the hypothesis (2.2) implies (3.3), our proposition guarantees that the formula (3.4) makes sense for any function x in Ω, and this formula defines a mapping M of Ω into E. As in the proof of Theorem 2.1, we can use the hypothesis (2.2) to show that M is a mapping of Ω into itself.

Now, we will prove that the mapping M is a contraction. For this purpose, let us consider two arbitrary functions x and $\tilde{x}$ in Ω. In view of (3.4), we have $(Mx)(t) = (M\tilde{x})(t) = \phi(t)$ for $-r \le t \le 0$, and consequently

$$\max_{-r\le t\le 0}\left|(Mx)(t) - \left(M\tilde{x}\right)(t)\right| = 0. \tag{3.34}$$

Furthermore, by using (2.10), from (3.4) we obtain for $t > 0$,

$$\begin{aligned}\left|p(t)(Mx)'(t) - p(t)\left(M\tilde{x}\right)'(t)\right| &= \left|\int_t^\infty \left[f\left(s, x_s, p(s)x'(s)\right) - f\left(s, \tilde{x}_s, p(s)\tilde{x}'(s)\right)\right]ds\right| \\ &\le \int_t^\infty \left|\left[f\left(s, x_s, p(s)x'(s)\right) - f\left(s, \tilde{x}_s, p(s)\tilde{x}'(s)\right)\right]\right|ds \\ &\le \int_t^\infty K(s)\max\left\{\left\|x_s - \tilde{x}_s\right\|, \left|p(s)x'(s) - p(s)\tilde{x}'(s)\right|\right\}ds.\end{aligned} \tag{3.35}$$

This gives

$$\sup_{t>0}\left|p(t)(Mx)'(t) - p(t)\left(M\tilde{x}\right)'(t)\right| \le \int_{0+}^\infty K(t)\max\left\{\left\|x_t - \tilde{x}_t\right\|, \left|p(t)x'(t) - p(t)\tilde{x}'(t)\right|\right\}dt. \tag{3.36}$$

By the definition of the norm $\|\cdot\|_E$ in E, the last inequality together with (3.34) implies that

$$\left\|Mx - M\tilde{x}\right\|_E \le \int_{0+}^\infty K(t)\max\left\{\left\|x_t - \tilde{x}_t\right\|, \left|p(t)x'(t) - p(t)\tilde{x}'(t)\right|\right\}dt. \tag{3.37}$$

From the definition of Ω, it follows that $x(t) = \tilde{x}(t) = \phi(t)$ for $-r \le t \le 0$, and so

$$\left|x(t) - \tilde{x}(t)\right| = 0 \quad \text{for } -r \le t \le 0. \tag{3.38}$$

Moreover, in view of the fact that $x(0) = \tilde{x}(0) = \phi(0) = 0$, we get for $t > 0$,

$$\begin{aligned}\left|x(t) - \tilde{x}(t)\right| &= \left|\int_0^t \left[x'(s) - \tilde{x}'(s)\right]ds\right| \\ &\le \int_0^t \left|\left[x'(s) - \tilde{x}'(s)\right]\right|ds \\ &= \int_0^t \frac{1}{p(s)}\left|p(s)x'(s) - p(s)\tilde{x}'(s)\right|ds.\end{aligned} \tag{3.39}$$

But, by the definition of the norm $\|\cdot\|_E$, we have

$$\left|p(t)x'(t) - p(t)\tilde{x}'(t)\right| \le \left\|x - \tilde{x}\right\|_E \quad \text{for every } t > 0. \tag{3.40}$$

Thus, we obtain

$$\left|x(t) - \tilde{x}(t)\right| \le P(t)\left\|x - \tilde{x}\right\|_E \quad \text{for every } t > 0. \tag{3.41}$$

This inequality and (3.38) can be written as

$$\left|x(t) - \tilde{x}(t)\right| \le \mu(t)\left\|x - \tilde{x}\right\|_E \quad \forall t \ge -r, \tag{3.42}$$

where the function μ is defined by

$$\mu(t) = 0 \quad \text{for } -r \le t \le 0, \qquad \mu(t) = P(t) \quad \text{for } t > 0. \tag{3.43}$$

Hence,

$$\left|x(t+\tau) - \tilde{x}(t+\tau)\right| \le \mu(t+\tau)\left\|x - \tilde{x}\right\|_E \quad \text{for } t > 0, \ -r \le \tau \le 0, \tag{3.44}$$

for example,

$$\left|x_t(\tau) - \tilde{x}_t(\tau)\right| \le \mu(t+\tau)\left\|x - \tilde{x}\right\|_E \quad \text{for } t > 0, \ -r \le \tau \le 0. \tag{3.45}$$

Therefore,

$$\max_{-r\le\tau\le 0}\left|x_t(\tau) - \tilde{x}_t(\tau)\right| \le \left[\max_{-r\le\tau\le 0}\mu(t+\tau)\right]\left\|x - \tilde{x}\right\|_E \quad \text{for } t > 0. \tag{3.46}$$

But since μ is nondecreasing on $[-r, \infty)$, we have

$$\max_{-r\le\tau\le 0}\mu(t+\tau) = \mu(t) = P(t) \ \text{for } t > 0. \tag{3.47}$$

So it holds that

$$\left\|x_t - \tilde{x}_t\right\| \le P(t)\left\|x - \tilde{x}\right\|_E \quad \text{for } t > 0. \tag{3.48}$$

Next, by using (3.48) and (3.40), from (3.37) we obtain

$$\begin{aligned}\|Mx - M\tilde{x}\|_E &\le \int_{0+}^{\infty} K(t) \max\left\{P(t)\|x - \tilde{x}\|_E, \|x - \tilde{x}\|_E\right\} dt \\ &= \left[\int_{0+}^{\infty} K(t) \max\{P(t), 1\} dt\right] \|x - \tilde{x}\|_E.\end{aligned} \tag{3.49}$$

By taking into account (2.11), we see that the mapping $M : \Omega \to \Omega$ is a contraction.

Finally, by the Banach contraction principle, the mapping $M : \Omega \to \Omega$ has a unique fixed point. Namely, there exists exactly one x in Ω such that $x = Mx$, for example, such that (1.9) is satisfied. So, by our lemma, the BVP (1.3)–(1.7) has exactly one solution x on the interval $(0, \infty)$ such that (2.12) holds. (Note that a solution x on the interval $(0, \infty)$ of the BVP (1.3)–(1.7) belongs to Ω if and only if x satisfies (2.12).) As in the proof of Theorem 2.1, we conclude that this unique solution x of the BVP (1.3)–(1.7) satisfies (2.4) and (2.5). In addition, for this solution x, (2.6) holds true, where the real number ξ_x is given by (2.7). The proof of the theorem is now complete. □

Acknowledgment

This work was supported by the NNSF of China (Grant no. 05046012).

References

[1] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.
[2] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
[3] C. Bai and J. Fang, "On positive solutions of boundary value problems for second-order functional differential equations on infinite intervals," *Journal of Mathematical Analysis and Applications*, vol. 282, no. 2, pp. 711–731, 2003.
[4] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations: Functional, Complex, and Nonlinear Analysis*, vol. 110 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1995.
[5] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, vol. 99, Springer, New York, NY, USA, 1993.
[6] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications. Vol. II: Functional, Partial, Abstract, and Complex Differential Equations*, vol. 55 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1969.
[7] N. Azbelev, V. Maksimov, and L. Rakhmatullina, *Introduction to the Theory of Linear Functional-Differential Equations*, vol. 3 of *Advanced Series in Mathematical Science and Engineering*, World Federation, Atlanta, Ga, USA, 1995.
[8] N. V. Azbelev and L. F. Rakhmatullina, "Theory of linear abstract functional-differential equations and applications," *Memoirs on Differential Equations and Mathematical Physics*, vol. 8, p. 102, 1996.
[9] R. P. Agarwal, Ch. G. Philos, and P. Ch. Tsamatos, "Global solutions of a singular initial value problem to second order nonlinear delay differential equations," *Mathematical and Computer Modelling*, vol. 43, no. 7-8, pp. 854–869, 2006.
[10] K. G. Mavridis, Ch. G. Philos, and P. Ch. Tsamatos, "Existence of solutions of a boundary value problem on the half-line to second order nonlinear delay differential equations," *Archiv der Mathematik*, vol. 86, no. 2, pp. 163–175, 2006.
[11] J. Schauder, "Der fixpunktsatz in funktionalraumen," *Studia Mathematica*, vol. 2, pp. 171–180, 1930.
[12] C. Avramescu, "Sur l'existence des solutions convergentes des systèmes d'équations différentielles non linéaires," *Annali di Matematica Pura ed Applicata*, vol. 81, pp. 147–168, 1969.
[13] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.