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Research Article

Global Behavior for a Diffusive Predator-Prey Model with Stage Structure and Nonlinear Density Restriction-II: The Case in $\mathbb{R}^1$

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A Holling type III predator-prey model with self- and cross-population pressure is considered. Using the energy estimate and Gagliardo-Nirenberg-type inequalities, the existence and uniform boundedness of global solutions to the model are dicussed. In addition, global asymptotic stability of the positive equilibrium point for the model is proved by Lyapunov function.

1. Introduction

This paper is a continuation of Part I [1]. In Section 3 of Part I, using the energy estimate and bootstrap arguments, the global existence of solutions for a Holling type III cross-diffusion predator-prey model with stage-structure has been discussed when the space dimension be less than 6. However, to obtain the L^∞ estimate for the population density w of predator species, there is not cross-diffusion for w in Part I.

All diffusive predator-prey systems behave, more or less, in the same way, for both semilinear and cross-diffusive models, at least for small values of the cross diffusivities. Consequently, all the available information for linear diffusive models is essential to realize the behavior of the most complicated cross-diffusive systems [2–17].

In this paper, we consider the following cross-diffusion system:

$$u_t = \left(du + \alpha_{11}u^2 + \alpha_{12}uv + \alpha_{13}uw\right)_{xx} + \beta v - au - bu^2 - cu^3 - \frac{u^2 w}{1+u^2},$$

$$v_t = \left(dv + \alpha_{21}uv + \alpha_{22}v^2 + \alpha_{23}vw\right)_{xx} + u - v, \quad 0 < x < 1,\ t > 0,$$

$$w_t = \left(d_3 w + \alpha_{31} u w + \alpha_{32} v w + \alpha_{33} w^2\right)_{xx} - k w - \gamma w^2 + \frac{\alpha u^2 w}{1+u^2},$$

$$u_x(x,t) = v_x(x,t) = w_x(x,t) = 0, \quad x = 0,1, \ t > 0,$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad 0 < x < 1, \tag{1.1}$$

where d, d_3, α_{ij} $(i,j = 1,2,3), \alpha, \beta, \gamma, a, b, c$, and k are positive constants. Also, d, d_3 are linear diffusion coefficients of u, v, w, respectively, while α_{ii} $(i = 1,2,3)$ are referred as self-diffusion pressures, and α_{ij} $(i \neq j, i, j = 1,2,3)$ are cross-diffusion pressures. If $\alpha_{12} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 0$, then (1.1) reduces to the system (1.4) of Part I.

Recently, the work in [18–20] studied the existence, uniform boundedness, and uniform convergence of global solutions for the Lotka-Volterra cross-diffusion models without stage-structure in the case that the space dimension $n = 1$. In this paper, we consider mainly the existence and uniform boundedness of global solutions for the model (1.1) with nonlinear density restriction and stage-structure. Moreover, global asymptotic stability of the positive equilibrium point for (1.1) is proved by an important lemma of [21]. The proof is complete and complement the uniform convergence theorem in [18–20].

2. Global Existence and Uniform Boundedness

For simplicity, denote $|\cdot|_{k,p} = \|\cdot\|_{W_p^k(0,1)}, |\cdot|_p = \|\cdot\|_{L^p(0,1)}$. The local existence result of solutions to (1.1) is an immediate consequence of a series of papers [22, 23] by Amann. Roughly speaking, if $u_0, v_0, w_0 \in W_p^1(0,1)$, $p > 1$, then (1.1) has a unique nonnegative solution $u, v, w \in C([0,T), W_p^1(0,1)) \bigcap C^\infty((0,T), C^\infty(0,1))$, where $T \leq +\infty$ is the maximal existence time for the solution. If (u,v,w) satisfies

$$\sup\left\{|u(\cdot,t)|_{1,p}, |v(\cdot,t)|_{1,p}, |w(\cdot,t)|_{1,p} : 0 < t < T\right\} < \infty, \tag{2.1}$$

then $T = +\infty$. If, in addition, $u_0, v_0, w_0 \in W_p^2(0,1)$, then $u, v, w \in C([0,\infty), W_p^2(0,1))$.

The main result in this section is as follows.

Theorem 2.1. *Let* $u_0, v_0, w_0 \in W_2^2(0,1)$, (u,v,w) *is the unique nonnegative solution of* (1.1) *in its maximal existence interval* $[0,T)$. *Assume that*

$$\begin{aligned} 8\alpha_{11}\alpha_{21}\alpha_{31} &> \alpha_{21}\alpha_{13}^2 + \alpha_{12}^2\alpha_{31}, \\ 8\alpha_{12}\alpha_{22}\alpha_{32} &> \alpha_{32}\alpha_{21}^2 + \alpha_{23}^2\alpha_{12}, \\ 8\alpha_{13}\alpha_{23}\alpha_{33} &> \alpha_{23}\alpha_{31}^2 + \alpha_{32}^2\alpha_{13}. \end{aligned} \tag{2.2}$$

Then there exists $t_0 > 0$ *and positive constants* M, M' *which depend on* d, d_3, α_{ij} $(i, j = 1, 2, 3)$, $\beta, a, b, c, k, \gamma, \alpha$, *such that*

$$\sup\left\{|u(\cdot,t)|_{1,2}, |v(\cdot,t)|_{1,2}, |w(\cdot,t)|_{1,2} : t \in (t_0, T)\right\} \leq M', \tag{2.3}$$

$$\max\{u(x,t), v(x,t), w(x,t) : 0 \leq x \leq 1,\ t_0 \leq t < T\} \leq M, \tag{2.4}$$

and $T = +\infty$. *In particular, if* $d, d_3 \geq 1$, $d_3/d \in [\underline{d}, \overline{d}]$, *where* $\underline{d} \leq 1$ *and* $\overline{d}$ *are positive constants, then* M', M *depend on* $\underline{d}, \overline{d}$, *but do not depend on* $d, d_3 \geq 1$.

The following Gagliardo-Nirenberg-type inequalities and corresponding corollary play an importance role in the proof of Theorem 2.1.

Theorem 2.2 (see [18]). *Let* $\Omega \subset R^n$ *be a bounded domain with* $\partial\Omega \in C^m$. *For every function* $u \in W_r^m(\Omega)$, $1 \leq q$, $r \leq \infty$, *the derivative* $D^j u$ $(0 \leq j < m)$ *satisfies the inequality*

$$\left|D^j u\right|_p \leq C\left(|D^m u|_r^a |u|_q^{1-a} + |u|_q\right), \tag{2.5}$$

provided one of the following three conditions is satisfied: (1) $r \leq q$, (2) $0 < n(r-q)/mrq < 1$, *or* (3) $n(r-q)/mrq = 1$, *and* $m - n/q$ *is not a nonnegative integer, where* $1/p = j/n + a(1/r - m/n) + (1-a)/q$, *for all* $a \in [j/m, 1)$, *and the positive constant* C *depends on* n, m, j, q, r, a.

Corollary 2.3. *There exists a positive constant* C *such that*

$$|u|_2 \leq C\left(|u_x|_2^{1/3} |u|_1^{2/3} + |u|_1\right), \quad \forall u \in W_2^1(0,1), \tag{2.6}$$

$$|u|_4 \leq C\left(|u_x|_2^{1/2} |u|_1^{1/2} + |u|_1\right), \quad \forall u \in W_2^1(0,1), \tag{2.7}$$

$$|u|_{7/2} \leq C\left(|u_x|_2^{10/21} |u|_1^{11/21} + |u|_1\right), \quad \forall u \in W_2^1(0,1), \tag{2.8}$$

$$|u_x|_2 \leq C\left(|u_{xx}|_2^{3/5} |u|_1^{2/5} + |u|_1\right), \quad \forall u \in W_2^2(0,1). \tag{2.9}$$

For simplicity, denote that C *is Sobolev embedding constant or other kind of absolute constant.* A_j, B_j, C_j *are some positive constants which depend on* α_{ij} $(i, j = 1, 2, 3)$, $\beta, a, b, c, k, \gamma, \alpha$. *Also,* K_j *are positive constants which depend on* α_{ij} $(i, j = 1, 2, 3)$, $\beta, a, b, c, k, \gamma, \alpha, d, d_3$. *When* $d, d_3 \geq 1$, K_j *do not depend on* d, d_3, *but on* $\underline{d}, \overline{d}$.

Proof of Theorem 2.1

Step 1. Estimate $|u|_1, |v|_1, |w|_1$. Firstly, taking integration of the first and second equations in (2.7) over the domain $[0,1]$, respectively, and combining the two integration equalities linearly, we have

$$\frac{d}{dt}\int_0^1 [u + (a+\beta)v]\,dx \le -a\int_0^1 v\,dx + \int_0^1 \left(\beta u - bu^2\right)dx. \tag{2.10}$$

From Young inequality and Hölder inequality, we can see

$$\frac{d}{dt}\int_0^1 [u + (a+\beta)v]\,dx \le C_1 - \frac{a}{a+\beta}\int_0^1 [u + (a+\beta)v]\,dx, \tag{2.11}$$

where $C_1 = (1/4b)(\beta + a/(a+\beta))^2$. From which it follows that there exists a constant $\tau_0 > 0$, such that

$$\int_0^1 u\,dx, \int_0^1 v\,dx \le M_0, \quad t \ge \tau_0, \tag{2.12}$$

where $M_0 = (2C_1(a+\beta)/a)\max\{(a+\beta)^{-1}, 1\}$.

Secondly, taking integration of the third equations in (2.7) over domain $[0,1]$, we have

$$\frac{d}{dt}\int_0^1 w\,dx \le (\alpha - k)\int_0^1 w\,dx - \gamma\left(\int_0^1 w\,dx\right)^2. \tag{2.13}$$

This implies that there exists a constant $\tilde{\tau}_0 > 0$, such that

$$\int_0^1 w\,dx \le \frac{2|\alpha - k|}{\gamma}, \quad t \ge \tilde{\tau}_0. \tag{2.14}$$

Let $M_1 = \max\{M_0, (2|\alpha - k|)/\gamma\}$, $\tau_1 = \max\{\tau_0, \tilde{\tau}_0\}$. Then

$$\int_0^1 u\,dx, \int_0^1 v\,dx, \int_0^1 w\,dx \le M_1, \quad t \ge \tau_1. \tag{2.15}$$

Moreover, there exists a positive constant M_1' which depends on $\beta, a, b, c, k, \gamma, \alpha$ and the L^1-norm of u_0, v_0, w_0, such that

$$\int_0^1 u\,dx, \int_0^1 v\,dx, \int_0^1 w\,dx \le M_1', \quad t \ge 0. \tag{2.15$'$}$$

Step 2. estimate $|u|_2, |v|_2$ and $|w|_2$. Multiplying the first three inequalities of Corollary 2.3 by u, v, w, respectively, and integrating over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx \\
&\quad \le -d\int_0^1 u_x^2 dx - \int_0^1 \left[(2\alpha_{11}u + \alpha_{12}v + \alpha_{13}w)u_x^2 + \alpha_{12}uu_xv_x + \alpha_{13}uu_xw_x\right]dx + \beta\int_0^1 uv dx, \\
&\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx \\
&\quad \le -d\int_0^1 v_x^2 dx - \int_0^1 \left[(\alpha_{21}u + 2\alpha_{22}v + \alpha_{23}w)v_x^2 + \alpha_{21}vu_xv_x + \alpha_{23}vv_xw_x\right]dx + \int_0^1 uv dx, \\
&\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx \\
&\quad \le -d_3\int_0^1 w_x^2 dx - \int_0^1 \left[(\alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w)w_x^2 + \alpha_{31}wu_xw_x + \alpha_{32}wv_xw_x\right]dx.
\end{aligned} \tag{2.16}$$

Let $d' = \min\{d, d_3\}$. By the above three inequalities and Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(u^2 + v^2 + w^2\right)dx \\
&\quad \le -d'\int_0^1 \left(u_x^2 + v_x^2 + w_x^2\right)dx - \int_0^1 q(u_x, v_x, w_x)dx + \left(\frac{\beta+1}{2} + \alpha\right)\int_0^1 \left(u^2 + v^2 + w^2\right)dx,
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
q(u_x, v_x, w_x) = {} & (2\alpha_{11}u + \alpha_{12}v + \alpha_{13}w)u_x^2 + (\alpha_{21}u + 2\alpha_{22}v + \alpha_{23}w)v_x^2 + (\alpha_{31}u + \alpha_{32}v + 2\alpha_{33}w)w_x^2 \\
& + (\alpha_{12}u + \alpha_{21}v)u_xv_x + (\alpha_{13}u + \alpha_{31}w)u_xw_x + (\alpha_{23}v + \alpha_{32}w)v_xw_x
\end{aligned} \tag{2.18}$$

is quadratic form of u_x, v_x, w_x. It is not hard to verify that $q(u_x, v_x, w_x)$ is positive definite if (2.2) holds. Moreover, if (2.2) holds, then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(u^2 + v^2 + w^2\right)d \le -d'\int_0^1 \left(u_x^2 + v_x^2 + w_x^2\right)dx + \left(\frac{\beta+1}{2} + \alpha\right)\int_0^1 \left(u^2 + v^2 + w^2\right)dx. \tag{2.19}$$

Now we proceed in the following two cases.

(i) It holds that $t \ge \tau_1$. By (2.6) and (2.15), we have $\int_0^1 u_x^2 dx \ge (1/CM_1^4)(\int_0^1 u^2 dx)^3 - M_1^2$, and

$$-d'\int_0^1 \left(u_x^2 + v_x^2 + w_x^2\right) dx \le 3d' M_1^2 - C_2 d' \left[\int_0^1 \left(u^2 + v^2 + w^2\right) dx\right]^3. \tag{2.20}$$

By (2.19) and (2.20), we can see that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(u^2 + v^2 + w^2\right) dx \\ &\quad \le -C_2 d' \left[\int_0^1 \left(u^2 + v^2 + w^2\right) dx\right]^3 + \left(\frac{\beta+1}{2} + \alpha\right)\int_0^1 \left(u^2 + v^2 + w^2\right) dx + 3d' M_1^2. \end{aligned} \tag{2.21}$$

Thus, there exists positive constants $\tau_2 > \tau_1$ and M_2 depending on $d, d_3, \beta, a, b, c, k, \gamma, \alpha$, such that

$$\int_0^1 u^2 dx, \int_0^1 v^2 dx, \int_0^1 w^2 dx \le M_2', \quad t \ge \tau_2. \tag{2.22}$$

Since the zero point of the right-hand side in (2.21) can be estimated by positive constants independent of d', when $d' \ge 1$. Thus M_2 do not depend on $d' \ge 1$.

(ii) $t \ge 0$. Repeating estimates in (i) by $(2.9)'$, we can obtain that there exists a positive constant M_2' depending on $d, d_3, \beta, a, b, c, k, \gamma, \alpha$ and the L^1, L^2-norm of u_0, v_0, w_0, such that

$$\int_0^1 u^2 dx, \int_0^1 v^2 dx, \int_0^1 w^2 dx \le M_2', \quad t \ge 0, \tag{2.22$'$}$$

when $d' \ge 1$, M_1' is independent of d'.

Step 3. Estimate $|u_x|_2, |v_x|_2, |w_x|_2$. Introduce the scaling that

$$\tilde{u} = \frac{u}{d_1}, \qquad \tilde{v} = \frac{v}{d_1}, \qquad \tilde{w} = \frac{w}{d_1}, \qquad \tilde{t} = d_1 t, \tag{2.23}$$

denote $\eta = d_3/d$, and redenote $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{t}$ by u, v, w, t, respectively. Then (2.7) reduces to

$$\begin{aligned} u_t &= P_{xx} + f(u, v, w), \quad 0 < x < 1,\ t > 0, \\ v_t &= Q_{xx} + g(u, v, w), \quad 0 < x < 1,\ t > 0, \\ w_t &= R_{xx} + h(u, v, w), \quad 0 < x < 1,\ t > 0, \\ u_x(x,t) &= v_x(x,t) = w_x(x,t) = 0, \quad x = 0, 1,\ t > 0, \\ u(x,0) &= \tilde{u}_0(x), \quad v(x,0) = \tilde{v}_0(x), \quad w(x,0) = \tilde{w}_0(x), \quad 0 < x < 1, \end{aligned} \tag{2.24}$$

where $P = u+\alpha_{11}u^2+\alpha_{12}uv+\alpha_{13}uw$, $Q = v+\alpha_{21}uv+\alpha_{22}v^2+\alpha_{23}vw$, $R = \eta w+\alpha_{31}uw+\alpha_{32}vw+\alpha_{33}w^2$, $f(u,v,w) = \beta d^{-1}v - ad^{-1}u - bu^2 - cdu^3 - (du^2w/(1+d^2u^2))$, $g(u,v,w) = d^{-1}(u-v)$, $h(u,v,w) = -kd^{-1}w - rw^2 + (\alpha du^2 w/(1+d^2u^2))$. We still proceed in following two cases.

(i) It holds that $t \geq \tau_2^* = d\tau_2$. From (2.15) and (2.22), we can easily obtain that

$$\begin{gathered}\int_0^1 u\,dx, \int_0^1 v\,dx, \int_0^1 w\,dx \leq M_1 d^{-1}, \\ \int_0^1 u^2dx, \int_0^1 v^2dx, \int_0^1 w^2dx \leq M_2 d^{-2}, \\ |P|_1, |Q|_1, |R|_1 \leq DK_1 d^{-1},\end{gathered} \tag{2.25}$$

where $K_1 = (2+\eta) + M_2 d^{-2}$, $D = \max\{M_1, \alpha_{11}+\alpha_{12}+\alpha_{13}, \alpha_{21}+\alpha_{22}+\alpha_{23}, \alpha_{31}+\alpha_{32}+\alpha_{33}\}$.

Multiply the first three equations in (2.24) by P_t, Q_t, R_t and integrate them over $[0,1]$, respectively, then adding up the three new equations, we have

$$\begin{aligned}\frac{1}{2}\overline{y}'(t) \leq& -\int_0^1 u_t^2dx - \int_0^1 v_t^2dx - \eta\int_0^1 w_t^2dx - \int_0^1 q(u_t,v_t,w_t)dx \\ &+ \int_0^1 \left[(1+2\alpha_{11}u+\alpha_{12}v+\alpha_{13}w)u_t f + \alpha_{12}uv_t f + \alpha_{13}uw_t f\right]dx \\ &+ \int_0^1 \left[\alpha_{21}vu_t g + (1+\alpha_{21}u+2\alpha_{22}v+\alpha_{23}w)v_t g + \alpha_{23}vw_t g\right]dx \\ &+ \int_0^1 \left[\alpha_{31}wu_t h + \alpha_{32}wv_t h + (\eta+\alpha_{31}u+\alpha_{32}v+2\alpha_{33}w)w_t h\right]dx,\end{aligned} \tag{2.26}$$

where $\overline{y} = \int_0^1 (P_x^2 + Q_x^2 + R_x^2)dx$. It is not hard to verify by (2.4) that there exists a positive constant C_3 depending only on α_{ij} $(i,j = 1,2,3)$, such that

$$q(u_t,v_t,w_t) \geq C_3(u+v+w)\left(u_t^2+v_t^2+w_t^2\right). \tag{2.27}$$

Thus,

$$\begin{aligned}\frac{1}{2}\overline{y}'(t) \leq& -\int_0^1 u_t^2dx - \int_0^1 v_t^2dx - \eta\int_0^1 w_t^2dx - C_3\int_0^1 (u+v+w)\left(u_t^2+v_t^2+w_t^2\right)dx \\ &+ \int_0^1 (1+2\alpha_{11}u+\alpha_{12}v+\alpha_{13}w)u_t f\,dx + \int_0^1 (1+\alpha_{21}u+2\alpha_{22}v+\alpha_{23}w)v_t g\,dx \\ &+ \int_0^1 (\eta+\alpha_{31}u+\alpha_{32}v+2\alpha_{33}w)w_t h\,dx + \int_0^1 \alpha_{12}uv_t f\,dx + \int_0^1 \alpha_{13}uw_t f\,dx \\ &+ \int_0^1 \alpha_{21}vu_t g\,dx + \int_0^1 \alpha_{23}vw_t g\,dx + \int_0^1 \alpha_{31}wu_t h\,dx + \int_0^1 \alpha_{32}wv_t h\,dx.\end{aligned} \tag{2.28}$$

Using Young inequality, Hölder inequality and (2.24), we can obtain the following estimates:

$$\int_0^1 u^3 dx \le \left(\int_0^1 u^7 dx\right)^{1/5}\left(\int_0^1 u^2 dx\right)^{4/5} \le M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5},$$

$$\int_0^1 u^4 dx \le \left(\int_0^1 u^7 dx\right)^{2/5}\left(\int_0^1 u^2 dx\right)^{3/5} \le M_2^{3/5} d^{-6/5}\left(\int_0^1 u^7 dx\right)^{2/5},$$

$$\int_0^1 u^5 dx \le \left(\int_0^1 u^7 dx\right)^{3/5}\left(\int_0^1 u^2 dx\right)^{2/5} \le M_2^{2/5} d^{-4/5}\left(\int_0^1 u^7 dx\right)^{3/5},$$

$$\int_0^1 u^6 dx \le \left(\int_0^1 u^7 dx\right)^{4/5}\left(\int_0^1 u^2 dx\right)^{1/5} \le M_2^{1/5} d^{-2/5}\left(\int_0^1 u^7 dx\right)^{4/5},$$

$$\int_0^1 uv dx \le \left(\int_0^1 u^2 dx\right)^{1/2}\left(\int_0^1 v^2 dx\right)^{1/2} \le M_2 d^{-2},$$

$$\int_0^1 u^2 v dx \le \left(\int_0^1 u^7 dx\right)^{1/5}\left(\int_0^1 u^2 dx\right)^{3/10}\left(\int_0^1 v^2 dx\right)^{1/2} \le M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5},$$

$$\int_0^1 u^3 v dx \le \left(\int_0^1 u^7 dx\right)^{2/5}\left(\int_0^1 u^2 dx\right)^{1/10}\left(\int_0^1 v^2 dx\right)^{1/2} \le M_2^{3/5} d^{-6/5}\left(\int_0^1 u^7 dx\right)^{2/5},$$

$$\int_0^1 u^6 v dx \le \frac{6}{7}\int_0^1 u^7 dx + \frac{1}{7}\int_0^1 v^7 dx \le \frac{6}{7}\int_0^1 \left(u^7 + v^7\right) dx,$$

$$\int_0^1 u^4 v dx \le \frac{1}{2}\int_0^1 u^2 v dx + \frac{1}{2}\int_0^1 u^6 v dx \le \frac{1}{2} M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5} + \frac{3}{7}\int_0^1 \left(u^7 + v^7\right) dx,$$

$$\int_0^1 u u_t dx \le \frac{1}{2\epsilon}\int_0^1 u dx + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx \le \frac{1}{2\epsilon} M_1 d^{-1} + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx,$$

$$\int_0^1 u^2 u_t dx \le \frac{1}{2\epsilon}\int_0^1 u^3 dx + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx \le \frac{1}{2\epsilon} M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5} + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx,$$

$$\int_0^1 u^3 u_t dx \le \frac{1}{2\epsilon}\int_0^1 u^5 dx + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx \le \frac{1}{2\epsilon} M_2^{2/5} d^{-4/5}\left(\int_0^1 u^7 dx\right)^{3/5} + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx,$$

$$\int_0^1 u^4 u_t dx \le \frac{1}{2\epsilon}\int_0^1 u^7 dx + \frac{\epsilon}{2}\int_0^1 u u_t^2 dx,$$

$$\int_0^1 u v u_t dx \le \frac{1}{2\epsilon}\int_0^1 u^2 v dx + \frac{\epsilon}{2}\int_0^1 v u_t^2 dx \le \frac{1}{2\epsilon} M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5} + \frac{\epsilon}{2}\int_0^1 v u_t^2 dx,$$

$$\int_0^1 u^2 v u_t dx \le \frac{1}{2\epsilon}\int_0^1 u^4 v dx + \frac{\epsilon}{2}\int_0^1 v u_t^2 dx$$

$$\leq \frac{1}{4\epsilon}M_2^{4/5}d^{-8/5}\left(\int_0^1 u^7dx\right)^{1/5} + \frac{3}{14\epsilon}\int_0^1\left(u^7+v^7\right)dx + \frac{\epsilon}{2}\int_0^1 vu_t^2dx,$$

$$\int_0^1 u^3vu_tdx \leq \frac{1}{2\epsilon}\int_0^1 u^6vdx + \frac{\epsilon}{2}\int_0^1 vu_t^2dx \leq \frac{3}{14\epsilon}\int_0^1\left(u^7+v^7\right)dx + \frac{\epsilon}{2}\int_0^1 vu_t^2dx. \tag{2.29}$$

Applying the above estimates and Gagliardo-Nirenberg-type inequalities to the terms on the right-hand side of (2.28), we have

$$-\int_0^1 u_t^2dx \leq -\frac{1}{2}\int_0^1 P_{xx}^2dx + \int_0^1 f^2dx,$$

$$-\int_0^1 v_t^2dx \leq -\frac{1}{2}\int_0^1 Q_{xx}^2dx + \int_0^1 g^2dx,$$

$$-\eta\int_0^1 w_t^2dx \leq -\frac{\eta}{2}\int_0^1 R_{xx}^2dx + \eta\int_0^1 h^2dx,$$

$$\begin{aligned}
\int_0^1 f^2dx &\leq \beta^2d^{-2}\int_0^1 v^2dx + a^2d^{-2}\int_0^1 u^2dx + b^2\int_0^1 u^4dx + 2bcd^2\int_0^1 u^5dx + c^2d^4\int_0^1 u^6dx \\
&\quad + 2abd^{-1}\int_0^1 u^3dx + 2ac\int_0^1 u^4dx + d^{-2}\int_0^1 w^2dx \\
&\quad + 2ad^{-2}\int_0^1 uwdx + 2bd^{-1}\int_0^1 u^2wdx + 2\int_0^1 u^3wdx \\
&\leq \left(a^2+\beta+1+2a\right)M_2d^{-4} + 2b(a+1)M_2^{3/5}d^{-13/5}\left(\int_0^1 u^7dx\right)^{1/5} \\
&\quad + \left(2ac+b^2+2\right)M_2^{3/5}d^{-6/5}\left(\int_0^1 u^7dx\right)^{2/5} \\
&\quad + 2bcM_2^{2/5}d^{1/5}\left(\int_0^1 u^7dx\right)^{3/5} + c^2M_2^{1/5}d^{8/5}\left(\int_0^1 u^7dx\right)^{4/5},
\end{aligned} \tag{2.30}$$

$$\int_0^1 g^2dx \leq d^{-2}\int_0^1\left(u^2+v^2\right)dx \leq 2M_2d^{-4},$$

$$\begin{aligned}
\eta\int_0^1 h^2dx &\leq d^{-2}\eta\left(k^2+\alpha^2\right)\int_0^1 w^2dx + 2kd^{-1}\gamma\eta\int_0^1 w^3dx + \gamma^2\eta\int_0^1 w^4dx \\
&\leq \eta\left(k^2+\alpha^2\right)M_2d^{-4} + 2k\gamma\eta M_2^{4/5}d_1^{-13/5}\left(\int_0^1 w^7dx\right)^{1/5} \\
&\quad + \gamma^2\eta M_2^{3/5}d^{-6/5}\left(\int_0^1 w^7dx\right)^{2/5}.
\end{aligned}$$

Thus

$$
\begin{aligned}
&-\int_0^1 u_t^2\,dx-\int_0^1 v_t^2\,dx-\eta\int_0^1 w_t^2\,dx \\
&\quad\le -\frac{1}{2}\int_0^1 P_{xx}^2\,dx-\frac{1}{2}\int_0^1 Q_{xx}^2\,dx-\frac{\eta}{2}\int_0^1 R_{xx}^2\,dx+C_4(2+\eta)M_2d^{-4} \\
&\qquad+C_5d^{-1}(1+\eta)M_2^{4/5}d^{-8/5}\left[\int_0^1\left(u^7+w^7\right)dx\right]^{1/5}+C_6(1+\eta)M_2^{3/5}d^{-6/5}\left[\int_0^1\left(u^5+w^7\right)dx\right]^{2/5} \\
&\qquad+C_7M_2^{2/5}d^{1/5}\left(\int_0^1 u^7\,dx\right)^{3/5}+C_8M_2^{1/5}d^{8/5}\left(\int_0^1 u^7\,dx\right)^{4/5}.
\end{aligned}
\tag{2.31}
$$

For the other terms on the right-hand side of (2.28), we have

$$
\begin{aligned}
\int_0^1 u_t f\,dx &\le \beta d^{-1}\left|\int_0^1 u_t v\,dx\right|+ad^{-1}\left|\int_0^1 uu_t\,dx\right|+b\left|\int_0^1 u^2u_t\,dx\right| \\
&\quad+cd\left|\int_0^1 u^3u_t\,dx\right|+d^{-1}\left|\int_0^1 wu_t\,dx\right| \\
&\le \frac{\beta^2+a^2+1}{2\epsilon}M_1d^{-3}+\frac{b^2}{2\epsilon}M_2^{4/5}d^{-8/5}\left(\int_0^1 u^7\,dx\right)^{1/5} \\
&\quad+\frac{c^2}{2\epsilon}M_2^{2/5}d^{6/5}\left(\int_0^1 u^7\,dx\right)^{3/5}+\frac{3}{2}\epsilon\int_0^1 uu_t^2\,dx+\frac{\beta d^{-1}}{2}\epsilon\int_0^1 vu_t^2\,dx+\frac{1}{2}\epsilon\int_0^1 wu_t^2\,dx, \\
2\alpha_{11}\int_0^1 uu_t f\,dx &\le 2\alpha_{11}\beta d^{-1}\left|\int_0^1 uu_t v\,dx\right|+2\alpha_{11}ad^{-1}\left|\int_0^1 u^2u_t\,dx\right| \\
&\quad+2\alpha_{11}^2 b\left|\int_0^1 u^3u_t\,dx\right|+2\alpha_{11}dc\left|\int_0^1 u^4u_t\,dx\right|+2\alpha_{11}d^{-1}\left|\int_0^1 uu_t w\,dx\right| \\
&\le \frac{\alpha_{11}^2\left(\beta^2+a^2+1\right)}{\epsilon}M_2^{4/5}d^{-18/5}\left(\int_0^1 u^7\,dx\right)^{1/5}+\frac{\alpha_{11}^2b^{2\,3}}{\epsilon}M_2^{2/5}d^{-4/5}\left(\int_0^1 u^7\,dx\right)^{3/5} \\
&\quad+\frac{\alpha_{11}^2c^2}{\epsilon}d^2\int_0^1 u^7\,dx+3\epsilon\int_0^1 uu_t^2\,dx+\epsilon\int_0^1 vu_t^2\,dx+\epsilon\int_0^1 wu_t^2\,dx,
\end{aligned}
$$

$$\alpha_{12}\int_0^1 vu_t f\,dx \le \alpha_{12}\beta d^{-1}\left|\int_0^1 u_t v^2 dx\right| + \alpha_{12} a d^{-1}\left|\int_0^1 uvu_t dx\right| + \alpha_{12}^2 b\left|\int_0^1 u^2 vu_t dx\right|$$
$$+ \alpha_{12} dc\left|\int_0^1 u^3 vu_t dx\right| + \alpha_{12} d^{-1}\left|\int_0^1 vwu_t dx\right|$$
$$\le \frac{\alpha_{12}^2}{2\epsilon}\left(a^2 d^{-2} + \frac{b^2}{2}\right) M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5}$$
$$+ \frac{\alpha_{12}^2(\beta^2+1)}{2\epsilon} M_2^{4/5} d^{-18/5}\left(\int_0^1 v^7 dx\right)^{1/5}$$
$$+ \frac{3\alpha_{12}^2}{7\epsilon}\left(\frac{b^2}{2} + c^2 d^2\right)\int_0^1 \left(u^7 + v^7\right) dx + \frac{5}{2}\epsilon\int_0^1 vu_t^2 dx,$$

$$\alpha_{13}\int_0^1 wu_t f\,dx \le \alpha_{13}\beta d^{-1}\left|\int_0^1 vwu_t dx\right| + \alpha_{13} a d^{-1}\left|\int_0^1 uwu_t dx\right| + \alpha_{13}^2 b\left|\int_0^1 u^2 wu_t dx\right|$$
$$+ \alpha_{13} dc\left|\int_0^1 u^3 wu_t dx\right| + \alpha_{13} d^{-1}\left|\int_0^1 w^2 u_t dx\right|$$
$$\le \frac{\alpha_{13}^2(a^2 d^{-2} + b^2/2)}{2\epsilon} M_2^{4/5} d^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5}$$
$$+ \frac{\alpha_{13}^2}{2\epsilon} M_2^{4/5} d^{-18/5}\left[\beta\left(\int_0^1 v^7 dx\right)^{1/5} + \left(\int_0^1 w^7 dx\right)^{1/5}\right]$$
$$+ \frac{3\alpha_{13}^2}{7\epsilon}\left(\frac{b^2}{2} + c^2 d^2\right)\int_0^1 \left(u^7 + w^7\right) dx + \frac{5}{2}\epsilon\int_0^1 wu_t^2 dx,$$

$$\int_0^1 v_t g\,dx \le d^{-1}\left|\int_0^1 uv_t dx\right| + d^{-1}\left|\int_0^1 vv_t dx\right| \le \frac{M_1 d^{-3}}{\epsilon} + \frac{\epsilon}{2}\int_0^1 \left(uv_t^2 + vv_t^2\right) dx,$$

$$\alpha_{21}\int_0^1 uv_t g\,dx \le \alpha_{21} d^{-1}\left|\int_0^1 u^2 v_t dx\right| + \alpha_{21} d^{-1}\left|\int_0^1 uvv_t dx\right|$$
$$\le \frac{\alpha_{21}^2}{\epsilon} M_2^{4/5} d^{-18/5}\left(\int_0^1 u^7 dx\right)^{1/5} + \frac{\epsilon}{2}\int_0^1 \left(uv_t^2 + vv_t^2\right) dx,$$

$$2\alpha_{22}\int_0^1 vv_t g\,dx \le 2\alpha_{22} d^{-1}\left|\int_0^1 uvv_t dx\right| + 2\alpha_{22} d^{-1}\left|\int_0^1 v^2 v_t dx\right|$$
$$\le \frac{\alpha_{22}\int_0^1 vv_t^2 dx}{\epsilon} M_2^{4/5} d^{-18/5}\left[\left(\int_0^1 u^7 dx\right)^{1/5} + \left(\int_0^1 v^7 dx\right)^{1/5}\right] + \epsilon\int_0^1 vv_t^2 dx,$$

$$\alpha_{23}\int_0^1 wv_t g\,dx \le \alpha_{23} d^{-1}\left|\int_0^1 uwv_t dx\right| + \alpha_{23} d^{-1}\left|\int_0^1 vwv_t dx\right|$$

$$\leq \frac{\alpha_{23}\int_0^1 v v_t^2 dx}{2\epsilon} M_2^{4/5} d^{-18/5} \left[\left(\int_0^1 u^7 dx \right)^{1/5} + \left(\int_0^1 v^7 dx \right)^{1/5} \right] + \epsilon \int_0^1 w v_t^2 dx,$$

$$\eta \int_0^1 w_t h dx \leq d^{-1} \eta (\alpha + k) \left| \int_0^1 w w_t dx \right| + \gamma \eta \left| \int_0^1 w^2 w_t dx \right|$$

$$\leq \frac{(\alpha + k)^2}{2\epsilon} \eta^2 d^{-3} M_1 + \frac{\gamma^2}{2\epsilon} \eta^2 M_2^{4/5} d^{-8/5} \left(\int_0^1 w^7 dx \right)^{1/5} + \epsilon \eta \int_0^1 w w_t^2 dx,$$

$$\alpha_{31} \int_0^1 u w_t h dx \leq (\alpha + k) d^{-1} \alpha_{31} \left| \int_0^1 u w w_t dx \right| + \alpha_{31} \gamma \left| \int_0^1 u w^2 w_t dx \right|$$

$$\leq \frac{\alpha_{31}^2}{2\epsilon} \left[(\alpha + k)^2 d^{-2} + \frac{\gamma^2}{2} \right] M_2^{4/5} d^{-8/5} \left[\left(\int_0^1 u^7 dx \right)^{1/5} + \left(\int_0^1 w^7 dx \right)^{1/5} \right]$$

$$+ \frac{3\alpha_{31}^2 \gamma^2}{14\epsilon} \int_0^1 \left(u^7 + w^7 \right) dx + \frac{1}{2} \epsilon \int_0^1 u w_t^2 dx + \frac{1}{2} \epsilon \int_0^1 w w_t^2 dx,$$

$$\alpha_{32} \int_0^1 v w_t h dx \leq (\alpha + k) d^{-1} \alpha_{32} \left| \int_0^1 v w w_t dx \right| + \alpha_{32} \gamma \left| \int_0^1 v w^2 w_t dx \right|$$

$$\leq \frac{\alpha_{32}^2}{2\epsilon} \left[(\alpha + k)^2 d^{-2} + \frac{\gamma^2}{2} \right] M_2^{4/5} d^{-8/5} \left[\left(\int_0^1 v^7 dx \right)^{1/5} + \left(\int_0^1 w^7 dx \right)^{1/5} \right]$$

$$+ \frac{3\alpha_{32}^2}{14\epsilon} \gamma^2 \int_0^1 \left(v^7 + w^7 \right) dx + \frac{1}{2} \epsilon \int_0^1 v w_t^2 dx + \frac{1}{2} \epsilon \int_0^1 w w_t^2 dx,$$

$$2\alpha_{33} \int_0^1 w w_t h dx \leq 2(\alpha + k) d^{-1} \alpha_{33} \left| \int_0^1 w^2 w_t dx \right| + 2\alpha_{33} \gamma \left| \int_0^1 w^3 w_t dx \right|$$

$$\leq \frac{(\alpha + k)^2 \alpha_{33}^2}{\epsilon} M_2^{4/5} d^{-18/5} \left(\int_0^1 w^7 dx \right)^{1/5} + \frac{\alpha_{33}^2 \gamma^2}{\epsilon} M_2^{2/5} d^{-4/5} \left(\int_0^1 w^7 dx \right)^{3/5}$$

$$+ \epsilon \int_0^1 w w_t^2 dx,$$

$$\alpha_{12} \int_0^1 u v_t f dx \leq \alpha_{12} \beta d^{-1} \left| \int_0^1 u v v_t dx \right| + \alpha_{12} a d^{-1} \left| \int_0^1 u^2 v_t dx \right| + \alpha_{12} b \left| \int_0^1 u^3 v_t dx \right|$$

$$+ \alpha_{12} c d \left| \int_0^1 u^4 v_t dx \right| + \alpha_{12} d^{-1} \left| \int_0^1 u w v_t dx \right|$$

$$\leq \frac{\alpha_{12}^2 (\beta^2 + a^2 + 1)}{2\epsilon} M_2^{4/5} d^{-18/5} \left(\int_0^1 u^7 dx \right)^{1/5} + \frac{\alpha_{12}^2 b^2}{2\epsilon} M_2^{2/5} d^{-4/5} \left(\int_0^1 u^7 dx \right)^{3/5}$$

$$+\frac{\alpha_{12}^2c^2}{2\epsilon}d^2\int_0^1 u^7dx+\frac{\epsilon}{2}\left[3\int_0^1 uv_t^2dx+\int_0^1 vv_t^2dx+\int_0^1 wv_t^2dx\right],$$

$$\alpha_{13}\int_0^1 uw_tf\,dx\le\alpha_{13}\beta d^{-1}\left|\int_0^1 uvw_tdx\right|+\alpha_{13}ad^{-1}\left|\int_0^1 u^2w_tdx\right|+\alpha_{13}b\left|\int_0^1 u^3w_tdx\right|$$

$$+\alpha_{13}^3cd\left|\int_0^1 u^4w_tdx\right|+\alpha_{13}d^{-1}\left|\int_0^1 uww_tdx\right|$$

$$\le\frac{\alpha_{13}^2(\beta^2+a^2+1)}{2\epsilon}M_2^{4/5}d^{-18/5}\left(\int_0^1 u^7dx\right)^{1/5}+\frac{\alpha_{13}^2b^2}{2\epsilon}M_2^{2/5}d^{-4/5}\left(\int_0^1 u^7dx\right)^{3/5}$$

$$+\frac{\alpha_{13}^2c^2}{2\epsilon}d^2\int_0^1 u^7dx+\frac{3}{2}\epsilon\int_0^1 uw_t^2dx+\frac{1}{2}\epsilon\int_0^1 vw_t^2dx+\frac{1}{2}\epsilon\int_0^1 ww_t^2dx,$$

$$\alpha_{21}\int_0^1 vu_tg\,dx\le\alpha_{21}d^{-1}\left|\int_0^1 uvu_tdx\right|+\alpha_{21}d^{-1}\left|\int_0^1 v^2u_tdx\right|$$

$$\le\frac{\alpha_{21}^2}{2\epsilon}M_2^{4/5}d^{-18/5}\left[\left(\int_0^1 u^7dx\right)^{1/5}+\left(\int_0^1 v^7dx\right)^{1/5}\right]+\epsilon\int_0^1 vu_t^2dx,$$

$$\alpha_{23}\int_0^1 vw_tg\,dx\le\alpha_{23}d^{-1}\left|\int_0^1 uvw_tdx\right|+\alpha_{23}d^{-1}\left|\int_0^1 v^2w_tdx\right|$$

$$\le\frac{\alpha_{23}^2}{2\epsilon}M_2^{4/5}d^{-18/5}\left[\left(\int_0^1 u^7dx\right)^{1/5}+\left(\int_0^1 v^7dx\right)^{1/5}\right]+\epsilon\int_0^1 vw_t^2dx,$$

$$\alpha_{31}\int_0^1 wu_th\,dx\le\alpha_{31}(\alpha+k)d^{-1}\left|\int_0^1 w^2u_tdx\right|+\alpha_{31}\gamma\left|\int_0^1 w^3u_tdx\right|$$

$$\le\frac{(\alpha+k)^2\alpha_{31}^2}{2\epsilon}M_2^{4/5}d^{-18/5}\left(\int_0^1 w^7dx\right)^{1/5}$$

$$+\frac{\alpha_{31}^2}{2\epsilon}\gamma^2M_2^{2/5}d^{-4/5}\left(\int_0^1 w^7dx\right)^{3/5}+\epsilon\int_0^1 wu_t^2dx,$$

$$\alpha_{32}\int_0^1 wv_th\,dx\le\alpha_{32}(\alpha+k)d^{-1}\left|\int_0^1 w^2v_tdx\right|+\alpha_{32}\gamma\left|\int_0^1 w^3v_tdx\right|$$

$$\le\frac{(\alpha^2+k^2)\alpha_{32}^2}{2\epsilon}M_2^{4/5}d^{-18/5}\left(\int_0^1 w^7dx\right)^{1/5}$$

$$+\frac{\alpha_{32}^2\gamma^2}{2\epsilon}M_2^{2/5}d^{-4/5}\left(\int_0^1 w^7dx\right)^{3/5}+\epsilon\int_0^1 wv_t^2dx. \tag{2.32}$$

Thus

$$\begin{aligned}
&\int_0^1 (1+2\alpha_{11}+\alpha_{12}v+\alpha_{13})u_t f\,dx+\int_0^1 (1+\alpha_{21}u+2\alpha_{22}v+\alpha_{23}w)v_t g\,dx \\
&\quad+\int_0^1 (\eta+\alpha_{31}u+\alpha_{32}v+2\alpha_{33}w)w_t h\,dx+\int_0^1 \alpha_{12}uv_t f\,dx+\int_0^1 \alpha_{13}uw_t f\,dx \\
&\quad+\int_0^1 \alpha_{21}vu_t g\,dx+\int_0^1 \alpha_{23}vw_t g\,dx+\int_0^1 \alpha_{31}wu_t h\,dx+\int_0^1 \alpha_{32}wv_t h\,dx \\
&\leq \lambda\epsilon\int_0^1 (u+v+w)\left(u_t^2+v_t^2+w_t^2\right)dx+\frac{C_9}{\epsilon}M_1 d^{-3}\left(2+\eta^2\right) \\
&\quad+\frac{C_{10}}{\epsilon}M_2^{4/5}d^{-8/5}\left(2+d^{-2}+\eta^2\right)\left[\int_0^1 (u^7+v^7+w^7)dx\right]^{1/5} \\
&\quad+\frac{C_{11}}{\epsilon}M_2^{4/5}d^{-8/5}\left(1+d^2\right)\left[\int_0^1 (u^7+v^7+w^7)dx\right]^{3/5} \\
&\quad+\frac{C_{12}}{\epsilon}\left(1+d^2\right)\int_0^1\left(u^7+v^7+w^7\right)dx,
\end{aligned} \tag{2.33}$$

where λ is a positive constant.

Note by (2.8) and (2.9) that $|P|_{7/2}^{7/2} \leq C(|P_x|_2^{5/3}|P|_1^{11/6}+|P|_1^{7/2})$, $|P_x|_2^{10/3} \leq B_1 K_1^{4/3} d^{-4/3}(|P_{xx}|_2^2+K_1^2 d^{-2})$, and

$$-\frac{1}{2}\int_0^1 P_{xx}^2 dx-\frac{1}{2}\int_0^1 Q_{xx}^2 dx-\frac{\eta}{2}\int_0^1 R_{xx}^2 dx \leq -B_2\min\{1,\eta\}K_1^{-4/3}d^{4/3}\overline{y}^{5/3}+K_1^2 d^{-2}(2+\eta). \tag{2.34}$$

Choose a small enough number $\epsilon>0$, such that $\lambda\epsilon<C_3$. According to (2.28)–(2.34), we have

$$\begin{aligned}
\frac{1}{2}y'(t) &\leq -A_1\min\{1,\eta\}K_1^{-4/3}y^{5/3}+A_2K_2y^{1/6}+A_3K_3y^{1/3}+A_4K_4y^{1/2} \\
&\quad+A_5K_5y^{2/3}+A_6K_6y^{5/6}+A_7K_7,
\end{aligned} \tag{2.35}$$

where $y=\int_0^1[(dP_x)^2+(dQ_x)^2+(dR_x)^2]dx$.

However, (2.35) implies that there exist positive constants $\tilde{\tau}_3>0$ and $\widetilde{M}_3$ depending on $d, d_3, \alpha_{ij}(i,j=1,2,3), \beta, a, b, c, k, \gamma, \alpha$, such that

$$\int_0^1 (dP_x)^2 dx, \int_0^1 (dQ_x)^2 dx, \int_0^1 (dR_x)^2 dx \leq \widetilde{M}_3, \quad t \geq \tilde{\tau}_3. \tag{2.36}$$

When $d, d_3 \geq 1, \eta \in [\underline{d}, \overline{d}]$, the coefficients of (2.35) can be estimated by constants depending on $\underline{d}, \overline{d}$, but not on d, d_3. Thus, when $d, d_3 \geq 1, \eta \in [\underline{d}, \overline{d}]$, $\widetilde{M}_3$ depends on α_{ij} $(i, j = 1, 2, 3), \beta, a, b, c, k, \gamma, \alpha, \underline{d}, \overline{d}$, and is irrelevant to $d, d_3 \geq 1$. Since

$$\begin{pmatrix} P_x \\ Q_x \\ R_x \end{pmatrix} = \begin{pmatrix} P_u & P_v & P_w \\ Q_u & Q_v & Q_w \\ R_u & R_v & R_w \end{pmatrix} \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}, \tag{2.37}$$

similar to (2.26) in [24], we have

$$|du_x| + |dv_x| + |dw_x| \leq D(|dP_x| + |dQ_x| + |dR_x|), \quad 0 < x < 1, \ t > 0, \tag{2.38}$$

where D is a positive constant only depending on η, α_{ij} $(i, j = 1, 2, 3)$. Scaling back with (2.22) to original variable u, v, w, t and combining (2.36),(2.38), there exist positive constants $\tau_3 > 0$ and M_3 depending on d, d_3, α_{ij} $(i, j = 1, 2, 3), \beta, a, b, c, k, \gamma, \alpha$, such that

$$\int_0^1 u_x^2 dx, \int_0^1 v_x^2 dx, \int_0^1 w_x^2 dx \leq M_3, \quad t \geq \tau_3. \tag{2.39}$$

In addition, when $d, d_3 \geq 1, \eta \in [\underline{d}, \overline{d}]$, M_3 is dependent of $\underline{d}, \overline{d}$, but independent of $d, d_3 \geq 1$.

(ii) It holds that $t \geq 0$. Replacing M_1, M_2 with M_1', M_2' in (2.24)–(2.34), we can obtain that there exists a positive constant M_3' depending on d, d_3, α_{ij} $(i, j = 1, 2, 3), \beta, a, b, c, k, \gamma, \alpha$ and the W_2^1-norm of u_0, v_0, w_0 such that

$$\int_0^1 u_x^2 dx, \int_0^1 v_x^2 dx, \int_0^1 w_x^2 dx \leq M_3', \quad t \geq 0. \tag{2.39$'$}$$

When $d, d_3 \geq 1, \eta \in [\underline{d}, \overline{d}]$, M_3' is dependent of $\underline{d}, \overline{d}$, but independent of $d, d_3 \geq 1$.

Concluding from (2.15), (2.22), (2.39), and *Sobolev* embedding theorem, there exists a positive constants $t_0 > 0$, M, M' depending on d, d_3, α_{ij} $(i, j = 1, 2, 3), \beta, a, b, c, k, \gamma, \alpha$, such that (2.3) and (2.4) are satisfied. Furthermore, when $d, d_3 \geq 1$, $\eta \in [\underline{d}, \overline{d}]$ and the time t is large enough, M, M' are dependent of α_{ij} $(i, j = 1, 2, 3), \beta, a, b, c, k, \gamma, \alpha, \underline{d}, \overline{d}$, but independent of $d, d_3 \geq 1$.

Similarly, according to (2.15$'$), (2.22$'$), (2.39$'$), we can see that there exists a positive constant M'' depending on d, d_3, α_{ij} $(i, j = 1, 2, 3), \beta, a, b, c, k, \gamma, \alpha$ and the initial functions u_0, v_0, w_0, such that

$$|u(\cdot, t)|_{1,2}, |v(\cdot, t)|_{1,2}, |w(\cdot, t)|_{1,2} \leq M'', \quad t \geq 0. \tag{2.40}$$

When $d, d_3 \geq 1$, $\eta \in [\underline{d}, \overline{d}]$, M'' is dependent of $\underline{d}$, $\overline{d}$, but independent of d, d_3. Thus $T = +\infty$. This completes proof of Theorem 2.1. □

3. Global Stability

From [1], we know that if

$$\alpha > k, \quad \beta > a, \quad \sqrt{\frac{k}{\alpha - k}} < m_0,$$
$$\frac{\beta - a - c}{2} + \frac{b^2}{8c} \leq \frac{b\sqrt{p_1}}{24c} + \frac{24(\beta - a)c^2}{3b^2 + 4c(\beta - a - c) - b\sqrt{p_1}}, \tag{H}$$

where ($p_1 = 9b^2 + 24c(\beta - a - c) \geq 0$), then (1.1) has the unique position equilibrium point $E^*(u^*, v^*, w^*)$.

Theorem 3.1. *Assume that all conditions in Theorem 2.1 and* (H) *are satisfied. Assume further that*

$$\frac{1}{\beta}\left(a + bu^* + cu^{*2}\right) > 2 + \frac{\left(u^{*2} + \sqrt{1 + u^{*2}}\right)^2}{8} + \frac{u^{*4}}{2}, \qquad \frac{\gamma}{\alpha} > \frac{1}{2(1 + u^{*2})^2}, \tag{3.1}$$

$$\begin{aligned} \frac{4}{\alpha\beta} w^* d^2 d_3 > {} & \frac{1}{\beta}\left(\alpha_{23} M^2 + \frac{1}{\alpha}\alpha_{32} w^*\right)^2 (d + 2\alpha_{11} M + \alpha_{12} M + \alpha_{13} M) \\ & + \left(\frac{1}{\beta}\alpha_{13} M^2 + \frac{1}{\alpha}\alpha_{31} w^*\right)^2 (d + \alpha_{21} M + 2\alpha_{22} M + \alpha_{23} M) \\ & + \frac{1}{\alpha}\left(\frac{1}{\beta}\alpha_{12} + \alpha_{21}\right)^2 M^2 w^* (d_3 + \alpha_{31} M + \alpha_{32} M + 2\alpha_{33} M) \end{aligned} \tag{3.2}$$

hold, where M *is the positive constant in* (2.4). *Then the unique positive equilibrium point* E^* *of* (1.1) *is globally asymptotically stable.*

Remark 3.2. Since M is independent of d, d_3 in the case of $d, d_3 \geq 1$, (3.2) is always satisfied if d and d_3 are big enough.

Proof. Define the Lyapunov function

$$H(u, v, w) = \frac{1}{2\beta}\int_0^1 (u - u^*)^2 dx + \frac{1}{2}\int_0^1 (v - v^*)^2 dx + \frac{1}{\alpha}\int_0^1 \left(w - w^* - w^* \ln\frac{w}{w^*}\right) dx. \tag{3.3}$$

Let (u,v,w) be any solution of (1.1) with initial functions $u_0(x), v_0(x), w_0(x) \geq (\not\equiv)0$. From the strong maximum principle for parabolic equations, it is not hard to verify that $u,v,w>0$ for $t>0$. Thus

$$\begin{aligned}
\frac{dH}{dt} \leq & -\int_0^1 \bigg[\frac{1}{\beta}(d+2\alpha_{11}u+\alpha_{12}v+\alpha_{13}w)u_x^2+(d+\alpha_{21}u+2\alpha_{22}v+\alpha_{23}w)v_x^2 \\
& \quad +\frac{1}{\alpha}(d_3+\alpha_{31}u+\alpha_{32}v+2\alpha_{33}w)\frac{w^*}{w^2}w_x^2+\left(\frac{1}{\beta}\alpha_{12}u+\alpha_{21}v\right)u_xv_x \\
& \quad +\left(\frac{1}{\beta}\alpha_{13}u+\frac{1}{\alpha}\alpha_{31}\frac{w^*}{w}\right)u_xw_x+\left(\alpha_{23}v+\frac{1}{\alpha}\alpha_{32}\frac{w^*}{w}\right)v_xw_x\bigg]dx \\
& -\int_0^1\bigg\{(u-u^*)^2\frac{1}{\beta}\left[a+b(u+u^*)+c\left(u^2+uu^*+u^{*2}\right)+\frac{w(u+u^*)}{(1+u^2)(1+u^{*2})}\right] \\
& \quad -2-\frac{1}{2}\left(\frac{u+u^*}{1+u^2}-u^{*2}\right)^2\bigg\}dx-\frac{1}{2}\int_0^1(v-v^*)^2dx \\
& -\int_0^1(w-w^*)^2\left[\frac{\gamma}{\alpha}-\frac{1}{2(1+u^{*2})^2}\right]dx.
\end{aligned} \tag{3.4}$$

The first integrand in the right hand of the above inequality is positive definite if

$$\begin{aligned}
& \frac{4}{\alpha\beta}w^*(d+2\alpha_{11}u+\alpha_{12}v+\alpha_{13}w)(d+\alpha_{21}u+2\alpha_{22}v+\alpha_{23}w)(d_3+\alpha_{31}u+\alpha_{32}v+2\alpha_{33}w) \\
& \quad +w^2\left(\frac{1}{\beta}\alpha_{12}u+\alpha_{21}v\right)\left(\frac{1}{\alpha}\alpha_{13}u+\frac{1}{\alpha}\alpha_{31}\frac{w^*}{w}\right)\left(\alpha_{23}v+\beta\alpha_{32}\frac{w^*}{w}\right) \\
& \quad >\frac{1}{\beta}\left(\alpha_{23}vw+\frac{1}{\alpha}\alpha_{32}w^*\right)^2(d+2\alpha_{11}u+\alpha_{12}v+\alpha_{13}w) \\
& \quad +\left(\frac{1}{\beta}\alpha_{13}uw+\frac{1}{\alpha}\alpha_{31}w^*\right)^2(d+\alpha_{21}u+2\alpha_{22}v+\alpha_{23}w) \\
& \quad +\frac{1}{\alpha}w^*\left(\frac{1}{\beta}\alpha_{12}u+\alpha_{21}v\right)^2(d_3+\alpha_{31}u+\alpha_{32}v+2\alpha_{33}w).
\end{aligned} \tag{3.5}$$

From the maximum-norm estimate in Theorem 2.1, (3.2) is a sufficient condition of (3.5). Thus when (3.1) holds, there exists a positive constant δ such that

$$\frac{dH(u,v,w)}{dt} \leq -\delta\int_0^1\left[(u-u^*)^2+(v-v^*)^2+(w-w^*)^2\right]dx. \tag{3.6}$$

By integration by parts, Hölder inequality and the maximum-norm estimate in Theorem 2.1, we can see that $d/dt(\int_0^1[(u-u^*)^2+(v-v^*)^2+(w-w^*)^2])dx$ is bounded from above. According to Lemma 3.1 in [1] and (3.6), we obtain

$$|u(\cdot,t)-u^*|_2 \longrightarrow 0, \quad |v(\cdot,t)-v^*|_2 \longrightarrow 0, \quad |w(\cdot,t)-w^*|_2 \longrightarrow 0, \quad (t \longrightarrow \infty). \tag{3.7}$$

Using Gagliardo-Nirenberg inequalities, we have $|u(\cdot,t)|_\infty \leq C|u|_{1,2}^{1/2}|u|_2^{1/2}$. Thus

$$|u(\cdot,t)-u^*|_\infty \longrightarrow 0, \quad |v(\cdot,t)-v^*|_\infty \longrightarrow 0, \quad |w(\cdot,t)-w^*|_\infty \longrightarrow 0, \quad (t \longrightarrow \infty). \tag{3.8}$$

That is, (u,v,w) converges uniformly to E^*. Since $H(u,v,w)$ is decreasing for $t>0$, E^* is globally asymptotically stable. □

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