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Research Article

Infinitely Many Solutions for a Boundary Value Problem with Discontinuous Nonlinearities

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The existence of infinitely many solutions for a Sturm-Liouville boundary value problem, under an appropriate oscillating behavior of the possibly discontinuous nonlinear term, is obtained. Several special cases and consequences are pointed out and some examples are presented. The technical approach is mainly based on a result of infinitely many critical points for locally Lipschitz functions.

1. Introduction

The aim of this paper is to establish infinitely many solutions for two-point boundary value problems with the nonlinear term possibly discontinuous. We immediately emphasize the following theorem which is a particular case of our main result (Theorem 3.1).

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded, and almost everywhere continuous function such that $\inf_{\mathbb{R}} f > 0$. Put $F(\xi) := \int_0^{\xi} f(t)dt$ for every $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{4} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}. \tag{1.1}$$

Then, for each $\lambda \in]8/\limsup_{\xi \to +\infty} F(\xi)/\xi^2, 2/\liminf_{\xi \to +\infty} F(\xi)/\xi^2[$, the problem

$$\begin{aligned} -u'' &= \lambda f(u) \quad \text{in }]0,1[\\ u(0) &= u(1) = 0 \end{aligned} \tag{$G^{1,0}_{f,\lambda}$}$$

admits a sequence of pairwise distinct positive weak solutions.

Clearly, when f is continuous in $\mathbb{R}$, the solutions in Theorem 1.1 are classical (in this case, it is enough to assume $\inf_{\mathbb{R}} f \geq 0$; see Corollary 3.5). Moreover, substituting $\xi \to +\infty$ with $\xi \to 0^+$, the same results hold and, in addition, the sequence of pairwise distinct positive solutions uniformly converges to zero (see Theorem 3.9 and Corollary 3.10).

When f is a continuous function, results of the existence of infinitely many solutions for problem $(G^{1,0}_{f,\lambda})$ are obtained, for example, in [1–7]. We observe that in the very interesting paper [6], the authors assume $\liminf_{\xi \to +\infty} F(\xi)/\xi^2 = 0$ and $\limsup_{\xi \to +\infty} F(\xi)/\xi^2 = +\infty$, which are conditions that imply our key assumption. Very recently, in [4], a more general condition than the previous assumption has been assumed, requiring in addition, however, that $\lim_{\xi \to +\infty} f(\xi) = +\infty$. Moreover, we also observe that the results in [1, 2] are obtained by using the important Variational Principle of Ricceri [8], which is, basically, the same as our tool. We emphasize that, also when f is a continuos function, our theorems in this paper and the results in [1–7] are mutually independent (see Remark 3.13 and Examples 3.11 and 3.12).

When the nonlinear term f is discontinuous, there have been many approaches to studying a nonlinear eigenvalue differential equation as it arises in physics problems, such as nonlinear elasticity theory, and mechanics, and engineering topics. Chang in [9] established the critical point theory for nondifferentiable functionals and presented some applications to partial differential equations with discontinuous nonlinearities. Next, Motreanu and Panagiotopoulos (see [10, Chapter 3]) studied the critical point theory for non-smooth functionals and in this framework, very recently, Marano and Motreanu, in [11], obtained an infinitely many critical points theorem, which extends the Variational Principle of Ricceri to non-smooth functionals, and applies this result to variational-hemivariational inequalities and semilinear elliptic eigenvalue problems with discontinuous nonlinearities.

In this paper, we present a more precise version of the infinitely many critical points theorem of Marano and Motreanu (Theorem 2.1), obtained by a completely different proof (see Remark 2.2) and, by using the previous theorem, we establish our main result (Theorem 3.1) on the existence of infinitely many solutions for a two-point boundary value problem with the Sturm-Liouville equation having discontinuous nonlinear term.

We explicitly observe that methods and techniques used in the proof of Theorem 3.1 can be applied to a wide class of nonlinear differential problems to investigate infinitely many solutions. The note is arranged as follows. In Section 2, we recall some basic definitions and our abstract framework, while Section 3 is devoted to infinitely many solutions for the Sturm-Liouville problem.

Finally, we point out that the existence of multiple solutions for nonlinear differential problems has been studied in several papers by using different techniques (see, e.g., [12, 13] and references therein).

2. Infinitely Many Critical Points

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $\Phi : X \to \mathbb{R}$ is called locally Lipschitz continuous when, to every $x \in X$, there corresponds a neighbourhood V_x of x and a constant $L_x \geq 0$ such that

$$\left|\Phi(z) - \Phi(w)\right| \leq L_x \|z - w\| \quad \forall z,\ w \in V_x. \tag{2.1}$$

If $x, z \in X$, we write $\Phi^\circ(x; z)$ for the generalized directional derivative of Φ at the point x along the direction z, that is,

$$\Phi^\circ(x;z) := \limsup_{w \to x,\, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}. \tag{2.2}$$

The generalized gradient of the function Φ in x, denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) := \left\{ x^* \in X^* : \langle x^*, z \rangle \le \Phi^\circ(x;z)\ \forall z \in X \right\}. \tag{2.3}$$

We say that $x \in X$ is a (generalized) critical point of Φ when

$$\Phi^\circ(x;z) \ge 0 \quad \forall z \in X, \tag{2.4}$$

that clearly signifies $0 \in \partial\Phi(x)$. When a non-smooth functional, $\Psi : X \to]-\infty, +\infty]$, is expressed as a sum of a locally Lipschitz function, $\Phi : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to]-\infty, +\infty]$, that is $\Psi := \Phi + j$, a (generalized) critical point of Ψ is every $u \in X$ such that

$$\Phi^\circ(u; v - u) + j(v) - j(u) \ge 0 \tag{2.5}$$

for all $v \in X$ (see [10, Chapter 3]).

Here, and in the sequel, X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D(j)$ is the effective dominion of j.

Write

$$\Psi := \Upsilon - j, \qquad I_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j. \tag{2.6}$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \ne \varnothing \tag{2.7}$$

for all $r > \inf_X \Phi$. Moreover, owing to (2.7) and provided $r > \inf_X \Phi$, we can define

$$\begin{gathered} \varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)\right) - \Psi(u)}{r - \Phi(u)}, \\ \gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r). \end{gathered} \tag{2.8}$$

Assuming also that Φ and Υ are locally Lipschitz functionals, we have the following result, which is a more precise version of [11, Theorem 1.1] (see Remark 2.2).

Theorem 2.1. *Under the above assumptions on* X, Φ *and* Ψ, *one has*

(a) *for every* $r > \inf_X \Phi$ *and every* $\lambda \in]0, 1/\varphi(r)[$, *the restriction of the functional* $I_\lambda = \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty, r[)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X.

(b) *if* $\gamma < +\infty$ *then, for each* $\lambda \in]0, 1/\gamma[$, *the following alternative holds: either*

  (b_1) I_λ *possesses a global minimum, or*

  (b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that* $\lim_{n\to+\infty}\Phi(u_n) = +\infty$.

(c) *if* $\delta < +\infty$ *then, for each* $\lambda \in]0, 1/\delta[$, *the following alternative holds: either*

  (c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

  (c_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ, *with* $\lim_{n\to+\infty}\Phi(u_n) = \inf_X \Phi$, *which weakly converges to a global minimum of* Φ.

Proof. Arguing as in the proof of [14, Theorem 3.1] we have (a). More precisely, let $1/\lambda > \varphi(r)$, then there is $\overline{u} \in D(j)$ such that $\Phi(\overline{u}) < r$ and $\Phi(\overline{u}) - \lambda\Psi(\overline{u}) < r - \lambda \sup_{\Phi(x)<r}\Psi(x)$. Moreover, put

$$M = \frac{r - \Phi(\overline{u})}{\lambda} + \Psi(\overline{u}). \tag{2.9}$$

Clearly,

$$\sup_{\Phi(x)<r}\Psi(x) < M. \tag{2.10}$$

Finally, put

$$\Psi_M(u) = \begin{cases} \Psi(u), & \text{if } \Psi(u) \le M \\ M, & \text{if } \Psi(u) > M. \end{cases} \tag{2.11}$$

Since, owing to [15, Corollary III.8] j is sequentially weakly lower semicontinuous, a simple computation shows that Ψ_M is sequentially weakly upper semicontinuous. Put $J = \Phi - \lambda\Psi_M$. Clearly J is a sequentially weakly lower semicontinuous functional and, as it is easy to see, it is also a coercive functional. Therefore (see, e.g., [16, Theorem 1.2]), it admits a global minimum u_0. If $J(u_0) = J(\overline{u})$, then $\overline{u}$ satisfies the conclusion.

Otherwise, assume $J(u_0) < J(\overline{u})$. In this case, we have that $\Psi(u_0) < M$. In fact, from $J(u_0) < J(\overline{u})$ one has $\Phi(u_0) - \lambda\Psi_M(u_0) < \Phi(\overline{u}) - \lambda\Psi_M(\overline{u})$. Hence, $\Phi(u_0) < \lambda\Psi_M(u_0) + \Phi(\overline{u}) - \lambda\Psi(\overline{u}) \le \lambda M + \Phi(\overline{u}) - \lambda\Psi(\overline{u}) = r$ and, from (2.10) one has $\Psi(u_0) < M$. Therefore, $\Phi(u_0) - \lambda\Psi(u_0) = \Phi(u_0) - \lambda\Psi_M(u_0) \le \Phi(u) - \lambda\Psi_M(u)$ for all $u \in X$ and, taking again (2.10) into account, $\Phi(u_0) - \lambda\Psi(u_0) \le \Phi(u) - \lambda\Psi(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$. Hence, u_0 satisfies the conclusion.

Let us prove (b). Pick $\lambda \in]0, 1/\gamma[$ and assume that (b_1) is not true. We will show that when $\gamma < +\infty$ and I_λ does not posses a global minimum in X, then I_λ admits a sequence of critical points. Let $a \in \mathbb{R}$ such that $\lambda < a < 1/\gamma$. From $\liminf_{r\to+\infty} \varphi(r) < 1/a$ there exists a sequence $\{r_n\}$ such that $\lim_{n\to+\infty} r_n = +\infty$ and $\varphi(r_n) < 1/a$ for all $n \in \mathbb{N}$. Put $r_{n_1} = r_1$. Owing to (a), one can find $u_1 \in \Phi^{-1}(]-\infty, r_{n_1}[)$ such that u_1 is a local minimum for I_λ. From our assumption, u_1 is not a global minimum for I_λ. Therefore, there exists $\overline{u}_1 \in X$ such that $I_\lambda(\overline{u}_1) < I_\lambda(u_1)$. Hence, $\overline{u}_1 \notin \Phi^{-1}(]-\infty, r_{n_1}[)$. Let $r_{n_2} \in \{r_n\}$ such that $r_{n_2} > \Phi(\overline{u}_1)$. Again from (a), there is $u_2 \in \Phi^{-1}(]-\infty, r_{n_2}[)$ such that u_2 is a local minimum for the functional I_λ. Taking into account that u_2 is a global minimum in $\Phi^{-1}(]-\infty, r_{n_2}[)$, we have that $I_\lambda(u_2) \leq I_\lambda(\overline{u}_1)$ and $I_\lambda(\overline{u}_1) < I_\lambda(u_1)$. Hence, $\Phi(u_2) > r_{n_1}$. Reasoning inductively we obtain a sequence $\{u_k\}$ of distinct critical points such that $\Phi(u_{k+1}) > r_{n_k}$ for all $k \in \mathbb{N}$. Hence, (b_2) holds.

Finally, we prove (c). Fix $\lambda \in]0, 1/\delta[$ and let $\overline{u} \in X$ such that $\Phi(\overline{u}) = \inf_X \Phi$. Assume that ($c_1$) does not hold, that is $\overline{u}$ is not a local minimum for I_λ. Consider $a \in \mathbb{R}$ such that $\lambda < a < 1/\delta$. By our assumption, $\liminf_{r\to(\inf_X \Phi)^+} \varphi(r) = \delta < 1/a$, hence there exists a decreasing sequence $\{r_n\}$ such that $\lim_{n\to+\infty} r_n = \inf_X \Phi$ and $\varphi(r_n) < 1/a$ for all $n \in \mathbb{N}$. Put $r_{n_1} = r_1$. Owing to (a), there exists $u_1 \in \Phi^{-1}(]-\infty, r_{n_1}[)$, which is a local minimum for I_λ. Therefore, $u_1 \neq \overline{u}$. Then, $\Phi(\overline{u}) < \Phi(u_1)$. Let $r_{n_2} \in \{r_n\}$ such that $\Phi(\overline{u}) < r_{n_2} < \Phi(u_1)$. Again from (a), there is $u_2 \in \Phi^{-1}(]-\infty, r_{n_2}[)$ which is a local minimum for the functional I_λ, with $\Phi(\overline{u}) < \Phi(u_2) < \Phi(u_1)$. Reasoning inductively we obtain a sequence $\{u_k\}$ of distinct local minima for I_λ such that $\Phi(\overline{u}) < \Phi(u_k) < r_{n_k}$ for all $k \in \mathbb{N}$. Hence, $\lim_{k\to+\infty} \Phi(u_k) = \inf_X \Phi$. Moreover, since $\{u_k\} \subseteq \Phi^{-1}(]-\infty, r_{n_1}[)$ and Φ is coercive, then it is bounded. Since X is reflexive, taking a subsequence if necessary, $\{u_k\}$ weakly converges to $u^* \in X$. From the weak sequential lower semicontinuity, one has $\Phi(u^*) \leq \lim_{k\to+\infty} \Phi(u_k) = \inf_X \Phi$, that is $\Phi(u^*) = \inf_X \Phi$. Hence, the conclusion is obtained. □

Remark 2.2. We explicitly observe that the proof here outlined is different from that proposed by Marano and Motreanu in [11]. Further we do not use the weak closure of the sub-levels $\Phi^{-1}(]-\infty, r[)$, for $r > \inf_X \Phi$.

3. Sturm-Liouville Boundary Value Problem

Consider the Sturm-Liouville boundary value problem

$$\begin{gathered} -(pu')' + qu = \lambda f(u) \quad \text{in }]0,1[\\ u(0) = u(1) = 0, \end{gathered} \tag{$G^{p,q}_{f,\lambda}$}$$

where $p, q \in L^\infty([0,1])$, $f : \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function and λ is a positive parameter.

Denoting D_f the set

$$D_f := \{z \in \mathbb{R} : f \text{ is discontinuous at } z\}, \tag{3.1}$$

we recall that f is said to be continuous almost everywhere if D_f is (Lebesgue) measurable and $m(D_f) = 0$. Moreover, if f is locally essentially bounded, we write

$$f^-(t) := \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-z|<\delta} f(z), \qquad f^+(t) := \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-z|<\delta} f(z) \tag{3.2}$$

for each $t \in \mathbb{R}$. We observe that f^- and f^+ are, respectively, lower semi-continuous and upper semi-continuous.

Assume that

$$p_0 := \operatorname*{ess\,inf}_{x\in[0,1]} p(x) > 0, \qquad q_0 := \operatorname*{ess\,inf}_{x\in[0,1]} q(x) \geq 0. \tag{3.3}$$

Let $W^{1,2}([0,1])$ be the Sobolev space endowed with the usual norm

$$\|u\|_* := \left(\int_0^1 |u'(x)|^2 dx + \int_0^1 |u(x)|^2 dx\right)^{1/2}. \tag{3.4}$$

As is customary, we denote by $W_0^{1,2}([0,1])$ the closure of $C_0^\infty([0,1])$ in $W^{1,2}([0,1])$. Moreover, a function $u : [0,1] \to \mathbb{R}$ is said to be a weak solution of $(G_{f,\lambda}^{p,q})$ if $u \in W_0^{1,2}([0,1])$ and

$$\begin{aligned}\int_0^1 p(x)u'(x)v'(x)dx &+ \int_0^1 q(x)u(x)v(x)dx \\ &= \lambda\int_0^1 f(u(x))v(x)dx \quad \forall v \in W_0^{1,2}([0,1]).\end{aligned} \tag{3.5}$$

We recall that $u \in AC([0,1])$ is a generalized solution of $(G_{f,\lambda}^{p,q})$ if $pu' \in AC([0,1])$, $u(0) = u(1)$ and

$$-(p(x)u'(x))' + q(x)u(x) = \lambda f(u(x)), \tag{3.6}$$

for almost every $x \in [0,1]$.

Clearly, the weak solutions of $(G_{f,\lambda}^{p,q})$ are also generalized solutions.

If f and q are continuous functions we recall that $u \in C^1([0,1])$ is a classical solution of $(G_{f,\lambda}^{p,q})$ if $pu' \in C^1([0,1])$, $u(0) = u(1)$ and

$$-(p(x)u'(x))' + q(x)u(x) = \lambda f(u(x)), \tag{3.7}$$

for every $x \in [0,1]$.

We recall that

$$(u,v) := \int_0^1 q(x)u(x)v(x)dx + \int_0^1 p(x)u'(x)v'(x)dx \quad u,v \in W_0^{1,2}([0,1]) \tag{3.8}$$

is an inner product that induces in $W_0^{1,2}([0,1])$ the norm

$$\|u\| := \left(\int_0^1 p(x) \left| u'(x) \right|^2 dx + \int_0^1 q(x) \left| u(x) \right|^2 dx \right)^{1/2}, \tag{3.9}$$

which is equivalent to the usual one.

It is well known that $W_0^{1,2}([0,1])$ is compactly embedded in $C^0([0,1])$ and in particular one has

$$\|u\|_\infty \leq \frac{1}{2\sqrt{p_0}} \|u\|, \tag{3.10}$$

for every $u \in W_0^{1,2}([0,1])$.

Consider $\Phi : W_0^{1,2}([0,1]) \to \mathbb{R}$ and $\Upsilon : W_0^{1,2}([0,1]) \to \mathbb{R}$ defined as follows

$$\Phi(u) := \frac{\|u\|^2}{2}, \qquad \Upsilon(u) := \int_0^1 F(u(x)) dx, \tag{3.11}$$

where

$$F(\xi) := \int_0^\xi f(t) dt, \tag{3.12}$$

for every $\xi \in \mathbb{R}$.

By standard arguments, one has that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous. Moreover, the Gâteaux derivative is the functional $\Phi'(u) \in (W_0^{1,2}([0,1]))^*$ given by

$$\Phi'(u)(v) = \int_0^1 p(x) u'(x) v'(x) dx + \int_0^1 q(x) u(x) v(x) dx, \tag{3.13}$$

for every $v \in W_0^{1,2}([0,1])$.

Moreover, Υ is locally Lipschitz continuous in $W_0^{1,2}([0,1])$. So it makes sense to consider the generalized directional derivative Υ°. Finally, by a standard argument, Υ is sequentially weakly continuous.

Now, put

$$\kappa := 3\frac{p_0}{\|q\|_\infty + 12\|p\|_\infty}, \tag{3.14}$$

$$A := \liminf_{\xi\to+\infty}\frac{\max_{|t|\le\xi}F(t)}{\xi^2},$$
$$B := \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}, \tag{3.15}$$

$$\lambda_1 := \frac{2}{3}\frac{\|q\|_\infty + 12\|p\|_\infty}{B},$$
$$\lambda_2 := 2\frac{p_0}{A}. \tag{3.16}$$

Our main result is the following theorem.

Theorem 3.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Put $F(\xi) := \int_0^\xi f(t)dt$ for every $\xi\in\mathbb{R}$ and assume that*

(i) $\int_0^\xi F(t)dt \ge 0$, *for every $\xi\ge 0$;*

(ii)

$$\liminf_{\xi\to+\infty}\frac{\max_{|t|\le\xi}F(t)}{\xi^2} < \kappa\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}, \tag{3.17}$$

where κ is given by (3.14);

(iii) *for almost every $x\in[0,1]$, for each $z\in D_f$ and for each $\lambda\in]\lambda_1,\lambda_2[$ (where λ_1, λ_2 are given by* (3.16)*) the condition*

$$\lambda f^-(z) - q(x)z \le 0 \le \lambda f^+(z) - q(x)z \tag{3.18}$$

implies $\lambda f(z) = q(x)z$.

Then, for each $\lambda\in]\lambda_1,\lambda_2[$, the problem $(G^{p,q}_{f,\lambda})$ possesses a sequence of weak solutions which is unbounded in $W_0^{1,2}([0,1])$.

Proof. Our aim is to apply Theorem 2.1(b). For this end, fix $\lambda\in]\lambda_1,\lambda_2[$ and denote by X the Banach space $W_0^{1,2}([0,1])$ endowed with the norm

$$\|u\| := \left(\int_0^1 p(x)\left|u'(x)\right|^2 dx + \int_0^1 q(x)\left|u(x)\right|^2 dx\right)^{1/2}. \tag{3.19}$$

For each $u \in X$, put

$$\begin{aligned} \Phi(u) &:= \frac{\|u\|^2}{2}, \qquad \Upsilon(u) := \int_0^1 F(u(x))\,dx, \qquad j(u) := 0 \\ \Psi(u) &:= \Upsilon(u) - j(u) = \Upsilon(u), \\ I_\lambda(u) &:= \Phi(u) - \lambda\Psi(u) = \Phi(u) - \lambda\Upsilon(u). \end{aligned} \tag{3.20}$$

Clearly, Φ is sequentially weakly lower semicontinuous and coercive, Υ is, in particular, sequentially weakly upper semicontinuous; moreover, they are locally Lipschitz functions and one has $I_\lambda^\circ(u;v) = \Phi'(u)(v) + \lambda(-\Upsilon)^\circ(u;v)$ for all $u, v \in X$.

Now, let $\{c_n\}$ be a real sequence such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\max_{|t|\le c_n} F(t)}{c_n^2} = A. \tag{3.21}$$

Put $r_n = 2p_0 c_n^2$ for all $n \in \mathbb{N}$. Taking (3.10) into account, one has $\max_{t\in[0,1]}|v(t)| \le c_n$ for all $v \in X$ such that $\|v\|^2 < 2r_n$. Hence, (taking also into account that the function $u_0(t) = 0$ for all $t \in [0,1]$ is such that $\|u_0\|^2 = 0 < 2r_n$) for all $n \in \mathbb{N}$ one has

$$\begin{aligned} \varphi(r_n) &= \inf_{\|u\|^2<2r_n} \frac{\sup_{\|v\|^2<2r_n}\int_0^1 F(v(x))\,dx - \int_0^1 F(u(x))\,dx}{r_n - \|u\|^2/2} \\ &\le \frac{\sup_{\|v\|^2<2r_n}\int_0^1 F(v(x))\,dx - \int_0^1 F(u_0(x))\,dx}{r_n - \|u_0\|^2/2} \\ &= \frac{\sup_{\|v\|^2<2r_n}\int_0^1 F(v(x))\,dx}{r_n} \\ &\le \frac{\max_{|t|\le c_n} F(t)}{r_n} \\ &= \frac{1}{2p_0}\frac{\max_{|t|\le c_n} F(t)}{c_n^2}. \end{aligned} \tag{3.22}$$

Therefore, since from assumption (ii) one has $A < +\infty$, we obtain

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{A}{2p_0} < +\infty. \tag{3.23}$$

Now we claim that the functional I_λ is unbounded from below. Let $\{d_n\}$ be a real sequence such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{F(d_n)}{d_n^2} = B. \tag{3.24}$$

For all $n \in \mathbb{N}$ define

$$w_n(x) := \begin{cases} 4d_n x & \text{if } x \in \left[0, \dfrac{1}{4}\right[\\ d_n & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right] \\ -4d_n(x-1) & \text{if } x \in \left]\dfrac{3}{4}, 1\right]. \end{cases} \tag{3.25}$$

Clearly, $w_n \in X$ and

$$\|w_n\|^2 \leq 8d_n^2\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right). \tag{3.26}$$

Therefore,

$$\begin{aligned} \Phi(w_n) - \lambda\Upsilon(w_n) &= \frac{\|w_n\|^2}{2} - \lambda\int_0^1 F(w_n(x))dx \\ &\leq 4d_n^2\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right) - \lambda\int_0^1 F(w_n(x))dx. \end{aligned} \tag{3.27}$$

Taking (i) into account, we have

$$\int_0^1 F(w_n(x))dx \geq \int_{1/4}^{3/4} F(d_n)dt = \frac{1}{2}F(d_n). \tag{3.28}$$

Then,

$$\Phi(w_n) - \lambda\Upsilon(w_n) \leq 4d_n^2\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right) - \frac{\lambda}{2}F(d_n), \tag{3.29}$$

for all $n \in \mathbb{N}$.

Now, if $B < +\infty$, we fix $\varepsilon \in](8/\lambda B)(\|q\|_\infty/12 + \|p\|_\infty), 1[$. From (3.24) there exists $\nu_\varepsilon \in \mathbb{N}$ such that

$$F(d_n) > \varepsilon B d_n^2, \tag{3.30}$$

for all $n > \nu_\varepsilon$. Therefore,

$$\begin{aligned}\Phi(w_n) - \lambda\Upsilon(w_n) &\le 4d_n^2\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right) - \frac{\lambda}{2}\varepsilon B d_n^2 \\ &= d_n^2\left[4\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right) - \frac{\lambda}{2}\varepsilon B\right]\end{aligned} \tag{3.31}$$

for all $n > \nu_\varepsilon$.

From the choice of ε, one has

$$\lim_{n\to+\infty}\left[\Phi(w_n) - \lambda\Upsilon(w_n)\right] = -\infty. \tag{3.32}$$

On the other hand, if $B = +\infty$, we fix $M > (8/\lambda)(\|q\|_\infty/12 + \|p\|_\infty)$ and, again from (3.24) there exists $\nu_M \in \mathbb{N}$ such that

$$F(d_n) > Md_n^2, \tag{3.33}$$

for all $n > \nu_M$. Therefore,

$$\begin{aligned}\Phi(w_n) - \lambda\Upsilon(w_n) &\le 4d_n^2\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right) - \frac{\lambda}{2}M d_n^2 \\ &= d_n^2\left[4\left(\frac{\|q\|_\infty}{12} + \|p\|_\infty\right) - \frac{\lambda}{2}M\right]\end{aligned} \tag{3.34}$$

for all $n > \nu_M$.

From the choice of M, also in this case, one has

$$\lim_{n\to+\infty}\left[\Phi(w_n) - \lambda\Upsilon(w_n)\right] = -\infty. \tag{3.35}$$

Hence, our claim is proved.

Since all assumptions of Theorem 2.1(b) are verified, the functional I_λ admits a sequence $\{u_n\}$ of generalized critical points such that $\lim_{n\to+\infty}\|u_n\| = +\infty$, that is $\{u_n\}$ is unbounded in X.

Now, we claim that the generalized critical points of I_λ are weak solutions for the problem $(G^{p,q}_{f,\lambda})$. To this end, let $u_0 \in X$ a generalized critical point of I_λ, that is $I^\circ_\lambda(u_0, v) \ge 0$, for all $v \in X$. From which, we obtain

$$\Phi'(u_0)(v) + \lambda(-\Upsilon)^\circ(u_0; v) \ge 0, \tag{3.36}$$

for all $v \in X$. Hence, $\Phi'(u_0)(v) \geq -\lambda(-\Upsilon)^\circ(u_0; v)$ for all $v \in X$, that is

$$-\int_0^1 \left[p(x)u_0'(x)v'(x) + q(x)u_0(x)v(x)\right]dx \leq \lambda(-\Upsilon)^\circ(u_0; v), \tag{3.37}$$

for all $v \in X$. Clearly, setting

$$L(v) := -\int_0^1 \left[p(x)u_0'(x)v'(x) + q(x)u_0(x)v(x)\right]dx \quad \forall v \in X, \tag{3.38}$$

L is a continuous and linear functional on X, for which (3.37) signifies $L \in \lambda\partial(-\Upsilon)(u_0)$. Now, since X is dense in $L^2([0,1])$, from [9, Theorem 2.2] one has $L(v) \leq \lambda(-\Upsilon)^\circ(u_0; v)$ for all $v \in L^2([0,1])$, so that L is continuous and linear on $L^2([0,1])$. Therefore, there is $h \in L^2([0,1])$ such that $L(v) = \int_0^1 h(x)v(x)dx$ for all $v \in L^2([0,1])$. From a standard result (see, e.g., [15, Example 2, page 219]) there is a unique $\overline{u} \in W^{2,2}([0,1]) \cap X$ such that $(p\overline{u}')' - q\overline{u} = h$. In particular, one has $\int_0^1[(p(x)\overline{u}'(x))' - q(x)\overline{u}(x)]v(x)dx = -\int_0^1[p(x)\overline{u}'(x)v'(x) + q(x)\overline{u}(x)v(x)]dx$ for all $v \in X$. Hence, $-\int_0^1[p(x)u_0'(x)v'(x) + q(x)u_0(x)v(x)]dx = L(v) = \int_0^1 h(x)v(x)dx = \int_0^1[(p(x)\overline{u}'(x))' - q(x)\overline{u}(x)]v(x)dx = -\int_0^1[p(x)\overline{u}'(x)v'(x) + q(x)\overline{u}(x)v(x)]dx$ for all $v \in X$, and since a continuous and linear functional on X is uniquely determined by a function in X (see [17, Theorem 5.9.3, page 295]), we have $\overline{u} = u_0$; so that, $u_0 \in W^{2,2}([0,1])$ and

$$\int_0^1 \left[(p(x)u_0'(x))' - q(x)u_0(x)\right]v(x)dx = -\int_0^1 \left[p(x)u_0'(x)v'(x) + q(x)u_0(x)v(x)\right]dx \tag{3.39}$$

for all $v \in X$. From (3.37) and (3.39) one has

$$\int_0^1 \left[(p(x)u_0'(x))' - q(x)u_0(x)\right]v(x)dx \leq \lambda(-\Upsilon)^\circ(u_0; v) \tag{3.40}$$

for all $v \in X$. Hence, [9, Corollary page 111] ensures that $(p(x)u_0'(x))' - q(x)u_0(x) \in [(-\lambda f)^-(u_0(x)), (-\lambda f)^+(u_0(x))]$ for almost every $x \in [0,1]$, that is $(p(x)u_0'(x))' \in [(-\lambda f)^-(u_0(x)) + q(x)u_0(x), (-\lambda f)^+(u_0(x)) + q(x)u_0(x)]$, for almost every $x \in [0,1]$. From which

$$-(p(x)u_0'(x))' \in \left[\lambda(f)^-(u_0(x)) - q(x)u_0(x),\ \lambda(f)^+(u_0(x)) - q(x)u_0(x)\right], \tag{3.41}$$

for almost every $x \in [0,1]$.

Now, since $m(D_f) = 0$, from [18, Lemma 1] we obtain $-(p(x)u_0'(x))' = 0$ for almost every $x \in u_0^{-1}(D_f)$. Hence, from (iii) we obtain $\lambda f(u_0(x)) - q(x)u_0(x) = 0$ for almost every $x \in u_0^{-1}(D_f)$. From which

$$-(p(x)u_0'(x))' + q(x)u_0(x) = \lambda f(u_0(x)), \tag{3.42}$$

for almost every $x \in u_0^{-1}(D_f)$. On the other hand, for almost every $x \in [0,1] \setminus u_0^{-1}(D_f)$, condition (3.41) reduces to

$$-\left(p(x)u_0'(x)\right)' + q(x)u_0(x) = \lambda f\left(u_0(x)\right). \tag{3.43}$$

Hence, our claim is proved and the assertion follows. □

Remark 3.2. If $q(x) = 0$ for all $x \in [0,1]$ assumption (iii) becomes

(iii′) for each $z \in D_f$, the condition

$$f^-(z) \le 0 \le f^+(z) \tag{3.44}$$

implies $f(z) = 0$.

If $z \in D_f$ is such that $0 \in [f^-(z), f^+(z)]$ and $f(z) = 0$, then (iii′) is verified. Otherwise, if $z \in D_f$ and there is a neighborhood V of z and a positive constant $m > 0$ such that either $f(t) > m$ for almost every $t \in V$, or $f(t) < -m$ for almost every $t \in V$, then (iii′) is verified. In particular, if $\inf_{\mathbb{R}} f > 0$ then (iii′) is verified for all $z \in D_f$.

If q is a nonzero function, $z \in D_f$ and $\lambda \in]\lambda_1, \lambda_2[$, (iii) is verified, e.g., when there is a neighborhood V of z and a positive constant $m > 0$ such that either $f(t) > (\|q\|_\infty/\lambda)z + m$ for almost every $t \in V$, or $f(t) < (q_0/\lambda)z - m$ for almost every $t \in V$. In particular, whenever $\lambda_1 > 0$ and one has $f(z) > (\|q\|_\infty/\lambda_1)z + m$ for some $m > 0$, then (iii) is verified.

Finally, since a function $u \in W_0^{1,2}([0,1])$ such that

$$-\left(p(x)u'(x)\right)' + q(x)u(x) \in \lambda\left[(f)^-\left(u(x)\right), (f)^+\left(u(x)\right)\right] \tag{3.45}$$

for almost every $x \in [0,1]$ is called multi-valued solution for $(G_{f,\lambda}^{p,q})$ (see [9]), we explicitly observe that, without assuming condition (iii), the same proof of Theorem 3.1 ensures a sequence of pairwise distinct multi-valued solutions to problem $(G_{f,\lambda}^{p,q})$.

Remark 3.3. The following condition

(ii′) there exist two real sequences $\{b_n\}$, $\{c_n\}$, with $b_n < \sqrt{(p_0/2(\|q\|_\infty/12 + \|p\|_\infty))}c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} c_n = +\infty$, such that

$$A_1 := \lim_{n\to+\infty} \frac{\max_{|t|\le c_n} F(t) - (1/2)F(b_n)}{2p_0c_n^2 - 4\left(\|q\|_\infty/12 + \|p\|_\infty\right)b_n^2} < \frac{\kappa}{2p_0} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} \tag{3.46}$$

is more general than condition (ii) of Theorem 3.1. In fact, from (ii′) we obtain (ii), by choosing $b_n = 0$ for all $n \in \mathbb{N}$.

Assuming in Theorem 3.1 condition (ii′) instead of condition (ii), for each $\lambda \in]\lambda_1, 1/A_1[$ the conclusion in Theorem 3.1 again holds. In fact, arguing as in the proof of

Theorem 3.1, one has $\varphi(r_n) = \inf_{\|u\|^2<2r_n}(\sup_{\|v\|^2<2r_n}\int_0^1 F(v(x))dx - \int_0^1 F(u(x))dx)/(r_n - \|u\|^2/2) \leq (\sup_{\|v\|^2<2r_n}\int_0^1 F(v(x))dx - \int_0^1 F(u_n(x))dx)/(r_n - \|u_n\|^2/2) \leq (\max_{|t|\leq c_n} F(t) - (1/2)F(b_n))/(2p_0 c_n^2 - 4(\|q\|_\infty/12 + \|p\|_\infty)b_n^2)$, by choosing

$$u_n(x) := \begin{cases} 4b_n x & \text{if } x \in \left[0, \dfrac{1}{4}\right[\\ b_n & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right] \\ -4b_n(x-1) & \text{if } x \in \left]\dfrac{3}{4}, 1\right]. \end{cases} \tag{3.47}$$

Among the consequences of Theorem 3.1 we point out the following results.

Corollary 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $p \in C^1([0,1])$, $q \in C^0([0,1])$. Assume that (i) and (ii) of Theorem 3.1 hold. Then, for each $\lambda \in]\lambda_1, \lambda_2[$, problem $(G_{f,\lambda}^{p,q})$ possesses a sequence of pairwise distinct classical solutions.*

If f is nonnegative, using the Strong Maximum Principle (see, e.g., [19, Theorem 8.19, page 198]) we can get the following two results.

Corollary 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, nonnegative function and $p \in C^1([0,1])$, $q \in C^0([0,1])$. Assume that*

(ii$_\text{a}$)

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < \kappa \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}. \tag{3.48}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, problem $(G_{f,\lambda}^{p,q})$ possesses a sequence of pairwise distinct positive classical solutions.

Corollary 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded, nonnegative and almost everywhere continuous function. Assume that (ii_a) of Corollary 3.5 and (iii) of Theorem 3.1 hold. Then, for each $\lambda \in]\lambda_1, \lambda_2[$, problem $(G_{f,\lambda}^{p,q})$ possesses a sequence of positive weak solutions which is unbounded in $W_0^{1,2}([0,1])$.*

Finally, we present the following result.

Corollary 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Assume that (i) and (iii) of Theorem 3.1 hold. Further, assume that*

(ii$_1$)

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} > \frac{2}{3}(\|q\|_\infty + 12\|p\|_\infty); \tag{3.49}$$

(ii_2)

$$\liminf_{\xi\to+\infty}\frac{\max_{|t|\le\xi}F(t)}{\xi^2}<2p_0. \tag{3.50}$$

Then, problem

$$\begin{gathered}-(pu')'+qu=f(u)\quad\text{in }]0,1[\\ u(0)=u(1)=0\end{gathered} \tag{$G^{p,q}_{f,1}$}$$

possesses a sequence of weak solutions which is unbounded in $W^{1,2}_0([0,1])$.

Remark 3.8. Clearly, also Theorem 1.1 in Introduction is a particular case of Theorem 3.1, taking Remark 3.2 and the Strong Maximum Principle into account.

Now, we present the other main result. First, put

$$\begin{aligned}A^*&:=\liminf_{\xi\to0^+}\frac{\max_{|t|\le\xi}F(t)}{\xi^2},\\ B^*&:=\limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^2},\end{aligned} \tag{3.51}$$

$$\begin{aligned}\lambda_1^*&:=\frac{2}{3}\frac{\|q\|_\infty+12\|p\|_\infty}{B^*},\\ \lambda_2^*&:=2\frac{p_0}{A^*}.\end{aligned} \tag{3.52}$$

Theorem 3.9. *Let $f:\mathbb{R}\to\mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Assume that*

(j) $\int_0^\xi F(t)dt\ge0$, *for every $\xi\ge0$;*

(jj)

$$\liminf_{\xi\to0^+}\frac{\max_{|t|\le\xi}F(t)}{\xi^2}<\kappa\limsup_{\xi\to0^+}\frac{F(\xi)}{\xi^2}, \tag{3.53}$$

where κ is given by (3.14)*;*

(jjj) *for almost every $x\in[0,1]$, each $z\in D_f$ and each $\lambda\in]\lambda_1^*,\lambda_2^*[$ (where λ_1^*,λ_2^* are given by* (3.52) *), the condition*

$$\lambda f^-(z)-q(x)z\le0\le\lambda f^+(z)-q(x)z \tag{3.54}$$

implies $\lambda f(z)=q(x)z$.

Then, for each $\lambda \in]\lambda_1^*, \lambda_2^*[$, problem $(G_{f,\lambda}^{p,q})$ possesses a sequence of pairwise distinct weak solutions, which strongly converges to zero in $W_0^{1,2}([0,1])$.

Proof. The proof is the same of Theorem 3.1 applying part (c) of Theorem 2.1 instead of part (b). □

Clearly, from Theorem 3.9 we obtain similar consequences to those of Theorem 3.1. Here, we present only one of them.

Corollary 3.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, nonnegative function and $p \in C^1([0,1])$, $q \in C^0([0,1])$. Assume that*

$(\mathrm{jj_a})$

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < \kappa \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2}. \tag{3.55}$$

Then, for each $\lambda \in]\lambda_1^*, \lambda_2^*[$, problem $(G_{f,\lambda}^{p,q})$ possesses a sequence of pairwise distinct positive classical solutions, which strongly converges to zero in $C^0([0,1])$.

Now, we present some examples of application of Theorem 3.1 for which the results in [1–4, 6, 7] cannot be applied (see Remark 3.13).

Example 3.11. Let q_0 be a nonnegative real constant, put

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}, \tag{3.56}$$

for every $n \in \mathbb{N}$ and define the nonnegative, continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(\xi) := \begin{cases} \dfrac{32(n+1)!^2\left[(n+1)!^2 - n!^2\right]}{\pi}\sqrt{\dfrac{1}{16(n+1)!^2} - \left(\xi - \dfrac{n!(n+2)}{2}\right)^2} & \text{if } \xi \in \bigcup_{n\in\mathbb{N}} [a_n, b_n] \\ 0 & \text{otherwise.} \end{cases} \tag{3.57}$$

One has $\int_{n!}^{(n+1)!} f(t)dt = \int_{a_n}^{b_n} f(t)dt = (n+1)!^2 - n!^2$ for every $n \in \mathbb{N}$. Then, one has $\lim_{n\to+\infty} F(b_n)/b_n^2 = 4$ and $\lim_{n\to+\infty} F(a_n)/a_n^2 = 0$. Therefore, by a simple computation, we have $\liminf_{\xi\to+\infty} F(\xi)/\xi^2 = 0$ and $\limsup_{\xi\to+\infty} F(\xi)/\xi^2 = 4$. Hence,

$$0 = \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < \frac{3}{q_0+12}4 = \frac{3}{q_0+12}\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}. \tag{3.58}$$

Owing to Corollary 3.5, for each $\lambda > (q_0 + 12)/6$ the problem

$$\begin{gathered} -u'' + q_0 u = \lambda f(u) \quad \text{in }]0,1[\\ u(0) = u(1) = 0 \end{gathered} \tag{$G^{1,q_0}_{f,\lambda}$}$$

possesses a sequence of pairwise distinct positive classical solutions.

Now, let $f^* : \mathbb{R} \to \mathbb{R}$ be the positive, continuous function defined as $f^*(\xi) = f(\xi) + 1$ for all $\xi \in \mathbb{R}$, where f is given by (3.57). Clearly, $\liminf_{\xi\to+\infty} F^*(\xi)/\xi^2 = 0$ and $\limsup_{\xi\to+\infty} F^*(\xi)/\xi^2 = 4$. Hence, again owing to Corollary 3.5, for each $\lambda > (q_0 + 12)/6$ the problem

$$\begin{gathered} -u'' + q_0 u = \lambda f^*(u) \quad \text{in }]0,1[\\ u(0) = u(1) = 0 \end{gathered} \tag{$\overline{G}^{1,q_0}_{f,\lambda}$}$$

possesses a sequence of pairwise distinct positive classical solutions.

Example 3.12. Let q_0 be a nonnegative real constant, put

$$a_1 := 2, \qquad a_{n+1} := (a_n)^{3/2}, \tag{3.59}$$

for every $n \in \mathbb{N}$ and $S := \bigcup_{n\geq 2}]a_{n+1} - 1, a_{n+1} + 1[$. Define the continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(t) := \begin{cases} (a_{n+1})^3 e^{1/(t-(a_{n+1}-1))(t-(a_{n+1}+1))+1} \dfrac{2(a_{n+1} - t)}{(t - (a_{n+1} - 1))^2 (t - (a_{n+1} + 1))^2} & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases} \tag{3.60}$$

For which, one has

$$\begin{aligned} F(\xi) &= \int_0^\xi f(t)dt \\ &= \begin{cases} (a_{n+1})^3 e^{1/(\xi-(a_{n+1}-1))(\xi-(a_{n+1}+1))+1} & \text{if } \xi \in S \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{3.61}$$

and $F(a_{n+1}) = (a_{n+1})^3$ for every $n \geq 2$. Hence, one has $\limsup_{\xi\to+\infty} F(\xi)/\xi^2 = +\infty$. On the other hand, by setting $x_n = a_{n+1} - 1$ for every $n \geq 2$, one has $\max_{\xi\in[-x_n,x_n]} F(\xi) = (a_n)^3$ for every $n \geq 2$. Therefore, one has $\lim_{n\to+\infty} \max_{\xi\in[-x_n,x_n]} F(\xi)/{x_n}^2 = 1$ and, by a simple computation, one has $\liminf_{\xi\to+\infty} \max_{t\in[-\xi,\xi]} F(t)/\xi^2 = 1$. Hence,

$$\liminf_{\xi\to+\infty} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^2} = 1 < \frac{3}{q_0 + 12} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty. \tag{3.62}$$

Owing to Corollary 3.4, for each $\lambda \in]0, 2[$ the problem $(G_{f,\lambda}^{1,q_0})$ possesses a sequence of pairwise distinct classical solutions.

Remark 3.13. In [3] the existence of infinitely many solutions for the problem

$$\begin{gathered} -u'' = f(u) \quad \text{in }]0,1[\\ u(0) = u(1) = 0 \end{gathered} \tag{3.63}$$

was studied under suitable assumptions on the function f, as $f(\xi) > 0$ for all ξ large enough. We explicitly observe that we cannot apply [3, Theorem 2.11] to problem $(G_{f,\lambda}^{1,q_0})$ of Example 3.11, even in the case $q_0 = 0$, since our function is not positive for all ξ large enough. The same remark for [4, Corollary 3.1] holds, since $\lim_{\xi \to +\infty} f(\xi) = +\infty$ (hence, in particular, $f(\xi) > 0$ for ξ large enough) is requested.

The same problem was studied in [1, 7]. Assumptions of [1, Theorem 2.1] imply that f is negative in suitable real intervals. Hence, [1, Theorem 2.1] cannot be applied to problem $(G_{f,\lambda}^{1,q_0})$ of Example 3.11. Moreover, assumptions in [7, Theorem 3.1], as $\inf\{t \in \mathbb{R} : f(t) > 0\} < 0$, cannot be applied to the function f of Example 3.11 since, in this case, one has $\inf\{t \in \mathbb{R} : f(t) > 0\} = 11/8$.

In [2, 6], the authors studied the existence of infinitely many weak solutions of the following autonomous Dirichlet problem

$$\begin{gathered} -\Delta_p u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{D}$$

where Ω is a bounded open subset of the Euclidean space $(\mathrm{R}^N, |\cdot|)$, $N \geq 1$, with boundary of class C^1, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, and f is a continuous function. In [6, Remark 3.3] the key assumption to obtain infinitely many solutions to (D) is: $\liminf_{\xi \to +\infty} F(\xi)/\xi^p = 0$ and $\limsup_{\xi \to +\infty} F(\xi)/\xi^p = +\infty$. Clearly, the function f in Example 3.11 does not satisfy this condition. Hence, we cannot apply [6, Theorem 1.1] to our problem $(G_{f,\lambda}^{1,q_0})$, even in the case $q_0 = 0$.

On the other hand, we cannot apply [2, Theorem 1.1] to $(\overline{G}_{f,\lambda}^{1,q_0})$, since one of the key assumptions is that function f is nonpositive in suitable real intervals. Another key assumption of [2, Theorem 1.1] is

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} < +\infty. \tag{3.64}$$

Hence, we cannot apply [2, Theorem 1.1] to $(G_{f,\lambda}^{1,q_0})$ in Example 3.12, even in the case $q_0 = 0$.

Clearly, we cannot apply [6, Remark 3.3], [3, Theorem 2.11], [4, Corollary 3.1], [7, Theorem 3.1] to $(G_{f,\lambda}^{1,q_0})$ in Example 3.12 for the same previous reasons.

The following example deals with a discontinuous function.

Example 3.14. Let $\{a_n\}$ and $\{b_n\}$ be the sequences defined as in the Example 3.11 and define the nonnegative (and discontinuous) function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(\xi) := \begin{cases} 2(n+1)!\left[(n+1)!^2 - n!^2\right] & \text{if } \xi \in \bigcup_{n\in\mathbb{N}} \left]a_n, b_n\right[\\ 0 & \text{otherwise.} \end{cases} \tag{3.65}$$

By a similar computation as in Example 3.11, we have

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 4, \qquad \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0. \tag{3.66}$$

From Corollary 3.6, for each $\lambda > 2$ the problem $(G^{1,0}_{f,\lambda})$ possesses a sequence of positive weak solutions which is unbounded in $W^{1,2}_0([0,1])$.

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