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Research Article

Exponential Stability and Estimation of Solutions of Linear Differential Systems of Neutral Type with Constant Coefficients

J. Baštinec,[1] J. Diblík,[1, 2] D. Ya. Khusainov,[3] and A. Ryvolová[1]

[1] *Department of Mathematics, Faculty of Electrical Engineering and Communication, Technická 8, Brno University of Technology, 61600 Brno, Czech Republic*
[2] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Veveří 331/95, Brno University of Technology, 60200 Brno, Czech Republic*
[3] *Department of Complex System Modeling, Faculty of Cybernetics, Taras, Shevchenko National University of Kyiv, Vladimirskaya Str., 64, 01033 Kyiv, Ukraine*

Correspondence should be addressed to J. Diblík, diblik@feec.vutbr.cz

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This paper investigates the exponential-type stability of linear neutral delay differential systems with constant coefficients using Lyapunov-Krasovskii type functionals, more general than those reported in the literature. Delay-dependent conditions sufficient for the stability are formulated in terms of positivity of auxiliary matrices. The approach developed is used to characterize the decay of solutions (by inequalities for the norm of an arbitrary solution and its derivative) in the case of stability, as well as in a general case. Illustrative examples are shown and comparisons with known results are given.

1. Introduction

This paper will provide estimates of solutions of linear systems of neutral differential equations with constant coefficients and a constant delay:

$$\dot{x}(t) = D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau), \tag{1.1}$$

where $t \geq 0$ is an independent variable, $\tau > 0$ is a constant delay, A, B, and D are $n \times n$ constant matrices, and $x : [-\tau, \infty) \to \mathbb{R}^n$ is a column vector-solution. The sign "$\cdot$" denotes the left-hand derivative. Let $\varphi : [-\tau, 0] \to \mathbb{R}^n$ be a continuously differentiable vector-function. The solution $x = x(t)$ of problem (1.1), (1.2) on $[-\tau, \infty)$ where

$$x(t) = \varphi(t), \qquad \dot{x}(t) = \dot{\varphi}(t), \qquad t \in [-\tau, 0] \tag{1.2}$$

is defined in the classical sense (we refer, e.g., to [1]) as a function continuous on $[-\tau, \infty)$ continuously differentiable on $[-\tau, \infty)$ except for points τp, $p = 0, 1, \ldots$, and satisfying (1.1) everywhere on $[0, \infty)$ except for points τp, $p = 0, 1, \ldots$.

The paper finds an estimate of the norm of the difference between a solution $x = x(t)$ of problem (1.1), (1.2) and the steady state $x(t) \equiv 0$ at an arbitrary moment $t \geq 0$.

Let $\mathcal{F}$ be a rectangular matrix. We will use the matrix norm:

$$\|\mathcal{F}\| := \sqrt{\lambda_{\max}(\mathcal{F}^T \mathcal{F})}, \tag{1.3}$$

where the symbol $\lambda_{\max}(\mathcal{F}^T \mathcal{F})$ denotes the maximal eigenvalue of the corresponding square symmetric positive semidefinite matrix $\mathcal{F}^T \mathcal{F}$. Similarly, $\lambda_{\min}(\mathcal{F}^T \mathcal{F})$ denotes the minimal eigenvalue of $\mathcal{F}^T \mathcal{F}$. We will use the following vector norms:

$$\begin{gathered}
\|x(t)\| := \sqrt{\sum_{i=1}^{n} x_i^2(t)}, \\
\|x(t)\|_\tau := \sup_{-r \leq s \leq 0} \{\|x(s+t)\|\}, \\
\|x(t)\|_{\tau,\beta} := \sqrt{\int_{t-r}^{t} e^{-\beta(t-s)} \|x(s)\|^2 \mathrm{d}s},
\end{gathered} \tag{1.4}$$

where β is a parameter.

The most frequently used method for investigating the stability of functional-differential systems is the method of Lyapunov-Krasovskii functionals [2, 3]. Usually, it uses positive definite functionals of a quadratic form generated from terms of (1.1) and the integral (over the interval of delay [4]) of a quadratic form. A possible form of such a functional is then

$$[x(t) - Dx(t-\tau)]^T H [x(t) - Dx(t-\tau)] + \int_{t-\tau}^{t} x^T(s) G x(s) \mathrm{d}s, \tag{1.5}$$

where H and G are suitable $n \times n$ positive definite matrices.

Regarding the functionals of the form (1.5), we should underline the following. Using a functional (1.5), we can only obtain propositions concerning the stability. Statements such as that the expression

$$\int_{t-\tau}^{t} x^T(s) G x(s) \mathrm{d}s \tag{1.6}$$

is bounded from above are of an integral type. Because the terms $[x(t) - Dx(t-\tau)]$ in (1.5) contain differences, they do not imply the boundedness of the norm of $x(t)$ itself.

Literature on the stability and estimation of solutions of neutral differential equations is enormous. Tracing previous investigations on this topic, we emphasize that a Lyapunov function $v(x) = x^T Hx$ has been used to investigate the stability of systems (1.1) in [5] (see [6] as well). The stability of linear neutral systems of type (1.1), but with different delays h_1 and h_2, is studied in [1] where a functional

$$\|x(t)\| + c_1 \int_{t-h_1}^{t} \|x(s)\| \mathrm{d}s + c_2 \int_{t-h_2}^{t} \|\dot{x}(s)\| \mathrm{d}s \tag{1.7}$$

is used with suitable constants c_1 and c_2. In [7, 8], functionals depending on derivatives are also suggested for investigating the asymptotic stability of neutral nonlinear systems. The investigation of nonlinear neutral delayed systems with two time dependent bounded delays in [9] to determine the global asymptotic and exponential stability uses, for example, functionals

$$\begin{gathered} x^T(t)Px(t) + \int_{-h_1}^{0} x^T(t+s)Qx(s)\mathrm{d}s + \int_{-h_2}^{0} \dot{x}^T(t+s)\dot{x}(t+s)\mathrm{d}s, \\ e^{2\gamma t}x^T(t)Px(t) + \int_{-h_1}^{0} e^{2\gamma(t+s)}x^T(t+s)Qx(s)\mathrm{d}s + \int_{-h_2}^{0} e^{2\gamma(t+s)}\dot{x}^T(t+s)\dot{x}(t+s)\mathrm{d}s, \end{gathered} \tag{1.8}$$

where P and Q are positive matrices and γ is a positive scalar.

Delay independent criteria of stability for some classes of delay neutral systems are developed in [10]. The stability of systems (1.1) with time dependent delays is investigated in [11]. For recent results on the stability of neutral equations, see [9, 12] and the references therein. The works in [12, 13] deal with delay independent criteria of the asymptotical stability of systems (1.1).

In this paper, we will use Lyapunov-Krasovskii quadratic type functionals of the dependent coordinates and their derivatives

$$V_0[x(t), t] = x^T(t)Hx(t) + \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s \tag{1.9}$$

and $V[x(t), t] = e^{pt}V_0[x(t), t]$, that is,

$$V[x(t), t] = e^{pt}\left[x^T(t)Hx(t) + \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^2(s)G_2\dot{x}^2(s)\right]\mathrm{d}s\right], \tag{1.10}$$

where x is a solution of (1.1), β and p are real parameters, the $n \times x$ matrices H, G_1, and G_2 are positive definite, and $t > 0$. The form of functionals (1.9) and (1.10) is suggested by the functionals (1.7)-(1.8). Although many approaches in the literature are used to judge the stability, our approach, among others, in addition to determining whether the system (1.1) is exponentially stable, also gives delay-dependent estimates of solutions in terms of the norms $\|x(t)\|$ and $\|\dot{x}(t)\|$ even in the case of instability. An estimate of the norm $\|\dot{x}(t)\|$ can be achieved by reducing the initial neutral system (1.1) to a neutral system having the same solution on the intervals indicated in which the "neutrality" is concentrated only on the initial interval. If, in the literature, estimates of solutions are given, then, as a rule, estimates of derivatives are not investigated.

To the best of our knowledge, the general functionals (1.9) and (1.10) have not yet been applied as suggested to the study of stability and estimates of solutions of (1.1).

2. Exponential Stability and Estimates of the Convergence of Solutions to Stable Systems

First we give two definitions of stability to be used later on.

Definition 2.1. The zero solution of the system of equations of neutral type (1.1) is called exponentially stable in the metric C^0 if there exist constants $N_i > 0$, $i = 1, 2$ and $\mu > 0$ such that, for an arbitrary solution $x = x(t)$ of (1.1), the inequality

$$\|x(t)\| \leq [N_1 \|x(0)\|_\tau + N_2 \|\dot{x}(0)\|_\tau] e^{-\mu t} \tag{2.1}$$

holds for $t > 0$.

Definition 2.2. The zero solution of the system of equations of neutral type (1.1) is called exponentially stable in the metric C^1 if it is stable in the metric C^0 and, moreover, there exist constants $R_i > 0$, $i = 1, 2$, and $\nu > 0$ such that, for an arbitrary solution $x = x(t)$ of (1.1), the inequality

$$\|\dot{x}(t)\| \leq [R_1 \|x(0)\|_\tau + R_2 \|\dot{x}(0)\|_\tau] e^{-\nu t} \tag{2.2}$$

holds for $t > 0$.

We will give estimates of solutions of the linear system (1.1) on the interval $(0, \infty)$ using the functional (1.9). Then it is easy to see that an inequality

$$\begin{aligned}
&\lambda_{\min}(H) \|x(t)\|^2 + \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s) G_1 x(s) + \dot{x}^T(s) G_2 x(s) \right] \mathrm{d}s \\
&\quad \leq V_0[x(t), t] \\
&\quad \leq \lambda_{\max}(H) \|x(t)\|^2 + \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s) G_1 x(s) + \dot{x}^T(s) G_2 x(s) \right] \mathrm{d}s
\end{aligned} \tag{2.3}$$

holds on $(0, \infty)$. We will use an auxiliary $3n \times 3n$-dimensional matrix:

$$\begin{aligned}
S &= S(\beta, G_1, G_2, H) \\
&:= \begin{pmatrix} -A^T H - HA - G_1 - A^T G_2 A & -HB - A^T G_2 B & -HD - A^T G_2 D \\ -B^T H - B^T G_2 A & e^{-\beta\tau} G_1 - B^T G_2 B & -B^T G_2 D \\ -D^T H - D^T G_2 A & -D^T G_2 B & e^{-\beta\tau} G_2 - D^T G_2 D \end{pmatrix},
\end{aligned} \tag{2.4}$$

depending on the parameter β and the matrices G_1, G_2, H. Next we will use the numbers

$$\varphi(H) := \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \qquad \varphi_1(G_1, H) := \frac{\lambda_{\max}(G_1)}{\lambda_{\min}(H)}, \qquad \varphi_2(G_2, H) := \frac{\lambda_{\max}(G_2)}{\lambda_{\min}(H)}. \tag{2.5}$$

The following lemma gives a representation of the linear neutral system (1.1) on an interval $[(m-1)\tau, m\tau]$ in terms of a delayed system derived by an iterative process. We will adopt the customary notation $\sum_{i=k+s}^{k} \mathcal{O}(i) = 0$ where k is an integer, s is a positive integer, and $\mathcal{O}$ denotes the function considered independently of whether it is defined for the arguments indicated or not.

Lemma 2.3. *Let m be a positive integer and $t \in [(m-1)\tau, m\tau)$. Then a solution $x = x(t)$ of the initial problem* (1.1), (1.2) *is a solution of the delayed system*

$$\dot{x}(t) = D^m \dot{x}(t - m\tau) + Ax(t) + (DA + B)\sum_{i=1}^{m-1} D^{i-1} x(t - i\tau) + D^{m-1} Bx(t - m\tau) \tag{2.6}$$

for $t \in [(m-1)\tau, m\tau)$ where $x(t - m\tau) = \varphi(t - m\tau)$ and $\dot{x}(t - m\tau) = \dot{\varphi}(t - m\tau)$.

Proof. For $m = 1$ the statement is obvious. If $t \in [\tau, 2\tau)$, replacing t by $t - \tau$, system (1.1) will turn into

$$\dot{x}(t - \tau) = D\dot{x}(t - 2\tau) + Ax(t - \tau) + Bx(t - 2\tau). \tag{2.7}$$

Substituting (2.7) into (1.1), we obtain the following system of equations:

$$\dot{x}(t) = D^2 \dot{x}(t - 2\tau) + Ax(t) + (DA + B)x(t - \tau) + DBx(t - 2\tau), \tag{2.8}$$

where $t \in [\tau, 2\tau)$. If $t \in [2\tau, 3\tau)$, replacing t by $t - \tau$ in (2.7), we get

$$\dot{x}(t - 2\tau) = D\dot{x}(t - 3\tau) + Ax(t - 2\tau) + Bx(t - 3\tau). \tag{2.9}$$

We do one more iteration substituting (2.9) into (2.8), obtaining

$$\begin{aligned} \dot{x}(t) = {} & D^3 \dot{x}(t - 3\tau) + Ax(t) + (DA + B)x(t - \tau) \\ & + D(DA + B)x(t - 2\tau) + D^2 Bx(t - 3\tau) \end{aligned} \tag{2.10}$$

for $t \in [2\tau, 3\tau)$. Repeating this procedure $(m-1)$-times, we get the equation

$$\dot{x}(t) = D^m \dot{x}(t - m\tau) + Ax(t) + (DA + B)\sum_{i=1}^{m-1} D^{i-1} x(t - i\tau) + D^{m-1} Bx(t - m\tau) \tag{2.11}$$

for $t \in [(m-1)\tau, m\tau)$ coinciding with (2.6). $\square$

Remark 2.4. The advantage of representing a solution of the initial problem (1.1), (1.2) as a solution of (2.6) is that, although (2.6) remains to be a neutral system, its right-hand side does not explicitly depend on the derivative $\dot{x}(t)$ for $t \in [0, m\tau]$ depending only on the derivative of the initial function on the initial interval $[-\tau, 0)$.

Now we give a statement on the stability of the zero solution of system (1.1) and estimates of the convergence of the solution, which we will prove using Lyapunov-Krasovskii functional (1.9).

Theorem 2.5. *Let there exist a parameter $\beta > 0$ and positive definite matrices G_1, G_2, H such that matrix S is also positive definite. Then the zero solution of system* (1.1) *is exponentially stable in the metric C^0. Moreover, for the solution $x = x(t)$ of* (1.1), (1.2) *the inequality*

$$\|x(t)\| \leq \left[\sqrt{\varphi(H)}\|x(0)\| + \sqrt{\tau\varphi_1(G_1, H)}\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2, H)}\ \|\dot{x}(0)\|_\tau\right] e^{-\gamma t/2} \tag{2.12}$$

holds on $(0, \infty)$ *where* $\gamma \leq \gamma_0 := \min(\beta, \lambda_{\min}(S)/\lambda_{\max}(H))$.

Proof. Let $t > 0$. We will calculate the full derivative of the functional (1.9) along the solutions of system (1.1). We obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t), t] = {} & [D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau)]^T Hx(t) \\
& + x^T(t)H[D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau)] \\
& + \left[x^T(t)G_1x(t) - e^{-\beta\tau}x^T(t-\tau)G_1x(t-\tau)\right] \\
& + \left[\dot{x}^T(t)G_2\dot{x}(t) - e^{-\beta\tau}\dot{x}^T(t-\tau)G_2\dot{x}(t-\tau)\right] \\
& - \beta\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s.
\end{aligned} \tag{2.13}$$

For $\dot{x}(t)$, we substitute its value from (1.1) to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t), t] = {} & [D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau)]^T Hx(t) \\
& + x^T(t)H[D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau)] \\
& + \left[x^T(t)G_1x(t) - e^{-\beta\tau}x^T(t-\tau)G_1x(t-\tau)\right] \\
& + [D\dot{x}(t-\tau) + Ax(t) + B\mathrm{x}(t-\tau)]^T G_2[D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau)] \\
& - e^{-\beta\tau}\dot{x}^T(t-\tau)G_2\dot{x}(t-\tau) \\
& - \beta\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s.
\end{aligned} \tag{2.14}$$

Now it is easy to verify that the last expression can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t),t] = -\left(x^T(t), x^T(t-\tau), \dot{x}^T(t-\tau)\right)$$
$$\times \begin{pmatrix} -A^TH - HA - G_1 - A^TG_2A & -HB - A^TG_2B & -HD - A^TG_2D \\ -B^TH - B^TG_2A & e^{-\beta\tau}G_1 - B^TG_2B & -B^TG_2D \\ -D^TH - D^TG_2A & -D^TG_2B & e^{-\beta\tau}G_2 - D^TG_2D \end{pmatrix}$$
$$\times \begin{pmatrix} x(t) \\ x(t-\tau) \\ \dot{x}(t-\tau) \end{pmatrix} - \beta\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s \tag{2.15}$$

or

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t),t] = {}& -\left(x^T(t), x^T(t-\tau), \dot{x}^T(t-\tau)\right) \times S \times \begin{pmatrix} x(t) \\ x(t-\tau) \\ \dot{x}(t-\tau) \end{pmatrix} \\ & - \beta\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s. \end{aligned} \tag{2.16}$$

Since the matrix S was assumed to be positive definite, for the full derivative of Lyapunov-Krasovskii functional (1.9), we obtain the following inequality:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t),t] \le {}& -\lambda_{\min}(S)\left[\|x(t)\|^2 + \|x(t-\tau)\|^2 + \|\dot{x}(t-\tau)\|^2\right] \\ & - \beta\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s. \end{aligned} \tag{2.17}$$

We will study the two possible cases (depending on the positive value of β): either

$$\beta > \frac{\lambda_{\min}(S)}{\lambda_{\max}(H)} \tag{2.18}$$

is valid or

$$\beta \le \frac{\lambda_{\min}(S)}{\lambda_{\max}(H)} \tag{2.19}$$

holds.

(1) Let (2.18) be valid. From (2.3) follows that

$$-\|x(t)\|^2 \le -\frac{1}{\lambda_{\max}(H)} V_0[x(t),t] + \frac{1}{\lambda_{\max}(H)} \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right] \mathrm{d}s. \tag{2.20}$$

We use this expression in (2.17). Since $\lambda_{\min}(S) > 0$, we obtain (omitting terms $\|x(t-\tau)\|^2$ and $\|\dot{x}(t-\tau)\|^2$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V_0[x(t),t] \le{}& \lambda_{\min}(S) \\ &\times \left[-\frac{1}{\lambda_{\max}(H)} V_0[x(t),t] + \frac{1}{\lambda_{\max}(H)} \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right] \mathrm{d}s\right] \\ &- \beta \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right] \mathrm{d}s\end{aligned} \tag{2.21}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0[x(t),t] \le -\frac{\lambda_{\min}(S)}{\lambda_{\max}(H)} V_0[x(t),t] - \left[\beta - \frac{\lambda_{\min}(S)}{\lambda_{\max}(H)}\right] \int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right] \mathrm{d}s. \tag{2.22}$$

Due to (2.18) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0[x(t),t] \le -\frac{\lambda_{\min}(S)}{\lambda_{\max}(H)} V_0[x(t),t]. \tag{2.23}$$

Integrating this inequality over the interval $(0,t)$, we get

$$V_0[x(t),t] \le V_0[x(0),0] \exp\left(-\frac{\lambda_{\min}(S)}{\lambda_{\max}(H)} \cdot t\right) \le V_0[x(0),0] e^{-\gamma_0 t}. \tag{2.24}$$

(2) Let (2.19) be valid. From (2.3) we get

$$-\int_{t-\tau}^{t} e^{-\beta(t-s)} \left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right] \mathrm{d}s \le -V_0[x(t),t] + \lambda_{\max}(H)\ \|x(t)\|^2. \tag{2.25}$$

We substitute this expression into inequality (2.17). Since $\lambda_{\min}(S) > 0$, we obtain (omitting terms $\|x(t-\tau)\|^2$ and $\|\dot{x}(t-\tau)\|^2$)

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0[x(t),t] \le -\lambda_{\min}(S)\|x(t)\|^2 + \beta\left[-V_0[x(t),t] + \lambda_{\max}(H)\|x(t)\|^2\right] \tag{2.26}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t),t] \le -\beta V_0[x(t),t] - \left(\lambda_{\min}(S) - \beta\lambda_{\max}(H)\right)\|x(t)\|^2. \tag{2.27}$$

Since (2.19) holds, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0[x(t),t] \le -\beta V_0[x(t),t]. \tag{2.28}$$

Integrating this inequality over the interval $(0,t)$, we get

$$V_0[x(t),t] \le V_0[x(0),0]e^{-\beta t} \le V_0[x(0),0]e^{-\gamma_0 t}. \tag{2.29}$$

Combining inequalities (2.24), (2.29), we conclude that, in both cases (2.18), (2.19), we have

$$V_0[x(t),t] \le V_0[x(0),0]e^{-\gamma_0 t} \le V_0[x(0),0]e^{-\gamma t} \tag{2.30}$$

and, obviously (see (1.9)),

$$V_0[x(0),0] \le \lambda_{\max}(H)\|x(0)\|^2 + \lambda_{\max}(G_1)\|x(0)\|^2_{\tau,\beta} + \lambda_{\max}(G_2)\|\dot{x}(0)\|^2_{\tau,\beta}. \tag{2.31}$$

We use inequality (2.30) to obtain an estimate of the convergence of solutions of system (1.1). From (2.3) follows that

$$\|x(t)\|^2 \le \frac{1}{\lambda_{\min}(H)}\left[\lambda_{\max}(H)\|x(0)\|^2 + \lambda_{\max}(G_1)\|x(0)\|^2_{\tau,\beta} + \lambda_{\max}(G_2)\|\dot{x}(0)\|^2_{\tau,\beta}\right]e^{-\gamma t} \tag{2.32}$$

or (because $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for nonnegative a and b)

$$\|x(t)\| \le \left[\sqrt{\varphi(H)}\|x(0)\| + \sqrt{\varphi_1(G_1,H)}\|x(0)\|_{\tau,\beta} + \sqrt{\varphi_2(G_2,H)}\|\dot{x}(0)\|_{\tau,\beta}\right]e^{-\gamma t/2}. \tag{2.33}$$

The last inequality implies

$$\|x(t)\| \le \left[\sqrt{\varphi(H)}\|x(0)\| + \sqrt{\tau\varphi_1(G_1,H)}\|x(0)\|_{\tau} + \sqrt{\tau\varphi_2(G_2,H)}\|\dot{x}(0)\|_{\tau}\right]e^{-\gamma t/2}. \tag{2.34}$$

Thus inequality (2.12) is proved and, consequently, the zero solution of system (1.1) is exponentially stable in the metric C^0. □

Theorem 2.6. *Let the matrix D be nonsingular and $\|D\| < 1$. Let the assumptions of Theorem 2.5 with $\gamma < (2/\tau)\ln(1/\|D\|)$ and $\gamma \le \gamma_0$ be true. Then the zero solution of system* (1.1) *is exponentially stable in the metric C^1. Moreover, for a solution $x = x(t)$ of* (1.1), (1.2), *the inequality*

$$\|\dot{x}(t)\| \le \left[\left(\frac{\|B\|}{\|D\|} + M\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\right)\|x(0)\|_\tau + \left(1 + M\sqrt{\tau\varphi_2(G_2,H)}\right)\|\dot{x}(t)\|_\tau\right] e^{-\gamma\tau/2} \tag{2.35}$$

where

$$M = \|A\| + \|DA + B\| e^{\gamma\tau/2}\left(1 - \|D\| e^{\gamma\tau/2}\right)^{-1} \tag{2.36}$$

holds on $(0,\infty)$.

Proof. Let $t > 0$. Then the exponential stability of the zero solution in the metric C^0 is proved in Theorem 2.5. Now we will show that the zero solution is exponentially stable in the metric C^1 as well. As follows from Lemma 2.3, for derivative $\dot{x}(t)$, the inequality

$$\|\dot{x}(t)\| \le \|D\|^m \|\dot{x}(0)\|_\tau + \|D\|^{m-1}\|B\|\|x(0)\|_\tau + \|A\|\|x(t)\| + \|DA + B\|\sum_{i=1}^{m-1}\|D\|^{i-1}\|x(t - i\tau)\| \tag{2.37}$$

holds if $t \in [(m-1)\tau, m\tau)$. We estimate $\|x(t)\|$ and $\|x(t - i\tau)\|$ using (2.12) and inequality $\|x(0)\| \le \|x(0)\|_\tau$. We obtain

$$\begin{aligned}
\|\dot{x}(t)\| \le{}& \|D\|^m \|\dot{x}(0)\|_\tau + \|D\|^{m-1}\|B\|\|x(0)\|_\tau \\
&+ \|A\|\left[\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2,H)}\|\dot{x}(0)\|_\tau\right] e^{-\gamma t/2} \\
&+ \|DA + B\|\|D\|^{-1}\left[\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H}\right)\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2,H)}\|\dot{x}(0)\|_\tau\right] \\
&\times\left[\sum_{i=1}^{m-1}\|D\|^i e^{\gamma i\tau/2}\right] e^{-\gamma t/2}.
\end{aligned} \tag{2.38}$$

Since

$$\sum_{i=1}^{m-1}\|D\|^i e^{\gamma i\tau/2} < \sum_{i=1}^{\infty}\|D\|^i e^{\gamma i\tau/2} = \frac{\|D\| e^{\gamma\tau/2}}{1 - \|D\| e^{\gamma\tau/2}}, \tag{2.39}$$

inequality (2.38) yields

$$\begin{aligned}
\|\dot{x}(t)\| \le{}& \|D\|^m \|\dot{x}(0)\|_\tau + \|D\|^{m-1}\|B\|\|x(0)\|_\tau \\
&+ \left(\|A\| + \|DA+B\|\|D\|^{-1}\frac{\|D\|e^{\gamma\tau/2}}{1-\|D\|e^{\gamma\tau/2}}\right) \\
&\times \left[\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2,H)}\|\dot{x}(0)\|_\tau\right]e^{-\gamma t/2} \\
={}& \|D\|^m \|\dot{x}(0)\|_\tau + \|D\|^{m-1}\|B\|\|x(0)\|_\tau \\
&+ M\left[\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2,H)}\|\dot{x}(0)\|_\tau\right]e^{-\gamma t/2}.
\end{aligned} \tag{2.40}$$

Because $t \in [(m-1)\tau, m\tau)$, we can estimate

$$\begin{aligned}
\|D\|^m &= \left(\frac{1}{\|D\|}\right)^{-m} < \left(\frac{1}{\|D\|}\right)^{-t/\tau} = \exp\left(-\frac{t}{\tau}\ln\frac{1}{\|D\|}\right), \\
\|D\|^{m-1} &= \frac{1}{\|D\|}\|D\|^m < \frac{1}{\|D\|}\exp\left(-\frac{t}{\tau}\ln\frac{1}{\|D\|}\right).
\end{aligned} \tag{2.41}$$

Then

$$\|D\|^m\|\dot{x}(0)\|_\tau + \|D\|^{m-1}\|B\|\|x(0)\|_\tau \le \left[\|\dot{x}(0)\|_\tau + \frac{\|B\|}{\|D\|}\|x(0)\|_\tau\right]\exp\left(-\frac{t}{\tau}\ln\frac{1}{\|D\|}\right). \tag{2.42}$$

Now we get from (2.40)

$$\begin{aligned}
\|\dot{x}(t)\| \le{}& \left[\|\dot{x}(0)\|_\tau + \frac{\|B\|}{\|D\|}\|x(0)\|_\tau\right]\exp\left(-\frac{t}{\tau}\ln\frac{1}{\|D\|}\right) \\
&+ M\left[\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2,H)}\|\dot{x}(0)\|_\tau\right]e^{-\gamma t/2}.
\end{aligned} \tag{2.43}$$

Since

$$\exp\left(-\frac{t}{\tau}\ln\frac{1}{\|D\|}\right) \le \exp\left(-\frac{\gamma t}{2}\right), \tag{2.44}$$

the last inequality implies

$$\begin{aligned}
\|\dot{x}(t)\| \le{}& \left[\left(\frac{\|B\|}{\|D\|} + M\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\right)\|x(0)\|_\tau \right.\\
&\left. + \left(1 + M\sqrt{\tau\varphi_2(G_2,H)}\right)\|\dot{x}(0)\|_\tau\right]e^{-\gamma t/2}.
\end{aligned} \tag{2.45}$$

The positive number m can be chosen arbitrarily large. Therefore, the last inequality holds for every $t > 0$. We have obtained inequality (2.35) so that the zero solution of (1.1) is exponentially stable in the metric C^1. □

3. Estimates of Solutions in a General Case

Now we will estimate the norms of solutions of (1.1) and the norms of their derivatives in the case of the assumptions of Theorem 2.5 or Theorem 2.6 being not necessarily satisfied. It means that the estimates derived will cover the case of instability as well. For obtaining such type of results we will use a functional of Lyapunov-Krasovskii in the form (1.10). This functional includes an exponential factor, which makes it possible, in the case of instability, to get an estimate of the "divergence" of solutions. Functional (1.10) is a generalization of (1.9) because the choice $p = 0$ gives $V[x(t), t] = V_0[x(t), t]$. For (1.10) the estimate

$$
\begin{aligned}
&e^{pt}\left[\lambda_{\min}(H)\|x(t)\|^2 + \int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^2(s)G_2\dot{x}^2(s)\right]\mathrm{d}s\right]\\
&\quad\le [V(t), t] \qquad (3.1)\\
&\quad\le e^{pt}\left[\lambda_{\max}(H)\|x(t)\|^2 + \int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^2(s)G_2\dot{x}^2(s)\right]\mathrm{d}s\right]
\end{aligned}
$$

holds. We define an auxiliary $3n \times 3n$ matrix

$$
\begin{aligned}
S^* &= S^*(\beta, G_1, G_2, H, p)\\
&:= \begin{pmatrix} -A^TH - HA - G_1 - A^TG_2A - pH & -HB - A^TG_2B & -HD - A^TG_2D \\ -B^TH - B^TG_2A & e^{-\beta\tau}G_1 - B^TG_2B & -B^TG_2D \\ -D^TH - D^TG_2A & -D^TG_2B & e^{-\beta\tau}G_2 - D^TG_2D \end{pmatrix} \qquad (3.2)
\end{aligned}
$$

depending on the parameters p, β and the matrices G_1, G_2, and H. The parameter p plays a significant role for the positive definiteness of the matrix S^*. Particularly, a proper choice of $p \ll 0$ can cause the positivity of S^*. In the following, $\varphi(H)$, $\varphi_1(G_1, H)$ and $\varphi_2(G_2, H)$, have the same meaning as in Part 2. The proof of the following theorem is similar to the proofs of Theorems 2.5 and 2.6 (and its statement in the case of $p = 0$ exactly coincides with the statements of these theorems). Therefore, we will restrict its proof to the main points only.

Theorem 3.1. (A) *Let p be a fixed real number, β a positive constant, and G_1, G_2, and H positive definite matrices such that the matrix S^* is also positive definite. Then a solution $x = x(t)$ of problem* (1.1), (1.2) *satisfies on* $(0, \infty)$ *the inequality*

$$
\|x(t)\| \le \left[\sqrt{\varphi(H)}\|x(0)\| + \sqrt{\tau\varphi_1(G_1, H)}\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2, H)}\|\dot{x}(0)\|_\tau\right]e^{-\gamma t/2}, \qquad (3.3)
$$

where $\gamma \le \gamma^* := \min(\beta, p + (\lambda_{\min}(S^*)/\lambda_{\max}(H)))$.

(B) *Let the matrix D be nonsingular and $\|D\| < 1$. Let all the assumptions of part (A) with $\gamma < (2/\tau)\ln(1/\|D\|)$ and $\gamma \le \gamma^*$ be true. Then the derivative of the solution $x = x(t)$ of problem* (1.1), (1.2) *satisfies on* $(0,\infty)$ *the inequality*

$$\begin{aligned}\|\dot{x}(t)\| \le \Bigg[&\left(\frac{\|B\|}{\|D\|} + M\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1,H)}\right)\right)\|x(0)\|_\tau \\ &+\left(1 + M\sqrt{\tau\varphi_2(G_2,H)}\right)\|\dot{x}(0)\|_\tau\Bigg]e^{-\gamma t/2},\end{aligned} \tag{3.4}$$

where M is defined by (2.36).

Proof. Let $t > 0$. We compute the full derivative of the functional (1.10) along the solutions of (1.1). For $\dot{x}(t)$, we substitute its value from (1.1). Finally we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V[x(t),t] = &-e^{pt}\left(x^T(t), x^T(t-\tau), \dot{x}^T(t-\tau)\right)\times S^* \\ &\times\begin{pmatrix} x(t) \\ x(t-\tau) \\ \dot{x}(t-\tau)\end{pmatrix} - e^{pt}(\beta - p)\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s.\end{aligned} \tag{3.5}$$

Since the matrix S^* is positive definite, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V[x(t),t] \le &-\lambda_{\min}(S^*)e^{pt}\left[\|x(t)\|^2 + \|x(t-\tau)\|^2 + \|\dot{x}(t-\tau)\|^2\right] \\ &- e^{pt}(\beta - p)\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s.\end{aligned} \tag{3.6}$$

Now we will study the two possible cases: either

$$\beta - p > \frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)} \tag{3.7}$$

is valid or

$$\beta - p \le \frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)} \tag{3.8}$$

holds.

(1) Let (3.7) be valid. Since $\lambda_{\min}(S^*) > 0$, from inequality (3.1) follows that

$$\begin{aligned}-e^{pt}\|x(t)\|^2 \le &-\frac{1}{\lambda_{\max}(H)}V[x(t),t] \\ &+ \frac{e^{pt}}{\lambda_{\max}(H)}\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s.\end{aligned} \tag{3.9}$$

We use this inequality in (3.6). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}V[x(t),t] \le -\frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)}V[x(t),t] - e^{pt}\left(\beta - p - \frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)}\right) \times \int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s. \tag{3.10}$$

From inequality (3.7) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V[x(t),t] \le -\frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)}V[x(t),t]. \tag{3.11}$$

Integrating this inequality over the interval $(0,t)$, we get

$$V[x(t),t] \le V[x(0),0]\exp\left(-\frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)}t\right) \le V[x(0),0]e^{-(\gamma^*-p)t}. \tag{3.12}$$

(2) Let (3.8) be valid. From inequality (3.1) we get

$$-e^{pt}\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s \le -V[x(t),t] + e^{pt}\lambda_{\max}(H)\|x(t)\|^2. \tag{3.13}$$

We use this inequality in (3.6) again. Since $\lambda_{\min}(S^*) > 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V[x(t),t] \le -(\beta-p)V[x(t),t] - \left[\lambda_{\min}(S^*) - (\beta-p)\lambda_{\max}(H)\right]e^{pt}\|x(t)\|^2. \tag{3.14}$$

Because the inequality (3.8) holds, we have

$$\frac{d}{dt}V[x(t),t] \le -(\beta-p)V[x(t),t]. \tag{3.15}$$

Integrating this inequality over the interval $(0,t)$, we get

$$V[x(t),t] \le V[x(0),0]e^{-(\beta-p)t} \le V[x(0),0]e^{-(\gamma^*-p)t}. \tag{3.16}$$

Combining inequalities (3.12), (3.16), we conclude that, in both cases (3.7), (3.8), we have

$$V[x(t),t] \le V[x(0),0]e^{-(\gamma^*-p)t}. \tag{3.17}$$

From this, it follows

$$
\begin{aligned}
&e^{pt}\left[x^T(t)Hx(t)+\int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s)+\dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s\right]\\
&\qquad\le\left[x^T(0)Hx(0)+\int_{-\tau}^{0} e^{\beta s}\left[x^T(s)G_1x(s)+\dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s\right]e^{-(\gamma^*-p)t},\\
&e^{pt}\lambda_{\min}(H)\|x(t)\|^2\\
&\qquad\le\left[\lambda_{\max}(H)\|x(0)\|^2+\lambda_{\max}(G_1)\|x(0)\|^2_{\beta,\tau}+\lambda_{\max}(G_2)\|\dot{x}(0)\|^2_{\beta,\tau}\right]e^{-(\gamma^*-p)t}.
\end{aligned}
\tag{3.18}
$$

From the last inequality we derive inequality (3.3) in a way similar to that of the proof of Theorem 2.5. The inequality to estimate the derivative (3.4) can be obtained in much the same way as in the proof of Theorem 2.6. □

Remark 3.2. As can easily be seen from Theorem 3.1, part (A), if

$$
p+\frac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)}>0,
\tag{3.19}
$$

we deal with an exponential stability in the metric C^0. If, moreover, part (B) holds and (3.19) is valid, then we deal with an exponential stability in the metric C^1.

4. Examples

In this part we consider two examples. Auxiliary numerical computations were performed by using MATLAB & SIMULINK R2009a.

Example 4.1. We will investigate system (1.1) where $n=2$, $\tau=1$,

$$
D=\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix},\qquad A=\begin{pmatrix}-1 & 0.1\\ 0.1 & -1\end{pmatrix},\qquad B=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix},
\tag{4.1}
$$

that is, the system

$$
\begin{aligned}
\dot{x}_1(t)&=0.5\dot{x}_1(t-1)-x_1(t)+0.1x_2(t)+0.1x_1(t-1),\\
\dot{x}_2(t)&=0.5\dot{x}_2(t-1)+0.1x_1(t)-x_2(t)+0.1x_2(t-1),
\end{aligned}
\tag{4.2}
$$

with initial conditions (1.2). Set $\beta=0.1$ and

$$
G_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad G_2=\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix},\qquad H=\begin{pmatrix}2 & 0.1\\ 0.1 & 5\end{pmatrix}.
\tag{4.3}
$$

For the eigenvalues of matrices G_1, G_2, and H, we get $\lambda_{\min}(G_1) = \lambda_{\max}(G_1) = 1$, $\lambda_{\min}(G_2) \doteq 0.5858$, $\lambda_{\max}(G_2) \doteq 3.4142$, $\lambda_{\min}(H) \doteq 1.9967$, and $\lambda_{\max}(H) \doteq 5.0033$. The matrix $S = S(\beta, G_1, G_2, H)$ takes the form

$$S \doteq \begin{pmatrix} 2.1500 & -1.1100 & -0.1100 & 0.0600 & -0.5500 & 0.3000 \\ -1.1100 & 6.1700 & 0.0800 & -0.2100 & 0.4000 & -1.0500 \\ -0.1100 & 0.0800 & 0.8948 & -0.0100 & -0.0500 & -0.0500 \\ 0.0600 & -0.2100 & -0.0100 & 0.8748 & -0.0500 & -0.1500 \\ -0.5500 & 0.4000 & -0.0500 & -0.0500 & 0.6548 & 0.6548 \\ 0.3000 & -1.0500 & -0.0500 & -0.1500 & 0.6548 & 1.9645 \end{pmatrix} \tag{4.4}$$

and $\lambda_{\min}(S) \doteq 0.1445$. Because all the eigenvalues are positive, matrix S is positive definite. Since all conditions of Theorem 2.5 are satisfied, the zero solution of system (4.2) is asymptotically stable in the metric C^0. Further we have

$$\begin{gathered} \varphi(H) \doteq \frac{5.0033}{1.9967} \doteq 2.5058, \qquad \varphi_1(G_1, H) \doteq \frac{1}{1.9967} \doteq 0.5008, \\ \varphi_2(G_2, H) \doteq \frac{3.4142}{1.9967} \doteq 1.7099, \qquad \gamma_0 = \min\left(0.1, \frac{0.1445}{5.0033}\right) \doteq \min(0.1, 0.0289) = 0.0289, \\ \|A\| = 1.1, \qquad \|B\| = 0.1, \qquad \|D\| = 0.5, \qquad \|DA + B\| = 0.45, \qquad M = 2.0266. \end{gathered} \tag{4.5}$$

Since $\gamma_0 < (2/\tau)\ln(1/\|D\|) \doteq 1.3863$, all conditions of Theorem 2.6 are satisfied and, consequently, the zero solution of (4.2), (35) is asymptotically stable in the metric C^1. Finally, from (2.12) and (2.35) follows that the inequalities

$$\begin{aligned} \|x(t)\| &\le \left[\sqrt{2.5058}\|x(0)\| + \sqrt{0.5008}\|x(0)\|_1 + \sqrt{1.7099}\ \|\dot{x}(0)\|_1\right] e^{-0.0289t/2} \\ &\doteq [1.5830\|x(0)\| + 0.7077\|x(0)\|_1 + 1.3076\ \|\dot{x}(0)\|_1] e^{-0.0289t/2}, \\ \|\dot{x}(t)\| &\le \left[\left(0.2 + 2.0266\left(\sqrt{2.5058} + \sqrt{0.5008}\right)\right)\|x(0)\|_1 \right. \\ &\quad \left. + \left(1 + 2.0266\sqrt{1.7099}\right)\|\dot{x}(0)\|_1\right] e^{-0.0289t/2} \\ &\doteq [4.8422\|x(0)\|_1 + 3.6500\|\dot{x}(0)\|_1] e^{-0.0289t/2} \end{aligned} \tag{4.6}$$

hold on $(0, \infty)$.

Example 4.2. We will investigate system (1.1) where $n = 2$, $\tau = 1$,

$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0.6213 \\ 0.6213 & 0 \end{pmatrix}, \tag{4.7}$$

that is, the system

$$\dot{x}_1(t) = 0.1\dot{x}_1(t-1) - 3x_1(t) - 2x_2(t) + 0.6213x_2(t-1),$$
$$\dot{x}_2(t) = 0.1\dot{x}_2(t-1) + 1x_1(t) + 0.6213x_1(t-1), \tag{4.8}$$

with initial conditions (1.2). Set $\beta = 0.1$ and

$$G_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}. \tag{4.9}$$

For the eigenvalues of matrices G_1, G_2, and H, we get $\lambda_{\min}(G_1) \doteq 0.0764$, $\lambda_{\max}(G_1) \doteq 0.5236$, $\lambda_{\min}(G_2) = \lambda_{\max}(G_2) = 0.1$ $\lambda_{\min}(H) = 0.2$, and $\lambda_{\max}(H) = 1$. The matrix $S = S(\beta, G_1, G_2, H)$ takes the form

$$S \doteq \begin{pmatrix} 1.3000 & 1.1000 & -0.3106 & -0.1864 & -0.0300 & -0.0500 \\ 1.1000 & 1.1000 & -0.3728 & -0.1243 & -0.0200 & -0.0600 \\ -0.3106 & -0.3728 & 0.4138 & 0.0905 & 0 & -0.0062 \\ -0.1864 & -0.1243 & 0.0905 & 0.0519 & -0.0062 & 0 \\ -0.0300 & -0.0200 & 0 & -0.0062 & 0.0895 & 0 \\ -0.0500 & -0.0600 & -0.0062 & 0 & 0 & 0.0895 \end{pmatrix} \tag{4.10}$$

and $\lambda_{\min}(S) \doteq 0.00001559$. Because all eigenvalues are positive, matrix S is positive definite. Since all conditions of Theorem 2.5 are satisfied, the zero solution of system (4.8) is asymptotically stable in the metric C^0. Further we have

$$\varphi(H) = \frac{1}{0.2} = 5, \qquad \varphi_1(G_1, H) \doteq \frac{0.5236}{0.2} \doteq 2.618, \qquad \varphi_2(G_2, H) = \frac{0.1}{0.2} = 0.5,$$
$$\gamma_0 \doteq \min(0.1,\ 0.00001559) = 0.00001559,$$
$$\|A\| \doteq 3.7025, \qquad \|B\| \doteq 0.6213, \qquad \|D\| = 0.1, \qquad \|DA + B\| \doteq 0.8028, \qquad M \doteq 4.5945. \tag{4.11}$$

Since $\gamma_0 < (2/\tau)\ln(1/\|D\|) = 2\ln 10 \doteq 4.6052$, all conditions of Theorem 2.6 are satisfied and, consequently, the zero solution of (4.8) is asymptotically stable in the metric C^1. Finally, from

(2.12) and (2.35) follows that the inequalities

$$\begin{aligned}
\|x(t)\| &\le \left[\sqrt{5}\|x(0)\| + \sqrt{2.618}\|x(0)\|_1 + \sqrt{0.5}\ \|\dot{x}(0)\|_1\right] e^{-0.00001559t/2} \\
&\doteq [2.2361\|x(0)\| + 1.6180\|x(0)\|_1 + 0.7071\ \|\dot{x}(0)\|_1] e^{-0.00001559t/2}, \\
\|\dot{x}(t)\| &\le \Big[\left(6.213 + 4.5945\left(\sqrt{5} + \sqrt{2.618}\right)\right)\ \|x(0)\|_1 \\
&\quad + \left(1 + 4.5945\sqrt{0.5}\right)\|\dot{x}(0)\|_1\Big] e^{-0.00001559t/2} \\
&\doteq [23.9206\|x(0)\|_1 + 4.2488\|\dot{x}(0)\|_1] e^{-0.00001559t/2}
\end{aligned} \tag{4.12}$$

hold on $(0, \infty)$.

Remark 4.3. In [12] an example can be found similar to Example 4.2 with the same matrices A, D, arbitrary constant positive τ, and with a matrix

$$B = B_\alpha = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \tag{4.13}$$

where α is a real parameter. The stability is established for $|\alpha| < 0.4$. In recent paper [13], the stability of the same system is even established for $|\alpha| < 0.533$.

Comparing these particular results with Example 4.2, we see that, in addition to stability, our results imply the exponential stability in the metric C^0 as well as in the metric C^1. Moreover, we are able to prove the exponential stability (in C^0 as well as in C^1) in Example 4.2 with the matrix $B = B_\alpha$ for $|\alpha| \le 0.6213$ and for an arbitrary constant delay τ. The latter statement can be explained easily—for an arbitrary positive τ, we set $\beta = 0.1/\tau$. Calculating the characteristic equation for the matrix S where B is changed by B_α we get

$$P_6(\lambda) := \sum_{i=0}^{6} p_i(\alpha)\lambda^i = 0, \tag{4.14}$$

where

$$\begin{gathered}
p_6(\alpha) = -1, \\
p_5(\alpha) = -0.2\alpha^2 - 3.1219, \\
p_4(\alpha) = -0.01\alpha^4 - 1.3105\alpha^2 + 2.0830, \\
p_3(\alpha) = -0.0998\alpha^4 + 0.5717\alpha^2 - 0.4943, \\
p_2(\alpha) = -0.0366\alpha^4 - 0.096828\alpha^2 + 0.053858, \\
p_1(\alpha) = -0.004204382\alpha^4 + 0.0073\alpha^2 - 0.0028, \\
p_0(\alpha) = -0.00015392\alpha^4 - 0.00020116\alpha^2 + 0.000059723.
\end{gathered} \tag{4.15}$$

It is easy to verify that $(-1)^i p_i(\alpha) > 0$ for $i = 0, 1, \ldots, 6$ and $|\alpha| \le 0.6213$, and for the equation

$$P_6^*(\lambda) = P_6(-\lambda) = \sum_{i=0}^{6} p_i^*(\alpha)\lambda^i = 0, \tag{4.16}$$

we have $p_i^*(\alpha) = (-1)^i p_i(\alpha) > 0$. Then, due to the symmetry of the real matrix S, we conclude that, by Descartes' rule of signs, all eigenvalues of S (i.e., all roots of $P_6(\lambda) = 0$) are positive. This means that the exponential stability (in the metric C^0 as well as in the metric C^1) for $|\alpha| \le 0.6213$ is proved. Finally, we note that the variation of α within the interval indicated or the choice $\beta = 0.1/\tau$ does not change the exponential stability having only influence on the form of the final inequalities for $\|x(t)\|$ and $\|\dot{x}(t)\|$.

5. Conclusions

In this paper we derived statements on the exponential stability of system (1.1) as well as on estimates of the norms of its solutions and their derivatives in the case of exponential stability and in the case of exponential stability being not guaranteed. To obtain these results, special Lyapunov functionals in the form (1.9) and (1.10) were utilized as well as a method of constructing a reduced neutral system with the same solution on the intervals indicated as the initial neutral system (1.1). The flexibility and power of this method was demonstrated using examples and comparisons with other results in this field. Considering further possibilities along these lines, we conclude that, to generalize the results presented to systems with bounded variable delay $\tau = \tau(t)$, a generalization is needed of Lemma 2.3 to the above reduced neutral system. This can cause substantial difficulties in obtaining results which are easily presentable. An alternative would be to generalize only the part of the results related to the exponential stability in the metric C^0 and the related estimates of the norms of solutions in the case of exponential stability and in the case of the exponential stability being not guaranteed (omitting the case of exponential stability in the metric C^1 and estimates of the norm of a derivative of solution). Such an approach will probably permit a generalization to variable matrices ($A = A(t)$, $B = B(t)$, $D = D(t)$) and to a variable delay ($\tau = \tau(t)$) or to two different variable delays. Nevertheless, it seems that the results obtained will be very cumbersome and hardly applicable in practice.

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References

[1] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, vol. 463 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1999.

[2] N. N. Krasovskii, *Some Problems of Theory of Stability of Motion*, Fizmatgiz, Moscow, Russia, 1959.

[3] N. N. Krasovskiĭ, *Stability of Motion. Applications of Lyapunov's Second Method to Differential systems and Equations with Delay*, Translated by J. L. Brenner, Stanford University Press, Stanford, Calif, USA, 1963.

[4] D. G. Korenevskiĭ, *Stability of Dynamical Systems Under Random Perturbations of Parameters. Algebraic Criteria*, Naukova Dumka, Kiev, Ukraine, 1989.

[5] D. Ya. Khusainov and E. A. Yunkova, "Investigation of the stability of linear systems of neutral type by the method of Lyapunov functions," *Differentsial cprime nye Uravneniya*, vol. 24, no. 4, pp. 613–621, 1988, Translated in *Differential Equations*, vol. 24, no. 4, pp. 424–431.

[6] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Pynamics*, vol. 74 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1992.

[7] V. Kolmanovskiĭ and A. Myshkis, *Applied Theory of Functional-Differential Equations*, vol. 85 of *Mathematics and its Applications (Soviet Series)*, Kluwer Academic Publishers Group, Dordrecht, the Netherlands, 1992.

[8] V. Kolmanovskiĭ and V. Nosov, *Stability of Functional Differential Equations*, vol. 180 of *Mathematics in Science and Engineering*, Academic Press, Harcourt Brace Jovanovich, London, UK, 1986.

[9] L. Mei-Gin, "Stability analysis of neutral-type nonlinear delayed systems: an LMI approach," *Journal of Zhejiang University A*, vol. 7, supplement 2, pp. 237–244, 2006.

[10] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*, Control Engineering, Birkhuser, Boston, Mass, USA, 2003.

[11] X. Liao, L. Wang, and P. Yu, *Stability of Dynamical Systems*, vol. 5 of *Monograph Series on Nonlinear Science and Complexity*, Elsevier, Amsterdam, the Netherlands, 2007.

[12] Ju.-H. Park and S. Won, "A note on stability of neutral delay-differential systems," *Journal of the Franklin Institute*, vol. 336, no. 3, pp. 543–548, 1999.

[13] X.-x. Liu and B. Xu, "A further note on stability criterion of linear neutral delay-differential systems," *Journal of the Franklin Institute*, vol. 343, no. 6, pp. 630–634, 2006.