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Research Article

Eigenvalue Problem and Unbounded Connected Branch of Positive Solutions to a Class of Singular Elastic Beam Equations

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This paper investigates the eigenvalue problem for a class of singular elastic beam equations where one end is simply supported and the other end is clamped by sliding clamps. Firstly, we establish a necessary and sufficient condition for the existence of positive solutions, then we prove that the closure of positive solution set possesses an unbounded connected branch which bifurcates from $(0,\theta)$. Our nonlinearity $f(t,u,v,w)$ may be singular at u,v, $t=0$ and/or $t=1$.

1. Introduction

Singular differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer, and so on. Therefore, singular boundary value problems have been investigated extensively in recent years (see [1–4] and references therein).

This paper investigates the following fourth-order nonlinear singular eigenvalue problem:

$$\begin{gathered} u^{(4)}(t)=\lambda f(t,u(t),u'(t),u''(t)), \quad t\in(0,1), \\ u(0)=u'(1)=u''(0)=u'''(1)=0, \end{gathered} \tag{1.1}$$

where $\lambda\in(0,+\infty)$ is a parameter and f satisfies the following hypothesis:

(H) $f\in C((0,1)\times(0,+\infty)\times(0,+\infty)\times(-\infty,0],[0,+\infty))$, and there exist constants α_i, β_i, N_i, $i=1,2,3$ $(-\infty<\alpha_1\le 0\le\beta_1<+\infty,\ -\infty<\alpha_2\le 0\le\beta_2<+\infty,\ 0\le\alpha_3\le\beta_3<1,$

$\sum_{i=1}^{3} \beta_i < 1$; $0 < N_i \le 1$, $i = 1,2,3$) such that for any $t \in (0,1)$, $u, v \in (0,+\infty)$, $w \in (-\infty, 0]$, f satisfies

$$\begin{aligned} c^{\beta_1} f(t,u,v,w) &\le f(t,cu,v,w) \le c^{\alpha_1} f(t,u,v,w), \quad \forall 0 < c \le N_1, \\ c^{\beta_2} f(t,u,v,w) &\le f(t,u,cv,w) \le c^{\alpha_2} f(t,u,v,w), \quad \forall 0 < c \le N_2, \\ c^{\beta_3} f(t,u,v,w) &\le f(t,u,v,cw) \le c^{\alpha_3} f(t,u,v,w), \quad \forall 0 < c \le N_3. \end{aligned} \tag{1.2}$$

Typical functions that satisfy the above sublinear hypothesis (H) are those taking the form

$$f(t,u,v,w) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} p_{i,j,k}(t) u^{r_i} v^{s_j} w^{\sigma_k}, \tag{1.3}$$

where $p_{i,j,k}(t) \in C[(0,1),(0,+\infty)]$, $r_i, s_j \in R$, $0 \le \sigma_k < 1$, $\max\{r_i, 0\} + \max\{s_j\} + \sigma_k < 1$, $i = 1,2,\ldots,m_1$, $j = 1,2,\ldots,m_2$, $k = 1,2,\ldots,m_3$. The hypothesis (H) is similar to that in [5, 6].

Because of the extensive applications in mechanics and engineering, nonlinear fourth-order two-point boundary value problems have received wide attentions (see [7–12] and references therein). In mechanics, the boundary value problem (1.1) (BVP (1.1) for short) describes the deformation of an elastic beam simply supported at left and clamped at right by sliding clamps. The term u'' in f represents bending effect which is useful for the stability analysis of the beam. BVP (1.1) has two special features. The first one is that the nonlinearity f may depend on the first-order derivative of the unknown function u, and the second one is that the nonlinearity $f(t,u,v,w)$ may be singular at u, v, $t = 0$ and/or $t = 1$.

In this paper, we study the existence of positive solutions and the structure of positive solution set for the BVP (1.1). Firstly, we construct a special cone and present a necessary and sufficient condition for the existence of positive solutions, then we prove that the closure of positive solution set possesses an unbounded connected branch which bifurcates from $(0,\theta)$. Our analysis mainly relies on the fixed point theorem in a cone and the fixed point index theory.

By singularity of f, we mean that the function f in (1.1) is allowed to be unbounded at the points $u = 0$, $v = 0$, $t = 0$, and/or $t = 1$. A function $u(t) \in C^2[0,1] \cap C^4(0,1)$ is called a (positive) solution of the BVP (1.1) if it satisfies the BVP (1.1) ($u(t) > 0$, $-u''(t) > 0$ for $t \in (0,1]$ and $u'(t) > 0$ for $t \in [0,1)$). For some $\lambda \in (0,+\infty)$, if the BVP (1.1) has a positive solution u, then λ is called an eigenvalue and u is called corresponding eigenfunction of the BVP (1.1).

The existence of positive solutions of BVPs has been studied by several authors in the literature; for example, see [7–20] and the references therein. Yao [15, 18] studied the following BVP:

$$\begin{gathered} u^{(4)}(t) = f(t,u(t),u'(t)), \quad t \in [0,1] \setminus E, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{gathered} \tag{1.4}$$

where $E \subset [0,1]$ is a closed subset and $\text{mes}E = 0$, $f \in C(([0,1] \setminus E) \times [0,+\infty) \times [0,+\infty), [0,+\infty))$. In [15], he obtained a sufficient condition for the existence of positive solutions of BVP (1.4) by using the monotonically iterative technique. In [13, 18], he applied Guo-Krasnosel'skii's fixed point theorem to obtain the existence and multiplicity of positive solutions of BVP (1.4) and the following BVP:

$$u^{(4)}(t) = f(t,u(t)), \quad t \in [0,1],$$
$$u(0) = u'(0) = u(1) = u''(1) = 0. \tag{1.5}$$

These differ from our problem because $f(t,u,v)$ in (1.4) cannot be singular at $u = 0$, $v = 0$ and the nonlinearity f in (1.5) does not depend on the derivatives of the unknown functions.

In this paper, we first establish a necessary and sufficient condition for the existence of positive solutions of BVP (1.1) for any $\lambda > 0$ by using the following Lemma 1.1. Efforts to obtain necessary and sufficient conditions for the existence of positive solutions of BVPs by the lower and upper solution method can be found, for example, in [5, 6, 21–23]. In [5, 6, 22, 23] they considered the case that f depends on even order derivatives of u. Although the nonlinearity f in [21] depends on the first-order derivative, where the nonlinearity f is increasing with respect to the unknown function u. Papers [24, 25] derived the existence of positive solutions of BVPs by the lower and upper solution method, but the nonlinearity $f(t,u)$ does not depend on the derivatives of the unknown functions, and $f(t,u)$ is decreasing with respect to u.

Recently, the global structure of positive solutions of nonlinear boundary value problems has also been investigated (see [26–28] and references therein). Ma and An [26] and Ma and Xu [27] discussed the global structure of positive solutions for the nonlinear eigenvalue problems and obtained the existence of an unbounded connected branch of positive solution set by using global bifurcation theorems (see [29, 30]). The terms $f(u)$ in [26] and $f(t,u,u'')$ in [27] are not singular at $t = 0, 1$, $u = 0$, $u'' = 0$. Yao [14] obtained one or two positive solutions to a singular elastic beam equation rigidly fixed at both ends by using Guo-Krasnosel'skii's fixed point theorem, but the global structure of positive solutions was not considered. Since the nonlinearity $f(t,u,v,w)$ in BVP (1.1) may be singular at $u, v, t = 0$ and/or $t = 1$, the global bifurcation theorems in [29, 30] do not apply to our problem here. In Section 4, we also investigate the global structure of positive solutions for BVP (1.1) by applying the following Lemma 1.2.

The paper is organized as follows: in the rest of this section, two known results are stated. In Section 2, some lemmas are stated and proved. In Section 3, we establish a necessary and sufficient condition for the existence of positive solutions. In Section 4, we prove that the closure of positive solution set possesses an unbounded connected branch which comes from $(0,\theta)$.

Finally we state the following results which will be used in Sections 3 and 4, respectively.

Lemma 1.1 (see [31]). *Let X be a real Banach space, let K be a cone in X, and let Ω_1, Ω_2 be bounded open sets of E, $\theta \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Suppose that $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is completely continuous such that one of the following two conditions is satisfied:*

(1) $\|T(x)\| \le \|x\|$, $x \in K \cap \partial\Omega_1$; $\|T(x)\| \ge \|x\|$, $x \in K \cap \partial\Omega_2$.

(2) $\|T(x)\| \ge \|x\|$, $x \in K \cap \partial\Omega_1$; $\|T(x)\| \le \|x\|$, $x \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 1.2 (see [32]). *Let M be a metric space and $(a,b) \subset R^1$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy*

$$a < \cdots < a_n < \cdots < a_1 < b_1 < \cdots < b_n < \cdots < b,$$
$$\lim_{n\to+\infty} a_n = a, \qquad \lim_{n\to+\infty} b_n = b. \tag{1.6}$$

Suppose also that $\sum = \{C_n : n = 1,2,\ldots\}$ is a family of connected subsets of $R^1 \times M$, satisfying the following conditions:

(1) *$C_n \cap (\{a_n\} \times M) \neq \emptyset$ and $C_n \cap (\{b_n\} \times M) \neq \emptyset$ for each n.*

(2) *For any two given numbers α and β with $a < \alpha < \beta < b$, $(\bigcup_{n=1}^{\infty} C_n) \cap ([\alpha,\beta] \times M)$ is a relatively compact set of $R^1 \times M$.*

Then there exists a connected branch C of $\limsup_{n\to+\infty} C_n$ such that

$$C \cap (\{\lambda\} \times M) \neq \emptyset, \quad \forall \lambda \in (a,b), \tag{1.7}$$

where $\limsup_{n\to+\infty} C_n = \{x \in M$: there exists a sequence $x_{n_i} \in C_{n_i}$ such that $x_{n_i} \to x,\ (i \to \infty)\}$.

2. Some Preliminaries and Lemmas

Let $E = \{u \in C^2[0,1] : u(0) = 0,\ u'(1) = 0,\ u''(0) = 0\}$, $\|u\|_2 = \max\{\|u\|, \|u'\|, \|u''\|\}$, then $(E, \|\cdot\|_2)$ is a *Banach* space, where $\|u\| = \max_{t\in[0,1]}|u(t)|$. Define

$$P = \left\{u \in E : u(t) \geq \left(t - \frac{t^2}{2}\right)\|u\|,\ u'(t) \geq \frac{1}{2}(1-t)\left\|u'\right\|,\ -u''(t) \geq t\left\|u''\right\|,\ t \in [0,1]\right\}. \tag{2.1}$$

It is easy to conclude that P is a cone of E. Denote

$$P_r = \{u \in P : \|u\|_2 < r\}; \qquad \partial P_r = \{u \in P : \|u\|_2 = r\}. \tag{2.2}$$

Let

$$G_0(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1, \end{cases}$$
$$G(t,s) = \int_0^1 G_0(t,r) G_0(r,s)\,dr. \tag{2.3}$$

Then $G(t,s)$ is the Green function of homogeneous boundary value problem

$$u^{(4)}(t) = 0, \quad t \in (0,1),$$
$$u(0) = u'(1) = u''(0) = u'''(1) = 0,$$
$$G(t,s) = \begin{cases} \dfrac{s^3}{3} + \dfrac{s(t^2 - s^2)}{2} + st(1-t), & 0 \le s \le t \le 1, \\ \dfrac{t^3}{3} + \dfrac{t(s^2 - t^2)}{2} + ts(1-s), & 0 \le t \le s \le 1, \end{cases}$$
$$G_1(t,s) =: G_t'(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ \dfrac{s^2}{2} - \dfrac{t^2}{2} + s(1-s), & 0 \le t \le s \le 1, \end{cases}$$
$$G_2(t,s) =: -G_t''(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Lemma 2.1. *$G(t,s)$, $G_1(t,s)$, and $G_2(t,s)$ have the following properties:*

(1) *$G(t,s) > 0$, $G_i(t,s) > 0$, $i = 1,2$, for all $t,s \in (0,1)$.*

(2) *$G(t,s) \le s(t - t^2/2)$, $G_1(t,s) \le s(1-t)$, $G_2(t,s) \le t$ (or s), for all $t,s \in [0,1]$.*

(3) *$\max_{t\in[0,1]} G(t,s) \le (1/2)s$, $\max_{t\in[0,1]} G_i(t,s) \le s$, $i = 1,2$, for all $s \in [0,1]$.*

(4) *$G(t,s) \ge (s/2)(t - t^2/2)$, $G_1(t,s) \ge (s/2)(1-t)$, $G_2(t,s) \ge st$, for all $t,s \in [0,1]$.*

Proof. From (2.4), it is easy to obtain the property (2.18).

We now prove that property (2) is true. For $0 \le s \le t \le 1$, by (2.4), we have

$$G(t,s) = \frac{s^3}{3} + \frac{st^2}{2} - \frac{s^3}{2} + st - st^2 \le st - \frac{st^2}{2} = s\left(t - \frac{t^2}{2}\right),$$
$$G_1(t,s) = s(1-t), \qquad G_2(t,s) \le t(\text{or } s). \tag{2.5}$$

For $0 \le t \le s \le 1$, by (2.4), we have

$$G(t,s) = \frac{t^3}{3} - \frac{t^3}{2} + ts - \frac{ts^2}{2} \le st - \frac{st^2}{2} = s\left(t - \frac{t^2}{2}\right),$$
$$G_1(t,s) = s - \frac{t^2}{2} - \frac{s^2}{2} \le s - ts = s(1-t), \qquad G_2(t,s) \le t \text{ (or } s). \tag{2.6}$$

Consequently, property (2) holds.

From property (2), it is easy to obtain property (3).

We next show that property (4) is true. From (2.4), we know that property (4) holds for $s = 0$.

For $0 < s \leq 1$, if $s \leq t \leq 1$, then

$$\frac{G(t,s)}{s} = t - \frac{t^2}{2} - \frac{s^2}{6} = \frac{1}{2}\left[t - \frac{t^2}{2} + \left(t - \frac{t^2}{2} - \frac{s^2}{3}\right)\right] \geq \frac{1}{2}\left[t - \frac{t^2}{2} + \left(t - \frac{t^2}{2} - \frac{t^2}{3}\right)\right] > \frac{1}{2}\left(t - \frac{t^2}{2}\right),$$
$$\frac{G_1(t,s)}{s} = (1-t) \geq \frac{1}{2}(1-t), \qquad G_2(t,s) \geq st; \tag{2.7}$$

if $0 \leq t \leq s$, then

$$\frac{G(t,s)}{s} \geq t - \frac{t^2}{6} - \frac{ts}{2} = \frac{1}{2}\left[t - \frac{t^2}{3} + (t - ts)\right] \geq \frac{1}{2}\left(t - \frac{t^2}{3}\right) \geq \frac{1}{2}\left(t - \frac{t^2}{2}\right),$$
$$\frac{G_1(t,s)}{s} \geq 1 - \frac{t}{2} - \frac{s}{2} \geq \frac{1}{2}(1-t), \qquad G_2(t,s) \geq st. \tag{2.8}$$

Therefore, property (4) holds. □

Lemma 2.2. *Assume that $u \in P \setminus \{\theta\}$, then $\|u\|_2 = \|u''\|$ and*

$$\frac{1}{4}\|u'\| \leq \|u\| \leq \|u'\|, \qquad \frac{1}{2}\|u''\| \leq \|u'\| \leq \|u''\|. \tag{2.9}$$

$$\frac{1}{8}\left(t - \frac{t^2}{2}\right)\|u\|_2 \leq u(t) \leq \left(t - \frac{t^2}{2}\right)\|u\|_2, \qquad \frac{1}{4}(1-t)\|u\|_2 \leq u'(t) \leq (1-t)\|u\|_2,$$
$$t\|u\|_2 \leq -u''(t) \leq \|u\|_2, \quad \forall t \in [0,1]. \tag{2.10}$$

Proof. Assume that $u \in P \setminus \{\theta\}$, then $u'(t) \geq 0$, $-u''(t) \geq 0$, $t \in [0,1]$, so

$$\|u\| = \max_{t\in[0,1]} \int_0^t u'(s)ds = \int_0^1 u'(s)ds \leq \|u'\|,$$
$$\|u\| = \max_{t\in[0,1]} \int_0^t u'(s)ds = \int_0^1 u'(s)ds \geq \frac{1}{2}\|u'\| \int_0^1 (1-s)ds = \frac{1}{4}\|u'\|,$$
$$\|u'\| = \max_{t\in[0,1]} \int_t^1 -u''(s)ds = \int_0^1 -u''(s)ds \leq \|u''\|,$$
$$\|u'\| = \max_{t\in[0,1]} \int_t^1 -u''(s)ds = \int_0^1 -u''(s)ds \geq \int_0^1 s\|u''\|\, ds = \frac{1}{2}\|u''\|. \tag{2.11}$$

Therefore, (2.9) holds. From (2.9), we get

$$\|u\|_2 = \max\{\|u\|, \|u'\|, \|u''\|\} = \|u''\|. \tag{2.12}$$

By (2.9) and the definition of P, we can obtain that

$$
\begin{aligned}
u(t) &= \int_0^1 G_0(t,s)(-u''(s))ds \le \left(\int_0^t s ds + \int_t^1 t ds\right)\|u''\| = \left(t-\frac{t^2}{2}\right)\|u''\| = \left(t-\frac{t^2}{2}\right)\|u\|_2, \\
&\qquad \forall t \in [0,1], \\
u(t) &\ge \left(t-\frac{t^2}{2}\right)\|u\| \ge \frac{1}{8}\left(t-\frac{t^2}{2}\right)\|u\|_2, \quad \forall t \in [0,1], \\
u'(t) &= \int_t^1 -u''(s)ds \le (1-t)\|u''\| = (1-t)\|u\|_2, \\
u'(t) &\ge \frac{1}{2}(1-t)\|u'\| \ge \frac{1}{4}(1-t)\|u\|_2, \quad \forall t \in [0,1], \\
t\|u\|_2 &= t\|u''\| \le -u''(t) \le \|u''\| = \|u\|_2, \quad \forall t \in [0,1].
\end{aligned}
\tag{2.13}
$$

Thus, (2.10) holds. □

For any fixed $\lambda \in (0,+\infty)$, define an operator T_λ by

$$
(T_\lambda u)(t) =: \lambda \int_0^1 G(t,s) f(s, u(s), u'(s), u''(s)) ds, \quad \forall u \in P \setminus \{\theta\}. \tag{2.14}
$$

Then, it is easy to know that

$$
(T_\lambda u)'(t) = \lambda \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s)) ds, \quad \forall u \in P \setminus \{\theta\}, \tag{2.15}
$$

$$
(T_\lambda u)''(t) = -\lambda \int_0^1 G_2(t,s) f(s, u(s), u'(s), u''(s)) ds, \quad \forall u \in P \setminus \{\theta\}. \tag{2.16}
$$

Lemma 2.3. *Suppose that (H) and*

$$
0 < \int_0^1 s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right) ds < +\infty \tag{2.17}
$$

hold. Then $T_\lambda(P \setminus \{\theta\}) \subset P$.

Proof. From (H), for any $t \in (0,1)$, $u, v \in (0,+\infty)$, $w \in (-\infty, 0]$, we easily obtain the following inequalities:

$$\begin{aligned}
&c^{\alpha_1} f(t,u,v,w) \le f(t,cu,v,w) \le c^{\beta_1} f(t,u,v,w), \quad \forall c \ge N_1^{-1},\\
&c^{\alpha_2} f(t,u,v,w) \le f(t,u,cv,w) \le c^{\beta_2} f(t,u,v,w), \quad \forall c \ge N_2^{-1},\\
&c^{\alpha_3} f(t,u,v,w) \le f(t,u,v,cw) \le c^{\beta_3} f(t,u,v,w), \quad \forall c \ge N_3^{-1}.
\end{aligned} \tag{2.18}$$

For every $u \in P \setminus \{\theta\}$, $t \in [0,1]$, choose positive numbers $c_1 \le \min\{N_1, (1/8)N_1\|u\|_2\}$, $c_2 \le \min\{N_2, (1/4)N_2\|u\|_2\}$, $c_3 \ge \max\{N_3^{-1}, N_3^{-1}\|u\|_2\}$. It follows from (H), (2.10), Lemma 2.1, and (2.17) that

$$\begin{aligned}
(T_\lambda u)(t) &= \lambda \int_0^1 G(t,s) f(s, u(s), u'(s), u''(s))\,ds\\
&\le \frac{1}{2}\lambda \int_0^1 s f\left(s, c_1 \frac{u(s)}{c_1(s-s^2/2)}\left(s - \frac{s^2}{2}\right), c_2 \frac{u'(s)}{c_2(1-s)}(1-s), (-1)c_3 \frac{u''(s)}{-c_3}\right) ds\\
&\le \frac{1}{2}\lambda \int_0^1 s c_1^{\alpha_1}\left(\frac{u(s)}{c_1(s-s^2/2)}\right)^{\beta_1} c_2^{\alpha_2}\left(\frac{u'(s)}{c_2(1-s)}\right)^{\beta_2} c_3^{\beta_3}\left(\frac{u''(s)}{-c_3}\right)^{\alpha_3} f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds\\
&\le \frac{1}{2}\lambda \int_0^1 s c_1^{\alpha_1}\left(\frac{\|u\|_2}{c_1}\right)^{\beta_1} c_2^{\alpha_2}\left(\frac{\|u\|_2}{c_2}\right)^{\beta_2} c_3^{\beta_3}\left(\frac{\|u\|_2}{c_3}\right)^{\alpha_3} f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds\\
&\le \frac{1}{2} c_1^{\alpha_1-\beta_1} c_2^{\alpha_2-\beta_2} c_3^{\beta_3-\alpha_3} \|u\|_2^{\beta_1+\beta_2+\alpha_3} \lambda \int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds < +\infty.
\end{aligned} \tag{2.19}$$

Similar to (2.19), from (H), (2.10), Lemma 2.1, and (2.17), for every $u \in P \setminus \{\theta\}$, $t \in [0,1]$, we have

$$\begin{aligned}
(T_\lambda u)'(t) &= \lambda \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s))\,ds\\
&\le \lambda \int_0^1 s f\left(s, c_1 \frac{u(s)}{c_1(s-s^2/2)}\left(s - \frac{s^2}{2}\right), c_2 \frac{u'(s)}{c_2(1-s)}(1-s), (-1)c_3 \frac{u''(s)}{-c_3}\right) ds\\
&\le c_1^{\alpha_1-\beta_1} c_2^{\alpha_2-\beta_2} c_3^{\beta_3-\alpha_3} \|u\|_2^{\beta_1+\beta_2+\alpha_3} \lambda \int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds < +\infty.
\end{aligned}$$

$$
\begin{aligned}
-(T_\lambda u)''(t) &= \lambda \int_0^1 G_2(t,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\le \lambda \int_0^1 s f\left(s, c_1 \frac{u(s)}{c_1(s-s^2/2)}\left(s-\frac{s^2}{2}\right), c_2 \frac{u'(s)}{c_2(1-s)}(1-s), (-1)c_3 \frac{u''(s)}{-c_3}\right)ds \\
&\le c_1^{\alpha_1-\beta_1} c_2^{\alpha_2-\beta_2} c_3^{\beta_3-\alpha_3} \|u\|_2^{\beta_1+\beta_2+\alpha_3} \lambda \int_0^1 s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds < +\infty.
\end{aligned}
\tag{2.20}
$$

Thus, T_λ is well defined on $P \setminus \{\theta\}$.

From (2.4) and (2.14)–(2.16), it is easy to know that

$$
\begin{gathered}
(T_\lambda u)(0) = 0, \qquad (T_\lambda u)'(1) = 0, \qquad (T_\lambda u)''(0) = 0, \\
\begin{aligned}
(T_\lambda u)(t) &= \lambda \int_0^1 G(t,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\ge \left(t-\frac{t^2}{2}\right)\lambda \int_0^1 \frac{1}{2} s f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\ge \left(t-\frac{t^2}{2}\right)\lambda \int_0^1 \max_{\tau\in[0,1]} G(\tau,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&= \left(t-\frac{t^2}{2}\right)\|T_\lambda u\|, \quad \forall t \in [0,1],\ u \in P \setminus \{\theta\}, \\
(T_\lambda u)'(t) &= \lambda \int_0^1 G_1(t,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\ge \frac{1}{2}(1-t)\lambda \int_0^1 s f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\ge \frac{1}{2}(1-t)\lambda \int_0^1 \max_{\tau\in[0,1]} G_1(\tau,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&= \frac{1}{2}(1-t)\big\|(T_\lambda u)'\big\|, \quad \forall t \in [0,1],\ u \in P \setminus \{\theta\}, \\
-(T_\lambda u)''(t) &= \lambda \int_0^1 G_2(t,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\ge t\lambda \int_0^1 s f\big(s,u(s),u'(s),u''(s)\big)ds \\
&\ge t\lambda \int_0^1 \max_{\tau\in[0,1]} G_2(\tau,s) f\big(s,u(s),u'(s),u''(s)\big)ds \\
&= t\big\|(T_\lambda u)''\big\|, \quad \forall t \in [0,1],\ u \in P \setminus \{\theta\}.
\end{aligned}
\end{gathered}
\tag{2.21}
$$

Therefore, $T(P \setminus \{\theta\}) \subset P$ follows from (2.21). □

Obviously, u^* is a positive solution of BVP (1.1) if and only if u^* is a positive fixed point of the integral operator T_λ in P.

Lemma 2.4. *Suppose that (H) and* (2.17) *hold. Then for any $R > r > 0$, $T_\lambda : \overline{P_R} \setminus P_r \to P$ is completely continuous.*

Proof. First of all, notice that T_λ maps $\overline{P_R} \setminus P_r$ into P by Lemma 2.3.

Next, we show that T_λ is bounded. In fact, for any $u \in \overline{P_R} \setminus P_r$, by (2.10) we can get

$$\frac{r}{8}\left(t-\frac{t^2}{2}\right) \le u(t) \le \left(t-\frac{t^2}{2}\right)R, \quad \frac{r}{4}(1-t) \le u'(t) \le (1-t)R, \quad rt \le -u''(t) \le R, \quad \forall t \in [0,1]. \tag{2.22}$$

Choose positive numbers $c_1 \le \min\{N_1, (r/8)N_1\}$, $c_2 \le \min\{N_2, (r/4)N_2\}$, $c_3 \ge \max\{N_3^{-1}, N_3^{-1}R\}$. This, together with (H), (2.22), (2.16), and Lemma 2.1 yields that

$$\begin{aligned}
\left|(T_\lambda u)''(t)\right| &= \lambda \int_0^1 G_2(t,s) f(s,u(s),u'(s),u''(s))ds \\
&\le \lambda \int_0^1 s f\left(s, c_1 \frac{u(s)}{c_1(s-s^2/2)}\left(s-\frac{s^2}{2}\right), c_2\frac{u'(s)}{c_2(1-s)}(1-s), (-1)c_3\frac{u''(s)}{-c_3}\right)ds \\
&\le \lambda \int_0^1 s c_1^{\alpha_1}\left(\frac{u(s)}{c_1(s-s^2/2)}\right)^{\beta_1} c_2^{\alpha_2}\left(\frac{u'(s)}{c_2(1-s)}\right)^{\beta_2} c_3^{\beta_3}\left(\frac{u''(s)}{-c_3}\right)^{\alpha_3} f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&\le c_1^{\alpha_1-\beta_1} c_2^{\alpha_2-\beta_2} c_3^{\beta_3-\alpha_3} R^{\beta_1+\beta_2+\alpha_3} \lambda \int_0^1 s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&< +\infty, \quad \forall t \in [0,1],\ u \in \overline{P_R} \setminus P_r.
\end{aligned} \tag{2.23}$$

Thus, T_λ is bounded on $\overline{P_R} \setminus P_r$.

Now we show that T_λ is a compact operator on $\overline{P_R} \setminus P_r$. By (2.23) and Ascoli-Arzela theorem, it suffices to show that $T_\lambda V$ is equicontinuous for arbitrary bounded subset $V \subset \overline{P_R} \setminus P_r$.

Since for each $u \in V$, (2.22) holds, we may choose still positive numbers $c_1 \le \min\{N_1, (r/8)N_1\}$, $c_2 \le \min\{N_2, (r/4)N_2\}$, $c_3 \ge \max\{N_3^{-1}, N_3^{-1}R\}$. Then

$$\begin{aligned}
\left|(T_\lambda u)'''(t)\right| &= \lambda \int_t^1 f(s,u(s),u'(s),u''(s))ds \\
&\le C_0 \int_t^1 f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&=: H(t), \quad t \in (0,1),
\end{aligned} \tag{2.24}$$

where $C_0 = \lambda c_1^{\alpha_1-\beta_1} c_2^{\alpha_2-\beta_2} c_3^{\beta_3-\alpha_3} R^{\beta_1+\beta_2+\alpha_3}$. Notice that

$$\begin{aligned}\int_0^1 H(t)dt &= C_0 \int_0^1 \int_t^1 f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds\, dt \\ &= C_0 \int_0^1 \int_0^s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) dt\, ds \\ &= C_0 \int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds < +\infty.\end{aligned} \tag{2.25}$$

Thus for any given $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$ and for any $u \in V$, we get

$$\left|(T_\lambda u)''(t_2) - (T_\lambda u)''(t_1)\right| \le \int_{t_1}^{t_2} \left|(T_\lambda u)'''(t)\right| dt \le \int_{t_1}^{t_2} H(t)dt. \tag{2.26}$$

From (2.25), (2.26), and the absolute continuity of integral function, it follows that $T_\lambda V$ is equicontinuous.

Therefore, $T_\lambda V$ is relatively compact, that is, T_λ is a compact operator on $\overline{P_R} \setminus P_r$.

Finally, we show that T_λ is continuous on $\overline{P_R} \setminus P_r$. Suppose $u_n,\ u \in \overline{P_R} \setminus P_r$, $n = 1, 2, \ldots$ and $\|u_n - u\|_2 \to 0$, $(n \to +\infty)$. Then $u_n''(t) \to u''(t)$, $u_n'(t) \to u'(t)$ and $u_n(t) \to u(t)$ as $n \to +\infty$ uniformly, with respect to $t \in [0,1]$. From (H), choose still positive numbers $c_1 \le \min\{N_1, (r/8)N_1\}$, $c_2 \le \min\{N_2, (r/4)N_2\}$, $c_3 \ge \max\{N_3^{-1}, N_3^{-1}R\}$. Then

$$\begin{gathered}0 \le f(t, u_n(t), u_n'(t), u_n''(t)) \le C_0 f\left(t, t - \frac{t^2}{2}, 1-t, -1\right), \quad t \in (0,1), \\ 0 \le G_2(t,s) f(s, u_n(s), u_n'(s), u_n''(s)) \le C_0 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right), \quad t \in [0,1],\ s \in (0,1).\end{gathered} \tag{2.27}$$

By (2.17), we know that $sf(s, s - s^2/2, 1-s, -1)$ is integrable on $[0,1]$. Thus, from the *Lebesgue* dominated convergence theorem, it follows that

$$\begin{aligned}&\lim_{n\to+\infty} \|(T_\lambda u_n) - (T_\lambda u)\|_2 = \lim_{n\to+\infty} \left\|(\mathrm{T}_\lambda u_n)'' - (T_\lambda u)''\right\| \\ &\quad\le \lim_{n\to+\infty} \lambda \int_0^1 s\left|f(s, u_n(s), u_n'(s), u_n''(s)) - f(s, u(s), u'(s), u''(s))\right| ds \\ &\quad= \lambda \int_0^1 s\left|\lim_{n\to+\infty} \left(f(s, u_n(s), u_n'(s), u_n''(s)) - f(s, u(s), u'(s), u_n''(s))\right)\right| ds \\ &\quad= 0.\end{aligned} \tag{2.28}$$

Thus, T_λ is continuous on $\overline{P_R} \setminus P_r$. Therefore, $T_\lambda : \overline{P_R} \setminus P_r \to P$ is completely continuous. □

3. A Necessary and Sufficient Condition for Existence of Positive Solutions

In this section, by using the fixed point theorem of cone, we establish the following necessary and sufficient condition for the existence of positive solutions for BVP (1.1).

Theorem 3.1. *Suppose* (H) *holds, then BVP* (1.1) *has at least one positive solution for any* $\lambda > 0$ *if and only if the integral inequality* (2.17) *holds.*

Proof. Suppose first that $u(t)$ be a positive solution of BVP (1.1) for any fixed $\lambda > 0$. Then there exist constants I_i $(i = 1,2,3,4)$ with $0 < I_i < 1 < I_{i+1}$, $i = 1,3$ such that

$$I_1\left(t-\frac{t^2}{2}\right) \le u(t) \le I_2\left(t-\frac{t^2}{2}\right), \quad I_3(1-t) \le u'(t) \le I_4(1-t), \quad t \in [0,1]. \tag{3.1}$$

In fact, it follows from $u^{(4)}(t) \ge 0$, $t \in (0,1)$ and $u(0) = u'(1) = u''(0) = u'''(1) = 0$, that $u'''(t) \le 0$ for $t \in (0,1]$ and $u''(t) \le 0$, $u'(t) \ge 0$ for $t \in [0,1]$. By the concavity of $u(t)$ and $u'(t)$, we have

$$\begin{gathered} u(t) \ge tu(1) + (1-t)u(0) = t\|u\| \ge \left(t-\frac{t^2}{2}\right)\|u\|, \\ u'(t) \ge tu'(1) + (1-t)u'(0) = (1-t)\left\|u'\right\|, \quad \forall t \in [0,1]. \end{gathered} \tag{3.2}$$

On the other hand,

$$\begin{aligned} u(t) &= \int_0^1 G_0(t,s)\left(-u''(s)\right)ds = \int_0^t s\left(-u''(s)\right)ds + \int_t^1 t\left(-u''(s)\right)ds \\ &\le \frac{t^2}{2}\left\|u''\right\| + t(1-t)\left\|u''\right\| = \left(t-\frac{t^2}{2}\right)\left\|u''\right\|, \\ u'(t) &= \int_t^1 -u''(s)ds \le (1-t)\left\|u''\right\|, \quad \forall t \in [0,1]. \end{aligned} \tag{3.3}$$

Let $I_1 = \min\{\|u\|, 1/2\}$, let $I_2 = I_4 = \max\{\|u''\|, 2\}$, and let $I_3 = \min\{\|u'\|, 1/2\}$, then (3.1) holds.

Choose positive numbers $c_1 \leq N_1 I_2^{-1}$, $c_2 \leq N_2 I_4^{-1}$, $c_3 \geq \max\{N_3^{-1}, N_3^{-1}\|u\|_2\}$. This, together with (H), (1.2), and (2.18) yields that

$$\begin{aligned}
f\left(t, t-\frac{t^2}{2}, 1-t, -1\right) &= f\left(t, c_1\frac{t-t^2/2}{c_1 u(t)}u(t), c_2\frac{1-t}{c_2 u'(t)}u'(t), \frac{1}{c_3}\frac{c_3}{-u''(t)}u''(t)\right)\\
&\leq c_1^{\alpha_1}\left(\frac{t-t^2/2}{c_1 u(t)}\right)^{\beta_1} c_2^{\alpha_2}\left(\frac{1-t}{c_2 u'(t)}\right)^{\beta_2}\left(\frac{1}{c_3}\right)^{\alpha_3}\left(\frac{c_3}{-u''(t)}\right)^{\beta_3} f\left(t, u(t), u'(t), u''(t)\right)\\
&\leq c_1^{\alpha_1}\left(\frac{1}{c_1 I_1}\right)^{\beta_1} c_2^{\alpha_2}\left(\frac{1}{c_2 I_3}\right)^{\beta_2}\left(\frac{1}{c_3}\right)^{\alpha_3}\left(-\frac{c_3}{u''(t)}\right)^{\beta_3} f\left(t, u(t), u'(t), u''(t)\right)\\
&= C^*\left(-u''(t)\right)^{-\beta_3} f\left(t, u(t), u'(t), u''(t)\right), \quad t \in (0,1),
\end{aligned} \tag{3.4}$$

where $C^* = c_1^{\alpha_1-\beta_1} c_2^{\alpha_2-\beta_2} c_3^{\beta_3-\alpha_3} I_1^{-\beta_1} I_3^{-\beta_2}$. Hence, integrating (3.4) from t to 1, we obtain

$$\lambda\int_t^1 \left(-u''(s)\right)^{\beta_3} f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \leq C^*\left(-u'''(t)\right), \quad t \in (0,1). \tag{3.5}$$

Since $-u''(t)$ increases on $[0,1]$, we get

$$\left(-u''(t)\right)^{\beta_3}\lambda\int_t^1 f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \leq C^*\left(-u'''(t)\right), \quad t \in (0,1), \tag{3.6}$$

that is,

$$\lambda\int_t^1 f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \leq C^*\frac{-u'''(t)}{\left(-u''(t)\right)^{\beta_3}}, \quad t \in (0,1). \tag{3.7}$$

Notice that $\beta_3 < 1$, integrating (3.7) from 0 to 1, we have

$$\lambda\int_0^1\int_t^1 f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds\,dt \leq C^*\left(1-\beta_3\right)^{-1}\left(-u''(1)\right)^{1-\beta_3}. \tag{3.8}$$

That is,

$$\lambda\int_0^1\int_0^s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)dt\,ds \leq C^*\left(1-\beta_3\right)^{-1}\left(-u''(1)\right)^{1-\beta_3}. \tag{3.9}$$

Thus,

$$\int_0^1 s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds < +\infty. \tag{3.10}$$

By an argument similar to the one used in deriving (3.5), we can obtain

$$\lambda \int_t^1 (-u''(s))^{\alpha_3} f\left(s, s - \frac{s^2}{2}, 1 - s, -1\right) ds \geq C_* (-u'''(t)), \quad t \in (0,1), \tag{3.11}$$

where $C_* = c_1^{\beta_1 - \alpha_1} c_2^{\beta_2 - \alpha_2} c_3^{\alpha_3 - \beta_3} I_2^{-\alpha_1} I_4^{-\alpha_2}$. So,

$$\lambda \int_t^1 f\left(s, s - \frac{s^2}{2}, 1 - s, -1\right) ds \geq C_* \|u\|_2^{-\alpha_3} (-u'''(t)), \quad t \in (0,1). \tag{3.12}$$

Integrating (3.12) from 0 to 1, we have

$$\lambda \int_0^1 \int_t^1 f\left(s, s - \frac{s^2}{2}, 1 - s, -1\right) ds\, dt \geq C_* \|u\|_2^{-\alpha_3} (-u''(1)). \tag{3.13}$$

That is,

$$\lambda \int_0^1 \int_0^s f\left(s, s - \frac{s^2}{2}, 1 - s, -1\right) dt\, ds \geq C_* \|u\|_2^{-\alpha_3} (-u''(1)). \tag{3.14}$$

So,

$$\int_0^1 s f\left(s, s - \frac{s^2}{2}, 1 - s, -1\right) ds > 0. \tag{3.15}$$

This and (3.10) imply that (2.17) holds.

Now assume that (2.17) holds, we will show that BVP (1.1) has at least one positive solution for any $\lambda > 0$. By (2.17), there exists a sufficient small $\delta > 0$ such that

$$\int_\delta^{1-\delta} s f\left(s, s - \frac{s^2}{2}, 1 - s, -1\right) ds > 0. \tag{3.16}$$

For any fixed $\lambda > 0$, first of all, we prove

$$\|T_\lambda u\|_2 \geq \|u\|_2, \quad \forall u \in \partial P_r, \tag{3.17}$$

where $0 < r \leq \min\{N_1, N_2, N_3, (\lambda \delta^{1+\beta_3} 2^{-3(\beta_1+\beta_2)} \int_\delta^{1-\delta} s f(s, s - s^2/2, 1 - s, -1) ds)^{1/(1-(\beta_1+\beta_2+\beta_3))}\}$.

Let $u \in \partial P_r$, then

$$\begin{gathered} \frac{r}{8}\left(t - \frac{t^2}{2}\right) \leq u(t) \leq r\left(t - \frac{t^2}{2}\right) \leq N_1\left(t - \frac{t^2}{2}\right), \qquad \frac{r}{4}(1-t) \leq u'(t) \leq r(1-t) \leq N_2(1-t), \\ \delta r \leq rt \leq -u''(t) \leq r \leq N_3, \quad \forall t \in [\delta, 1-\delta]. \end{gathered} \tag{3.18}$$

From Lemma 2.1, (3.18), and (H), we get

$$\begin{aligned}
\|T_\lambda u\|_2 = \left\|(T_\lambda u)''\right\| &\geq \lambda \max_{t\in[\delta,1-\delta]} \int_0^1 G_2(t,s) f\left(s,u(s),u'(s),u''(s)\right)ds \\
&\geq \delta\lambda \int_\delta^{1-\delta} s f\left(s, \frac{u(s)}{s-s^2/2}\left(s-\frac{s^2}{2}\right), \frac{u'(s)}{1-s}(1-s), (-1)\left(-u''(s)\right)\right)ds \\
&\geq \delta\lambda \int_\delta^{1-\delta} s\left(\frac{u(s)}{s-s^2/2}\right)^{\beta_1}\left(\frac{u'(s)}{1-s}\right)^{\beta_2}\left(-u''(s)\right)^{\beta_3} f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&\geq \delta\left(\frac{r}{8}\right)^{\beta_1}\left(\frac{r}{4}\right)^{\beta_2}(\delta r)^{\beta_3}\lambda \int_\delta^{1-\delta} s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&\geq \delta^{1+\beta_3} 2^{-3(\beta_1+\beta_2)} r^{\beta_1+\beta_2+\beta_3}\lambda \int_\delta^{1-\delta} s f\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&\geq r = \|u\|_2, \quad u \in \partial P_r.
\end{aligned} \tag{3.19}$$

Thus, (3.17) holds.

Next, we claim that

$$\|T_\lambda u\|_2 \leq \|u\|_2, \quad \forall u \in \partial P_R, \tag{3.20}$$

where $R \geq \max\{8N_1^{-1}, 4N_2^{-1}, (\lambda N_3^{\alpha_3-\beta_3}\int_0^1 s f(s, s-s^2/2, 1-s, -1)ds)^{1/(1-(\beta_1+\beta_2+\beta_3))}\}$.

Let $c = N_3/R$, then for $u \in \partial P_R$, we get

$$\begin{gathered}
N_1^{-1}\left(t-\frac{t^2}{2}\right) \leq \frac{R}{8}\left(t-\frac{t^2}{2}\right) \leq u(t) \leq R\left(t-\frac{t^2}{2}\right), \qquad N_2^{-1}(1-t) \leq \frac{R}{4}(1-t) \leq u'(t) \leq R(1-t), \\
-cu''(t) \leq c\|u\|_2 = cR = N_3, \quad \forall t \in [0,1].
\end{gathered} \tag{3.21}$$

Therefore, by Lemma 2.1 and (H), it follows that

$$\begin{aligned}
\left|(T_\lambda u)''(t)\right| &= \lambda \int_0^1 G_2(t,s) f\left(s,u(s),u'(s),u''(s)\right)ds \\
&\leq \lambda \int_0^1 s f\left(s, \frac{u(s)}{s-s^2/2}\left(s-\frac{s^2}{2}\right), \frac{u'(s)}{1-s}(1-s), (-1)\left(\frac{1}{c}\right)\left(-cu''(s)\right)\right)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \int_0^1 \left(\frac{u(s)}{s-s^2/2}\right)^{\beta_1} \left(\frac{u'(s)}{1-s}\right)^{\beta_2} \left(\frac{1}{c}\right)^{\beta_3} (-cu''(s))^{\alpha_3} f\left(s, s-\frac{s^2}{2}, 1-s, -1\right) ds \\
&\leq R^{\beta_1+\beta_2} \left(\frac{N_3}{R}\right)^{\alpha_3-\beta_3} R^{\alpha_3} \lambda \int_0^1 sf\left(s, s-\frac{s^2}{2}, 1-s, -1\right) ds \\
&= R^{\beta_1+\beta_2+\beta_3} (N_3)^{\alpha_3-\beta_3} \lambda \int_0^1 sf\left(s, s-\frac{s^2}{2}, 1-s, -1\right) ds \\
&\leq R = \|u\|_2, \quad u \in \partial P_R.
\end{aligned} \tag{3.22}$$

This implies that (3.20) holds.

By Lemmas 1.1 and 2.4, (3.17), and (3.20), we obtain that T_λ has a fixed point in $\overline{P_R} \setminus P_r$. Therefore, BVP (1.1) has a positive solution in $\overline{P_R} \setminus P_r$ for any $\lambda > 0$. □

4. Unbounded Connected Branch of Positive Solutions

In this section, we study the global continua results under the hypotheses (H) and (2.17). Let

$$L = \overline{\{(\lambda, u) \in (0, +\infty) \times (P \setminus \{\theta\}) : (\lambda, u) \quad \text{satisfies BVP (1.1)}\}}, \tag{4.1}$$

then, by Theorem 3.1, $L \cap (\{\lambda\} \times P) \neq \emptyset$ for any $\lambda > 0$.

Theorem 4.1. *Suppose (H) and* (2.17) *hold, then the closure L of positive solution set possesses an unbounded connected branch C which comes from $(0, \theta)$ such that*

(i) *for any $\lambda > 0$, $C \cap (\{\lambda\} \times P) \neq \emptyset$, and*

(ii) $\lim_{(\lambda, u_\lambda) \in C, \lambda \to 0^+} \|u_\lambda\|_2 = 0$, $\lim_{(\lambda, u_\lambda) \in C, \lambda \to +\infty} \|u_\lambda\|_2 = +\infty$.

Proof. We now prove our conclusion by the following several steps.

First, we prove that for arbitrarily given $0 < \lambda_1 < \lambda_2 < +\infty$, $L \cap ([\lambda_1, \lambda_2] \times P)$ is bounded. In fact, let

$$R = 2 \max \left\{ 8N_1^{-1}, 4N_2^{-1}, \left(\lambda_2 N_3^{\alpha_3-\beta_3} \int_0^1 sf\left(s, s-\frac{s^2}{2}, 1-s, -1\right) ds\right)^{1/(1-(\beta_1+\beta_2+\beta_3))} \right\}, \tag{4.2}$$

then for $u \in P \setminus \{\theta\}$ and $\|u\|_2 \geq R$, we get

$$\begin{gathered}
N_1^{-1}\left(t - \frac{t^2}{2}\right) \leq \frac{R}{8}\left(t - \frac{t^2}{2}\right) \leq u(t) \leq \left(t - \frac{t^2}{2}\right)\|u\|_2, \\
N_2^{-1}(1-t) \leq \frac{R}{4}(1-t) \leq u'(t) \leq (1-t)\|u\|_2, \quad \forall t \in [0,1].
\end{gathered} \tag{4.3}$$

Therefore, by Lemma 2.1 and (H), it follows that

$$\begin{aligned}
\|T_\lambda u\|_2 &\le \|T_{\lambda_2} u\|_2 \le \lambda_2 \int_0^1 s f(s, u(s), u'(s), u''(s)) ds \\
&\le \lambda_2 \int_0^1 s f\left(s, \frac{u(s)}{s - s^2/2}\left(s - \frac{s^2}{2}\right), \ \frac{u'(s)}{1-s}(1-s), (-1)\frac{\|u\|_2}{N_3}\frac{N_3}{\|u\|_2}(-u''(s))\right) ds \\
&\le \lambda_2 \|u\|_2^{\beta_1+\beta_2}\left(\frac{N_3}{\|u\|_2}\right)^{\alpha_3-\beta_3}\|u\|_2^{\alpha_3}\int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds \\
&= \lambda_2 \|u\|_2^{\beta_1+\beta_2+\beta_3}(N_3)^{\alpha_3-\beta_3}\int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds \\
&< \|u\|_2, \quad \forall \lambda \in [\lambda_1, \lambda_2].
\end{aligned} \tag{4.4}$$

Let

$$r = \frac{1}{2}\min\left\{N_1, N_2, N_3, \left(\lambda_1 \delta^{1+\beta_3} 2^{-3(\beta_1+\beta_2)}\int_\delta^{1-\delta} s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds\right)^{1/(1-(\beta_1+\beta_2+\beta_3))}\right\}, \tag{4.5}$$

where δ is given by (3.16). Then for $u \in P \setminus \{\theta\}$ and $\|u\|_2 \le r$, we get

$$\begin{gathered}
\frac{\|u\|_2}{8}\left(t - \frac{t^2}{2}\right) \le u(t) \le r\left(t - \frac{t^2}{2}\right) \le N_1\left(t - \frac{t^2}{2}\right); \qquad \frac{\|u\|_2}{4}(1-t) \le u'(t) \le r(1-t) \le N_2(1-t), \\
\delta\|u\|_2 \le t\|u\|_2 \le -u''(t) \le r \le N_3, \quad \forall t \in [\delta, 1-\delta].
\end{gathered} \tag{4.6}$$

Therefore, by Lemma 2.1 and (H), it follows that

$$\begin{aligned}
\|T_\lambda u\| &\ge \|T_{\lambda_1} u\| \ge \lambda_1 \max_{t\in[\delta,1-\delta]} \int_0^1 G_2(t,s) f(s, u(s), u'(s), u''(s)) ds \\
&\ge \delta\lambda_1 \int_\delta^{1-\delta} s f\left(s, \frac{u(s)}{s - s^2/2}\left(s - \frac{s^2}{2}\right), \ \frac{u'(s)}{1-s}(1-s), (-1)(-u''(s))\right) ds \\
&\ge \delta\lambda_1 \int_\delta^{1-\delta} s\left(\frac{u(s)}{s - s^2/2}\right)^{\beta_1}\left(\frac{u'(s)}{1-s}\right)^{\beta_2}(-u''(s))^{\beta_3} f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds
\end{aligned}$$

$$\begin{aligned}
&\geq \delta\left(\frac{\|u\|_2)}{8}\right)^{\beta_1}\left(\frac{\|u\|_2}{4}\right)^{\beta_2}(\delta\|u\|_2)^{\beta_3}\lambda_1\int_\delta^{1-\delta} sf\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&\geq \delta^{1+\beta_3}2^{-3(\beta_1+\beta_2)}\|u\|_2^{\beta_1+\beta_2+\beta_3}\lambda_1\int_\delta^{1-\delta} sf\left(s, s-\frac{s^2}{2}, 1-s, -1\right)ds \\
&> \|u\|_2, \quad u \in \partial P_r.
\end{aligned} \tag{4.7}$$

Therefore, $u = T_\lambda u$ has no positive solution in $([\lambda_1, \lambda_2] \times (P \setminus P_R)) \cup ([\lambda_1, \lambda_2] \times \overline{P_r})$. As a consequence, $L \cap ([\lambda_1, \lambda_2] \times P)$ is bounded.

By the complete continuity of T_λ, $L \cap ([\lambda_1, \lambda_2] \times P)$ is compact.

Second, we choose sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ satisfy

$$\begin{gathered}
0 < \cdots < a_n < \cdots < a_1 < b_1 < \cdots < b_n < \cdots, \\
\lim_{n\to+\infty} a_n = 0, \qquad \lim_{n\to+\infty} b_n = +\infty.
\end{gathered} \tag{4.8}$$

We are to prove that for any positive integer n, there exists a connected branch C_n of L satisfying

$$C_n \cap (\{a_n\} \times P) \neq \emptyset, \qquad C_n \cap (\{b_n\} \times P) \neq \emptyset. \tag{4.9}$$

Let n be fixed, suppose that for any $(b_n, u) \in L \cap (\{b_n\} \times P)$, the connected branch C_u of $L \cap ([a_n, b_n] \times P)$, passing through (b_n, u), leads to $C_u \cap (\{a_n\} \times P) = \emptyset$. Since C_u is compact, there exists a bounded open subset Ω_1 of $[a_n, b_n] \times P$ such that $C_u \subset \Omega_1$, $\overline{\Omega_1} \cap (\{a_n\} \times P) = \emptyset$, and $\overline{\Omega_1} \cap ([a_n, b_n] \times \{\theta\}) = \emptyset$, where $\overline{\Omega_1}$ and later $\partial\Omega_1$ denote the closure and boundary of Ω_1 with respect to $[a_n, b_n] \times P$. If $L \cap \partial\Omega_1 \neq \emptyset$, then C_{u} and $L \cap \partial\Omega_1$ are two disjoint closed subsets of $L \cap \overline{\Omega_1}$. Since $L \cap \overline{\Omega_1}$ is a compact metric space, there are two disjoint compact subsets M_1 and M_2 of $L \cap \overline{\Omega_1}$ such that $L \cap \overline{\Omega_1} = M_1 \cup M_2$, $C_u \subset M_1$, and $L \cap \partial\Omega_1 \subset M_2$. Evidently, $\gamma =: \mathrm{dist}(M_1,\ M_2) > 0$. Denoting by V the $\gamma/3$-neighborhood of M_1 and letting $\Omega_u = \Omega_1 \cap V$, then it follows that

$$C_u \subset \Omega_u, \qquad \overline{\Omega_u} \cap [(\{a_n\} \times P) \cup ([a_n,\ b_n] \times \{\theta\})] = \emptyset, \qquad L \cap \partial\Omega_u = \emptyset. \tag{4.10}$$

If $L \cap \partial\Omega_1 = \emptyset$, then taking $\Omega_u = \Omega_1$.

It is obvious that in $\{b_n\} \times P$, the family of $\{\Omega_u \cap (\{b_n\} \times P) : (b_n, u) \in L\}$ makes up an open covering of $L \cap (\{b_n\} \times P)$. Since $L \cap (\{b_n\} \times P)$ is a compact set, there exists a finite subfamily $\{\Omega_{u_i} \cap (\{b_n\} \times P) : (b_n, u_i) \in L\}_{i=1}^k$ which also covers $L \cap (\{b_n\} \times P)$. Let $\Omega = \bigcup_{i=1}^k \Omega_{u_i}$, then

$$L \cap (\{b_n\} \times P) \subset \Omega, \qquad \overline{\Omega} \cap [(\{a_n\} \times P) \cup ([a_n, b_n] \times \{\theta\})] = \emptyset, \qquad L \cap \partial\Omega = \emptyset. \tag{4.11}$$

Hence, by the homotopy invariance of the fixed point index, we obtain

$$i(T_{b_n}, \Omega \cap (\{b_n\} \times P), P) = i(T_{a_n}, \Omega \cap (\{a_n\} \times P), P) = 0. \tag{4.12}$$

By the first step of this proof, the construction of Ω, (4.4), and (4.7), it follows easily that there exist $0 < r_n < R_n$ such that

$$\left(\overline{\Omega} \cap (\{b_n\} \times P)\right) \cap (\{b_n\} \times P_{r_n}) = \emptyset, \qquad \left(\overline{\Omega} \cap (\{b_n\} \times P)\right) \subset (\{b_n\} \times P_{R_n}), \tag{4.13}$$

$$i(T_{b_n}, P_{r_n}, P) = 0, \tag{4.14}$$

$$i(T_{b_n}, P_{R_n}, P) = 1. \tag{4.15}$$

However, by the excision property and additivity of the fixed point index, we have from (4.12) and (4.14) that $i(T_{b_n}, P_{R_n}, P) = 0$, which contradicts (4.15). Hence, there exists some $(b_n, u) \in L \cap (\{b_n\} \times P)$ such that the connected branch C_u of $L \cap ([a_n, b_n] \times P)$ containing (b_n, u) satisfies that $C_u \cap (\{a_n\} \times P) \neq \emptyset$. Let C_n be the connected branch of L including C_u, then this C_n satisfies (4.9).

By Lemma 1.2, there exists a connected branch C^* of $\limsup_{n \to +\infty} C_n$ such that $C^* \cap (\{\lambda\} \times P) \neq \emptyset$ for any $\lambda > 0$. Noticing $\limsup_{n \to +\infty} C_n \subset L$, we have $C^* \subset L$. Let C be the connected branch of L including C^*, then $C \cap (\{\lambda\} \times P) \neq \emptyset$ for any $\lambda > 0$. Similar to (4.4) and (4.7), for any $\lambda > 0$, $(\lambda, u_\lambda) \in C$, we have, by (H), (4.2), (4.3), (4.5), (4.6), and Lemma 2.1,

$$\begin{aligned}
\|u_\lambda\|_2 = \|T_\lambda u_\lambda\|_2 &\le \lambda \int_0^1 s f\left(s, u_\lambda(s), u_\lambda'(s), u_\lambda''(s)\right) ds \\
&\le \lambda \|u_\lambda\|_2^{\beta_1+\beta_2} \left(\frac{N_3}{\|u_\lambda\|_2}\right)^{\alpha_3-\beta_3} \|u_\lambda\|_2^{\alpha_3} \int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds \\
&= \lambda \|u_\lambda\|_2^{\beta_1+\beta_2+\beta_3} (N_3)^{\alpha_3-\beta_3} \int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds \\
&\le \lambda R^{\beta_1+\beta_2+\beta_3} (N_3)^{\alpha_3-\beta_3} \int_0^1 s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\|u_\lambda\|_2 = \|T_\lambda u_\lambda\|_2 &\ge \lambda \max_{t \in [\delta, 1-\delta]} \int_0^1 G_2(t,s) f\left(s, u_\lambda(s), u_\lambda'(s), u_\lambda''(s)\right) ds \\
&\ge \lambda \delta \left(\frac{\|u_\lambda\|_2}{8}\right)^{\beta_1} \left(\frac{\|u_\lambda\|_2}{4}\right)^{\beta_2} (\delta \|u_\lambda\|_2)^{\beta_3} \int_\delta^{1-\delta} s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds \\
&\ge \lambda \delta^{1+\beta_3} 2^{-3(\beta_1+\beta_2)} \|u_\lambda\|_2^{\beta_1+\beta_2+\beta_3} \int_\delta^{1-\delta} s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds \\
&\ge \lambda \delta^{1+\beta_3} 2^{-3(\beta_1+\beta_2)} r^{\beta_1+\beta_2+\beta_3} \int_\delta^{1-\delta} s f\left(s, s - \frac{s^2}{2}, 1-s, -1\right) ds,
\end{aligned} \tag{4.17}$$

where δ is given by (3.16). Let $\lambda \to 0^+$ in (4.16) and $\lambda \to +\infty$ in (4.17), we have

$$\lim_{(\lambda,u_\lambda) \in C, \lambda \to 0^+} \|u_\lambda\|_2 = 0, \qquad \lim_{(\lambda,u_\lambda) \in C, \lambda \to +\infty} \|u_\lambda\|_2 = +\infty. \tag{4.18}$$

Therefore, Theorem 4.1 holds and the proof is complete. □

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