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Research Article

Strong Laws of Large Numbers for Arrays of Rowwise ρ^*-Mixing Random Variables

Meng-Hu Zhu

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Some strong laws of large numbers for arrays of rowwise ρ^*-mixing random variables are obtained. The result obtainted not only generalizes the result of Hu and Taylor (1997) to ρ^*-mixing random variables, but also improves it.

1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables. The Marcinkiewicz-Zygmund strong law of large numbers (SLLN) provides that

$$\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n}(X_i - EX_i) \longrightarrow 0 \quad \text{a.s. for } 1 \leq \alpha < 2,$$
$$\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n}X_i \longrightarrow 0 \quad \text{a.s. for } 0 < \alpha < 1 \tag{1.1}$$

if and only if $E|X|^\alpha < \infty$. The case $\alpha = 1$ is due to Kolmogorov. In the case of independence (but not necessarily identically distributed), Hu and Taylor [1] proved the following strong law of large numbers.

Theorem 1.1. *Let $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\psi(t)$ be a positive, even function such that $\psi(|t|)/|t|^p$ is an increasing function of $|t|$ and $\psi(|t|)/|t|^{p+1}$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{\psi(|t|)}{|t|^p}\uparrow, \quad \frac{\psi(|t|)}{|t|^{p+1}}\downarrow, \quad \text{as } |t|\uparrow \tag{1.2}$$

for some nonnegative integer p. If $p \geq 2$ and

$$EX_{ni} = 0,$$
$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(|X_{ni}|)}{\psi(a_n)} < \infty, \tag{1.3}$$
$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n}\right)^2\right)^{2k} < \infty,$$

where k is a positive integer, then

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \quad a.s. \tag{1.4}$$

Let nonempty sets $S, T \subset \mathcal{N}$, and define $\mathcal{F}_S = \sigma(X_k,\ k \in S)$, and the maximal correlation coefficient $\rho_n^* = \sup \operatorname{corr}(f,g)$ where the supremum is taken over all (S,T) with dist $(S,T) \geq n$ and all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$, and where $\operatorname{dist}(S,T) = \inf_{x\in S,\, y\in T}|x-y|$.

A sequence of random variables $\{X_n,\ n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^*-mixing if

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.5}$$

An array of random variables $\{X_{ni}; i \geq 1,\ n \geq 1\}$ is called rowwise ρ^*-mixing random variables if for every $n \geq 1$, $\{X_{ni}; i \geq 1\}$ is a ρ^*-mixing sequence of random variables.

As for ρ^*-mixing sequences of random variables, Bryc and Smoleński [2] established the moments inequality of partial sums. Peligrad [3] obtained a CLT. Peligrad [4] established an invariance principle. Peligrad and Gut [5] established the Rosenthal-type maximal inequality. Utev and Peligrad [6] obtained an invariance principle of nonstationary sequences.

The main purpose of this paper is to establish a strong law of large numbers for arrays of rowwise ρ^*-mixing random variables. The result obtained not only generalizes the result of Hu and Taylor [1] to ρ^*-mixing random variables, but also improves it.

2. Main results

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$.

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers (see Hsu and Robbins [7], Erdös [8]) states that

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty \tag{2.1}$$

is equivalent to $EX = 0$, $EX^2 < \infty$.

This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades. One of the most important results is Baum-Katz [9] law of large numbers, which states that for $p < 2$ and $r \geq p$,

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| > \varepsilon n^{1/p}) < \infty \tag{2.2}$$

if and only if $E|X|^r < \infty$, $r \geq 1$, and $EX = 0$.

There are many extensions in various directions. Some of them can be found by Chow and Lai in [10, 11], where the authors propose a two-sided estimate, and by Petrov in [12].

In order to prove our main result, we need the following lemma.

Lemma 2.1 (see Utev and Peligrad [6]). *Let $\{X_i, i \geq 1\}$ be a ρ^*-mixing sequence of random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right|^p \leq C \left\{ \sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}. \tag{2.3}$$

Theorem 2.2. *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise ρ^*-mixing random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\psi(t)$ be a positive, even function such that $\psi(|t|)/|t|$ is an increasing function of $|t|$ and $\psi(|t|)/|t|^p$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{\psi(|t|)}{|t|} \uparrow, \quad \frac{\psi(|t|)}{|t|^p} \downarrow, \quad \text{as } |t| \uparrow \tag{2.4}$$

for some nonnegative integer p. If $p \geq 2$ and

$$\begin{gathered} EX_{ni} = 0, \\ \sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(|X_{ni}|)}{\psi(a_n)} < \infty, \\ \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E \left(\frac{X_{ni}}{a_n} \right)^2 \right)^{v/2} < \infty, \end{gathered} \tag{2.5}$$

where v is a positive integer, $v \geq p$, then

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} P\left(\max_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} X_{ni} \right| > \varepsilon \right) < \infty. \tag{2.6}$$

Proof of Theorem 2.2. For all $i \geq 1$, define $X_i^{(n)} = X_{ni}I(|X_{ni}| \leq a_n)$, $T_j^{(n)} = (1/a_n)\sum_{i=1}^{j}(X_i^{(n)} - EX_i^{(n)})$, then for all $\varepsilon > 0$,

$$P\left(\max_{1\leq k\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{k}X_{ni}\right| > \varepsilon\right) \leq P\left(\max_{1\leq j\leq n}|X_{nj}| > a_n\right) + P\left(\max_{1\leq j\leq n}|T_j^{(n)}| > \varepsilon - \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right|\right). \tag{2.7}$$

First, we show that

$$\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

In fact, by $EX_{ni} = 0$, $\psi(|t|)/|t| \uparrow$ as $|t| \uparrow$ and $\sum_{n=1}^{\infty}\sum_{i=1}^{n}E(\psi(|X_{ni}|)/\psi(a_n)) < \infty$, then

$$\begin{aligned}
\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\right| &= \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_{ni}I(|X_{ni}| \leq a_n)\right| \\
&= \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{j}EX_{ni}I(|X_{ni}| > a_n)\right| \\
&\leq \sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}| > a_n)}{a_n} \\
&\leq \sum_{i=1}^{n}\frac{E\psi(|X_{ni}|)I(|X_{ni}| > a_n)}{\psi(a_n)} \\
&\leq \sum_{i=1}^{n}\frac{E\psi(|X_{ni}|)}{\psi(a_n)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.9}$$

From (2.7) and (2.8), it follows that for n large enough,

$$P\left(\max_{1\leq k\leq n}\left|\frac{1}{a_n}\sum_{i=1}^{k}X_{ni}\right| > \varepsilon\right) \leq \sum_{j=1}^{n}P(|X_{nj}| > a_n) + P\left(\max_{1\leq j\leq n}|T_j^{(n)}| > \frac{\varepsilon}{2}\right). \tag{2.10}$$

Hence, we need only to prove that

$$\begin{aligned}
I &=: \sum_{n=1}^{\infty}\sum_{j=1}^{n}P(|X_{nj}| > a_n) < \infty, \\
II &=: \sum_{n=1}^{\infty}P\left(\max_{1\leq j\leq n}|T_j^{(n)}| > \frac{\varepsilon}{2}\right) < \infty.
\end{aligned} \tag{2.11}$$

From the fact that $\sum_{n=1}^{\infty}\sum_{i=1}^{n} E(\psi(|X_{ni}|)/\psi(a_n)) < \infty$, it follows easily that

$$I = \sum_{n=1}^{\infty}\sum_{j=1}^{n} P(|X_{nj}| > a_n) \leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} E\frac{\psi(|X_{nj}|)}{\psi(a_n)} < \infty. \tag{2.12}$$

By $\nu \geq p$ and $\psi(|t|)/|t|^p \downarrow$ as $|t| \uparrow$, then $\psi(|t|)/|t|^\nu \downarrow$ as $|t| \uparrow$.

By Markov inequality, Lemma 2.1, and $\sum_{n=1}^{\infty}(\sum_{i=1}^{n} E(X_{ni}/a_n)^2)^{\nu/2} < \infty$, we have

$$\begin{aligned}
II &= \sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n} |T_j^{(n)}| > \frac{\varepsilon}{2}\right) \\
&\leq \sum_{n=1}^{\infty}\left(\frac{\varepsilon}{2}\right)^{-\nu} E\max_{1\leq j\leq n} |T_j^{(n)}|^\nu \\
&\leq C\sum_{n=1}^{\infty}\left(\frac{\varepsilon}{2}\right)^{-\nu}\frac{1}{a_n^\nu}\left[\left(\sum_{j=1}^{n} E|X_j^{(n)}|^2\right)^{\nu/2} + \sum_{j=1}^{n} E|X_j^{(n)}|^\nu\right] \\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n^\nu}\sum_{j=1}^{n} E|X_j^{(n)}|^\nu + C\sum_{n=1}^{\infty}\frac{1}{a_n^\nu}\left(\sum_{j=1}^{n} E|X_j^{(n)}|^2\right)^{\nu/2} \\
&= C\sum_{n=1}^{\infty}\frac{1}{a_n^\nu}\sum_{j=1}^{n} E|X_{nj}|^\nu I(|X_{nj}| \leq a_n) + C\sum_{n=1}^{\infty}\frac{1}{a_n^\nu}\left(\sum_{j=1}^{n} E|X_j^{(n)}|^2\right)^{\nu/2} \\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(|X_{ni}|)}{\psi(a_n)} + C\sum_{n=1}^{\infty}\frac{1}{a_n^\nu}\left[\sum_{j=1}^{n} E|X_j^{(n)}|^2\right]^{\nu/2} \\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(|X_{ni}|)}{\psi(a_n)} + C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n}\right)^2\right)^{\nu/2} < \infty.
\end{aligned} \tag{2.13}$$

Now we complete the proof of Theorem 2.2. □

Corollary 2.3. *Under the conditions of Theorem 2.2, then*

$$\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0 \ a.s. \tag{2.14}$$

Proof of Corollary 2.3. By Theorem 2.2, the Proof of Corollary 2.3 is obvious. □

Remark 2.4. Corollary 2.3 not only generalizes the result of Hu and Taylor [1] to ρ^*-mixing random variables, but also improves it.

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Meng-Hu Zhu: Department of Mathematics and Statistics, Zhejiang Gongshang University, Hangzhou 310035, China
Email address: zmhzju@163.com