

Research Article

On a Periodic Time-Dependent Model of Population Dynamics with Stage Structure and Impulsive Effects

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Received 12 June 2007; Revised 5 February 2008; Accepted 20 March 2008

Recommended by Xue-Zhong He

We consider a periodic time-dependent predator-prey system with stage structure and impulsive harvesting, in which the prey has a life history that takes them through two stages, immature and mature. A set of sufficient and necessary conditions which guarantee the permanence of the system is obtained. Finally, we give a brief discussion of our results.

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1. Introduction

In the natural world, there are many species whose individual members have a life history that takes them through two stages, immature and mature. In particular, we have mammalian populations and some amphibious animals in mind, which exhibit these two stages. From the view point of mathematics, the description of the stage structure of the population in the life history is also an interesting problem in population dynamics. The permanence and extinction of species are significant concepts for those stage-structured population dynamical systems. Recently, stage structure models have been studied by many authors [1–3]. This is not only because they are much more simple than the models governed by partial differential equations but also because they can exhibit phenomena similar to those of partial differential models [4], and many important physiological parameters can be incorporated. Much research has been devoted to the models concerning single-species population growth with the stage structure of immature and mature [5, 6]. Two species models with stage structure were investigated by Wang and Chen (1997), Magnusson (1999), Xiao and Chen (2003), as well as Cui and Song (2004). Also, Zhang, Chen, and Neumann (2000) proposed the following autonomous stage

structure predator-prey system:

$$\begin{aligned}x_1' &= \alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \beta_1 x_1 x_3, \\x_2' &= \beta x_1 - r_2 x_2, \\x_3' &= x_3(-r + k\beta_1 x_1 - \eta_1 x_3),\end{aligned}\tag{1.1}$$

where $\alpha, \beta, \beta_1, \eta, \eta_1, r, r_1, r_2$, and k are all positive constants, k is a digesting constant.

On the other hand, since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. So more realistic and interesting models should take into account the seasonality of the changing environment [7, 8]. This motivated Cui and Y. Takeuchi (2006) to consider the following periodically nonautonomous predator-prey model with stage structure for prey:

$$\begin{aligned}\dot{x}_1 &= a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p(t)\phi(t, x_1)x_1 y, \\ \dot{x}_2 &= c(t)x_1 - f(t)x_2^2, \\ \dot{y} &= y[-g(t) + h(t)\phi(t, x_1)x_1 - q(t)y],\end{aligned}\tag{1.2}$$

in which $a(t), b(t), c(t), d(t), f(t), g(t), h(t), p(t), q(t)$, and $\phi(t, x_1)$ are all continuous ω -periodic functions; $a(t), b(t), c(t), d(t), f(t), h(t), p(t), q(t)$ are all positive; $g(t), \phi(t, x_1)$ are nonnegative; x_1 and x_2 denote the density of immature and mature population (prey), respectively; and y is the density of the predator that only prey on x_1 (immature prey). They provided a set of sufficient and necessary conditions to guarantee the permanence of the above system.

Systems with impulsive effects describing evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. Processes of such type are studied in almost every domain of applied science. Impulsive equations [9, 10] have been recently used in population dynamics in relation to impulsive vaccination [11, 12], population ecology [13, 14], the chemotherapeutic treatment of disease [15], birth pulses [16], as well as the theory of the chemostat [17].

Let us assume that the predator population is affected by harvesting (e.g., fishing or hunting). Further, as we all know that the harvesting does not occur continuously, that is, the harvesting occurs in regular pulses, then let us assume that at some fixed moments, the predator population in system (1.2) is subject to a perturbation which incorporates the proportional decrease. After a perturbation at step $\tau_k > 0$ ($k \in \mathbf{N}$), the size of the population $y(\tau_k^+)$ becomes

$$y(\tau_k^+) = (1 - u_k)y(\tau_k),\tag{1.3}$$

where $y(\tau_k)$ is the size of the predator population at step τ_k before the impulsive perturbation, and $0 < u_k < 1$ represents the rate at which the predator is harvested.

In this article, we extend the model (1.2) to the case when the predator population is omnivorous and affected by impulsive effects, which is governed by the following system:

$$\begin{aligned}\dot{x}_1 &= a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p_1(t)\phi_1(t, x_1)x_1^2 y, \quad t \neq \tau_k, \\ \dot{x}_2 &= c(t)x_1 - f(t)x_2^2 - p_2(t)\phi_2(t, x_2)x_2^2 y, \quad t \neq \tau_k,\end{aligned}$$

$$\begin{aligned} \dot{y} &= y \left[g(t) + \sum_{i=1}^2 h_i(t) \phi_i(t, x_i) x_i^2 - q(t) y(t) \right], \quad t \neq \tau_k, \\ \Delta y(t) &= -u_k y(t), \quad t = \tau_k, \quad k \in \mathbf{N}, \end{aligned} \quad (1.4)$$

with the initial value conditions

$$x_1(0) = x_{10} \geq 0, \quad x_2(0) = x_{20} \geq 0, \quad y(0) = y_0 \geq 0, \quad (1.5)$$

in which x_1 and x_2 are the densities of immature and mature prey, respectively; and y is the density of the predator that can prey on x_1 and x_2 . When its favorite food is severely scarce, population y can eat other resources: $\Delta y(t) = y(t^+) - y(t)$. Also, there exists a positive integer q such that

$$\begin{aligned} u_{k+q} &= u_k, \quad \tau_{k+q} = \tau_k + \omega, \\ 0 &= \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots, \quad k \in \mathbf{N}. \end{aligned} \quad (1.6)$$

Here, $x_i^2 \phi_i(t, x_i)$, the number of the prey x_i consumed per predator in unit time, is called the predator functional response. We assume that there exists a positive constant L such that

$$0 < \phi_i(t, x_i) < L; \quad \frac{\partial}{\partial x_i} (x_i^2 \phi_i(t, x_i)) \geq 0 \quad \text{for } x_i > 0, \quad i = 1, 2. \quad (1.7)$$

The last condition in (1.7) implies that as the prey population increases, the consumption rate of prey consumed per predator increases. The birth rate of the immature prey population is proportional to the existing mature prey population with a proportionality function $a(t)$. For the immature prey population, the death rate is proportional to the existing immature prey population with a proportionality function $b(t)$. The variable parameter $d(t)$ represents that the immature prey population is density restriction. The transition rate from immature individuals to the mature individuals is assumed to be proportional to the existing immature population with proportionality coefficient $c(t)$. The death rate of the mature population is of a logistic nature with proportionality coefficient $f(t)$. Also, $p_i(t)$ and $h_i(t)$ ($i = 1, 2$) are the coefficients that relate to conversion rates of the immature and, respectively, mature prey biomass into predator biomass. The coefficients in (1.4) are all continuous ω -periodic for $t \geq 0$. In fact, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $p_i(t)$, $h_i(t)$, and $q(t)$ are all strictly positive, and $\phi_i(t, x_i)$ is nonnegative ($i = 1, 2$).

The organization of this paper is as follows. In Section 2, we provide some preliminary results which will be useful. In Section 3, we investigate the permanence and extinction of system (1.4) by using analysis technique. In the last section, we give a biological example and a brief discussion of our result.

2. Preliminary results

Before stating and proving our main results, we give the following definitions, notations, and lemmas which will be useful in the following section.

Let $\mathbf{R}_+ = [0, \infty)$, $\mathbf{R}_+^n = \{(x_1, \dots, x_n) : x_i > 0, i = 1, \dots, n\}$ and $\alpha(t)$ be a continuous ω -periodic function defined on $[0, +\infty)$, then we set

$$\mathbf{A}_\omega(\alpha) = \omega^{-1} \int_0^\omega \alpha(t) dt, \quad \alpha^U = \max_{t \in [0, \omega]} \alpha(t), \quad \alpha^L = \min_{t \in [0, \omega]} \alpha(t). \quad (2.1)$$

Lemma 2.1. *System (1.4) is dissipative.*

Proof. Define a function $V(t, x_1, x_2, y)$ such that

$$V(t, x_1, x_2, y) = h(x_1(t) + x_2(t)) + py(t), \quad (2.2)$$

in which $h = \max_{t \in [0, \omega]} \{h_1(t), h_2(t)\}$, $p = \min_{t \in [0, \omega]} \{p_1(t), p_2(t)\}$. After a simple computations, we have

$$D^+V|_{(4)} + \bar{w}V < (ha^U + \bar{w})x_2 - hf^Lx_2^2 + (hc^U + \bar{w} - hb^L)x_1 - hd^Lx_1^2 + (\bar{w} + pg^U)y - pq^Ly^2, \quad (2.3)$$

in which \bar{w} is a positive constant. Obviously, the right-hand side of the above inequality is bounded above for all $(x_1(t), x_2(t), y(t)) \in \mathbf{R}_+^3$. Hence,

$$D^+V|_{(4)} + \bar{w}V < \lambda, \quad (2.4)$$

where λ is a positive constant. When $t = \tau_k$, we get

$$V(\tau_k^+) \leq V(\tau_k). \quad (2.5)$$

According to Lemma 2.2 in [9, page 23], we derive that

$$V(t) = V(0)e^{-\bar{w}t} + \int_0^t \lambda e^{-\bar{w}(t-s)} ds < \frac{2\lambda}{\bar{w}}, \quad \text{as } t \rightarrow \infty, \quad (2.6)$$

which implies that system (1.4) is dissipative. This completes the proof. \square

Next, we consider the following two subsystems of system (1.4):

$$\dot{x}_1 = a(t)x_2 - b(t)x_1 - d(t)x_1^2, \quad (2.7)$$

$$\dot{x}_2 = c(t)x_1 - f(t)x_2^2,$$

$$\dot{y} = y[g(t) - q(t)y(t)], \quad t \neq \tau_k, \quad (2.8)$$

$$\Delta y(t) = -u_k y(t), \quad t = \tau_k, \quad k \in \mathbf{N}.$$

Lemma 2.2 (see [18]). *The system (2.7) has a positive ω -periodic solution $(x_1^*(t), x_2^*(t))$ which is globally asymptotically stable with respect to \mathbf{R}_+^2 .*

Lemma 2.3. *If the following conditions*

$$\mathbf{A}_\omega(g) > 0,$$

$$\prod_{k=1}^q (1 - u_k) > \exp(-\omega \mathbf{A}_\omega(g)) \quad (2.9)$$

hold, then system (2.8) has a unique ω -periodic solution:

$$\begin{aligned} \mathbf{y}^*(t) = & \left(\frac{\prod_{0 < \tau_k < t} (1/(1-u_k)) \exp\left(-\int_0^t g(\xi) d\xi\right) \int_0^\omega \prod_{s \leq \tau_k < \omega} (1/(1-u_k)) \exp\left(-\int_s^\omega g(\xi) d\xi\right) q(s) ds}{1 - \prod_{k=1}^q (1/(1-u_k)) \exp\left(-\int_0^\omega g(\xi) d\xi\right)} \right. \\ & \left. + \int_0^t \prod_{s \leq \tau_k < t} \frac{1}{1-u_k} \exp\left(-\int_s^t g(\xi) d\xi\right) q(s) ds \right)^{-1} \end{aligned} \quad (2.10)$$

and for every solution $\mathbf{y}(t)$ of system (2.8),

$$|\mathbf{y}(t) - \mathbf{y}^*(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (2.11)$$

Proof. Let $\mathbf{y}(t) = 1/z(t)$ and obtain the linear nonhomogeneous impulsive equation

$$\begin{aligned} \dot{z}(t) &= -g(t)z(t) + q(t), \quad t \neq \tau_k, \\ z(t^+) &= \frac{1}{1-u_k} z(t), \quad t = \tau_k. \end{aligned} \quad (2.12)$$

Let $W(t, s) = \prod_{s \leq \tau_k < t} (1/(1-u_k)) \exp(-\int_s^t g(\xi) d\xi)$ be the Cauchy matrix for the relevant homogeneous equation, then the solution of (2.12) has the form

$$z(t) = W(t, 0)z(0) + \int_0^t W(t, s)q(s)ds. \quad (2.13)$$

The solution $z(t)$ will be ω -periodic if $z(\omega) = z(0)$, or if

$$(1 - W(\omega, 0))z(0) = \int_0^\omega W(\omega, s)q(s)ds. \quad (2.14)$$

In view of conditions (2.9), (2.14) has a unique solution

$$z(0) = \frac{\int_0^\omega W(\omega, s)q(s)ds}{1 - W(\omega, 0)}. \quad (2.15)$$

Then, (2.13) has a unique ω -periodic solution

$$z^*(t) = \frac{W(t, 0) \int_0^\omega W(\omega, s)q(s)ds}{1 - W(\omega, 0)} + \int_0^t W(t, s)q(s)ds. \quad (2.16)$$

Hence, (2.8) has a unique ω -periodic solution

$$\begin{aligned} \mathbf{y}^*(t) &= \left(\frac{W(t, 0) \int_0^\omega W(\omega, s)q(s)ds}{1 - W(\omega, 0)} + \int_0^t W(t, s)q(s)ds \right)^{-1} \\ &= \left(\frac{\prod_{0 \leq \tau_k < t} (1/(1-u_k)) \exp\left(-\int_0^t g(\xi) d\xi\right) \int_0^\omega \prod_{s \leq \tau_k < \omega} (1/(1-u_k)) \exp\left(-\int_s^\omega g(\xi) d\xi\right) q(s) ds}{1 - \prod_{k=1}^q (1/(1-u_k)) \exp\left(-\int_0^\omega g(\xi) d\xi\right)} \right. \\ & \quad \left. + \int_0^t \prod_{s \leq \tau_k < t} \frac{1}{1-u_k} \exp\left(-\int_s^t g(\xi) d\xi\right) q(s) ds \right)^{-1}. \end{aligned} \quad (2.17)$$

From (2.12), we derive that

$$(z(t) - z^*(t))' = -g(t)(z(t) - z^*(t)), \quad \tau_k < t \leq \tau_{k+1}. \quad (2.18)$$

Let

$$\kappa \doteq \prod_{k=1}^q \frac{1}{1 - u_k} \exp \left(- \int_0^\omega g(t) dt \right). \quad (2.19)$$

Since conditions (2.9) hold, we note that $0 < \kappa < 1$. Then, from Lemma 2.1, the solution of (2.8) is governed by

$$\begin{aligned} |y(t) - y^*(t)| &\leq \frac{2\lambda}{\bar{\omega}} y^{*u} |z(0^+) - z^*(0^+)| \prod_{0 < \tau_k < t} \frac{1}{1 - u_k} \exp \left(- \int_0^t g(s) ds \right), \\ &\leq \frac{2\lambda}{\bar{\omega}} y^{*u} |z(0^+) - z^*(0^+)| \kappa^{[t/\omega]+1} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned} \quad (2.20)$$

in which λ and $\bar{\omega}$ are defined in Lemma 2.1. This completes the proof. \square

3. Permanence and extinction of system

Theorem 3.1. *System (1.4) is permanent if and only if*

$$\begin{aligned} \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) &> 0, \\ \prod_{k=1}^q (1 - u_k) &> \exp \left(- \omega \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) \right), \end{aligned} \quad (3.1)$$

in which $(x_1^*(t), x_2^*(t))$ is the positive ω -periodic solution of system (2.7).

We need the following lemmas to prove Theorem 3.1.

Lemma 3.2. *There exist positive constants M_x and M_y such that*

$$\limsup_{t \rightarrow \infty} x_i(t) \leq M_x \quad (i = 1, 2), \quad \limsup_{t \rightarrow \infty} y(t) \leq M_y. \quad (3.2)$$

Proof. In fact, from Lemmas 2.1 and 2.3, the proof of the lemma is obvious. This completes the proof. \square

Lemma 3.3. *There exists a positive constant ρ_x ($\rho_x < M_x$) such that*

$$\liminf_{t \rightarrow \infty} x_i(t) \geq \rho_x \quad (i = 1, 2). \quad (3.3)$$

Proof. From Lemma 3.2, one notes that there exists a positive constant T_0 such that

$$0 < y(t) \leq M_y \quad \text{for } t \geq T_0, \quad (3.4)$$

we then have

$$\begin{aligned} \dot{x}_1 &\geq a(t)x_2 - b(t)x_1 - (d(t) + p_1(t)LM_y)x_1^2, \\ \dot{x}_2 &\geq c(t)x_1 - (f(t) + p_2(t)LM_y)x_2^2 \end{aligned} \quad (3.5)$$

for $t \geq T_0$. According to Lemma 2.2, the following auxiliary equations:

$$\begin{aligned} \dot{u}_1 &= a(t)u_2 - b(t)u_1 - (d(t) + p_1(t)LM_y)u_1^2, \\ \dot{u}_2 &= c(t)u_1 - (f(t) + p_2(t)LM_y)u_2^2 \end{aligned} \quad (3.6)$$

has a globally asymptotically stable ω -periodic solution $(u_1^*(t), u_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.6) with $(u_1(0), u_2(0)) = (x_1(0), x_2(0))$. By the vector comparison theorem [19], we obtain

$$x_i(t) \geq u_i(t), \quad i = 1, 2 \quad \forall t \geq 0. \quad (3.7)$$

According to the global asymptotic stability of $(u_1^*(t), u_2^*(t))$, for any positive constant ε ($\leq \min_{t \in [0, \omega]} \{u_i^*(t)/3, i = 1, 2\}$), there exists a T_1 ($> T_0$) such that for all $t \geq T_1$,

$$|u_i(t) - u_i^*(t)| < \varepsilon, \quad i = 1, 2. \quad (3.8)$$

Hence, for all $t \geq T_1$, we derive that

$$u_i(t) \geq u_i^*(t) - \varepsilon, \quad i = 1, 2. \quad (3.9)$$

Let

$$\rho_x = \max_{t \in [0, \omega]} \left\{ \frac{u_i^*(t)}{2}, i = 1, 2 \right\}, \quad (3.10)$$

then

$$x_i(t) \geq \rho_x, \quad i = 1, 2. \quad (3.11)$$

Consequently,

$$\liminf_{t \rightarrow \infty} x_i(t) \geq \rho_x, \quad i = 1, 2. \quad (3.12)$$

This completes the proof. \square

Lemma 3.4. Suppose that (3.1) holds, then there exists a positive constant Q_y ($Q_y < M_y$) such that

$$\limsup_{t \rightarrow \infty} y(t) \geq Q_y. \quad (3.13)$$

Proof. In view of (3.1), we can choose a positive constant ε_0 such that

$$\begin{aligned} \mathbf{A}_\omega(\psi_{\varepsilon_0}(t)) &> 0, \\ \prod_{k=1}^q (1 - u_k) &> \exp(-\omega \mathbf{A}_\omega(\psi_{\varepsilon_0}(t))), \end{aligned} \quad (3.14)$$

in which

$$\psi_{\varepsilon_0}(t) = g(t) + \sum_{i=1}^2 h_i(t) \phi_i(t, (x_i^*(t) - \varepsilon_0)) (x_i^*(t) - \varepsilon_0)^2 - q(t) \varepsilon_0. \quad (3.15)$$

Consider the following system with a positive parameter μ :

$$\begin{aligned} \dot{x}_1 &= a(t)x_2 - b(t)x_1 - [d(t) + L\mu p_1(t)]x_1^2, \\ \dot{x}_2 &= c(t)x_1 - [f(t) + L\mu p_2(t)]x_2^2. \end{aligned} \quad (3.16)$$

By Lemma 2.2, system (3.16) has a positive ω -periodic solution $(x_{1\mu}^*(t), x_{2\mu}^*(t))$, which is globally asymptotically stable. Let $(x_{1\mu}(t), x_{2\mu}(t))$ be the solution of (3.16) with initial condition $x_{i\mu}(0) = x_i^*(0)$, $i = 1, 2$, where $(x_1^*(t), x_2^*(t))$ is the positive periodic solution of (2.7). Hence, for the above ε_0 , there exists $T_2 > T_1$ such that

$$|x_{i\mu}(t) - x_{i\mu}^*(t)| < \frac{\varepsilon_0}{4} \quad (3.17)$$

for $t \geq T_2$, $i = 1, 2$. According to the continuity of the solution in the parameter μ , we have $x_{i\mu}(t) \rightarrow x_i^*(t)$ ($i = 1, 2$) uniformly in $[T_2, T_2 + \omega]$ as $\mu \rightarrow 0$. Hence, for $\varepsilon_0 > 0$, there exists $\mu_0 = \mu_0(\varepsilon_0)$ ($0 < \mu_0 < \varepsilon_0$) such that

$$|x_{i\mu}(t) - x_i^*(t)| < \frac{\varepsilon_0}{4}, \quad 0 \leq \mu \leq \mu_0, \quad (3.18)$$

$t \in [T_2, T_2 + \omega]$, $i = 1, 2$. Thus, from (3.17) and (3.18), we get

$$|x_{i\mu}^*(t) - x_i^*(t)| < \frac{\varepsilon_0}{2}, \quad 0 \leq \mu \leq \mu_0, \quad (3.19)$$

$t \in [T_2, T_2 + \omega]$, $i = 1, 2$. Since $x_{i\mu}^*(t)$ and $x_i^*(t)$ are all ω -periodic, we have

$$|x_{i\mu}^*(t) - x_i^*(t)| < \frac{\varepsilon_0}{2}, \quad 0 \leq \mu \leq \mu_0, \quad (3.20)$$

$t \geq 0$, $i = 1, 2$.

Choose a constant μ_1 ($0 < \mu_1 < \mu_0$, $\mu_1 < \varepsilon_0$), from (3.20), we derive

$$x_{i\mu_1}^*(t) \geq x_i^*(t) - \frac{\varepsilon_0}{2}, \quad t \geq 0, \quad i = 1, 2. \quad (3.21)$$

Suppose that (3.13) is not true, then for the above ε_0 , there exists a $\nu \in \mathbf{R}_+^3$ such that

$$\limsup_{t \rightarrow \infty} y(t) < \mu_1, \quad (3.22)$$

where $(x_1(t), x_2(t), y(t))$ is the solution of (1.4) with the initial condition $(x_{10}, x_{20}, y_0) = \nu$. So there exists a constant $T_3 (> T_2)$ such that

$$y(t) < \mu_1, \quad t \geq T_3, \quad (3.23)$$

then we derive that

$$\begin{aligned} \dot{x}_1 &\geq a(t)x_2 - b(t)x_1 - (d(t) + L\mu_1 p_1(t))x_1^2, \\ \dot{x}_2 &\geq c(t)x_1 - (f(t) + L\mu_1 p_2(t))x_2^2 \end{aligned} \quad (3.24)$$

for $t \geq T_3$. Let $(x_{1\mu_1}, x_{2\mu_1})$ be the solution of (3.16) with $\mu = \mu_1$ and $(x_{1\mu_1}(T_3), x_{2\mu_1}(T_3)) = (x_1(T_3), x_2(T_3))$, then by the vector comparison theorem, we obtain

$$x_i(t) \geq x_{i\mu_1}(t), \quad i = 1, 2, \quad (3.25)$$

$t \geq T_3$. By the global asymptotic stability of $(x_{1\mu_1}^*(t), x_{2\mu_1}^*(t))$, for the given $\varepsilon_0 > 0$, there exists $T_4 > T_3$ such that

$$x_{i\mu_1}(t) > x_{i\mu_1}^*(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_4, \quad i = 1, 2, \quad (3.26)$$

and hence, by (3.21), we get

$$x_i(t) > x_i^*(t) - \varepsilon_0, \quad t \geq T_4, \quad i = 1, 2. \quad (3.27)$$

Since $0 < y(t) < \mu_1 < \varepsilon_0$ together with (1.4) and (1.7), we have

$$\begin{aligned} \dot{y} &\geq y \left[g(t) + \sum_{i=1}^2 h_i(t) \phi_i(t, (x_i^*(t) - \varepsilon_0)) (x_i^*(t) - \varepsilon_0)^2 - q(t) \varepsilon_0 \right], \quad t \neq \tau_k, \\ y(t^+) &= (1 - u_k) y(t), \quad t = \tau_k, \quad k \in \mathbf{N}. \end{aligned} \quad (3.28)$$

Hence, it follows from Lemma 2.2 in [9, page 23] that

$$y(t) \geq y_0 \prod_{0 < \tau_k < t} (1 - u_k) \exp \left(\int_0^t \psi_{\varepsilon_0}(s) ds \right), \quad (3.29)$$

that is,

$$y(t) \geq y_0 \left[\prod_{0 < \tau_k < \omega} (1 - u_k) \exp(\omega \mathbf{A}_\omega(\psi_{\varepsilon_0})) \right]^{[t/\omega]}. \quad (3.30)$$

By (3.14), we know that $y(t) \rightarrow \infty$ as $t \rightarrow \infty$, which leads to a contradiction. This completes the proof. \square

Lemma 3.5. *Assume that (3.1) holds, then there exists a positive constant δ_y ($\delta_y < M_y$) such that any solution (x_1, x_2, y) of system (1.4) with initial conditions satisfies*

$$\liminf_{t \rightarrow \infty} y(t) \geq \delta_y. \quad (3.31)$$

Proof. Suppose that (3.31) is not true, there must exist a time sequence $\{t_k^{(\hat{k})}\}_{k=1}^\infty \subset \mathbf{R}_+$, $\hat{k} \in \mathbf{Z}$ such that

$$\liminf_{t_k^{(\hat{k})} \rightarrow \infty} y(t_k^{(\hat{k})}) < \frac{\mathcal{Q}y}{(\hat{k}+1)^2}, \quad (3.32)$$

and by Lemma 3.4, we have $\limsup_{t_k^{(\hat{k})} \rightarrow \infty} y(t_k^{(\hat{k})}) \geq \mathcal{Q}y$, $\hat{k} = 1, 2, \dots$. Hence, for each \hat{k} , we choose two time sequences $\{s_q^{(\hat{k})}\}$ and $\{t_q^{(\hat{k})}\}$, satisfying $0 < s_1^{(\hat{k})} < t_1^{(\hat{k})} < s_2^{(\hat{k})} < t_2^{(\hat{k})} < \dots < s_q^{(\hat{k})} < t_q^{(\hat{k})} < \dots$ and $s_q^{(\hat{k})} \rightarrow \infty$ as $q \rightarrow \infty$, as well as

$$y(s_q^{(\hat{k})}) = \frac{\mathcal{Q}y}{\hat{k}+1}, \quad y(t_q^{(\hat{k})}) = \prod_{0 < \tau_k < t_q^{(\hat{k})}} (1 - u_k) \frac{\mathcal{Q}y}{(\hat{k}+1)^2}, \quad (3.33)$$

$$\prod_{0 < \tau_k < t_q^{(\hat{k})}} (1 - u_k) \frac{\mathcal{Q}y}{(\hat{k}+1)^2} < y(t) < \frac{\mathcal{Q}y}{\hat{k}+1}, \quad t \in (s_q^{(\hat{k})}, t_q^{(\hat{k})}). \quad (3.34)$$

By Lemma 3.2, for a given positive integer \hat{k} , there exists $\tilde{T}^{(\hat{k})} > 0$ such that $x_i(t) \leq M_x$ ($i = 1, 2$) and $y(t) \leq M_y$ for all $t \geq \tilde{T}^{(\hat{k})}$. In view of $s_q^{(\hat{k})} \rightarrow \infty$ as $q \rightarrow \infty$, there exists a positive integer $K_1^{(\hat{k})}$ such that $s_q^{(\hat{k})} > \tilde{T}^{(\hat{k})}$ as $q \geq K_1^{(\hat{k})}$. Hence, for any $t \geq \tilde{T}^{(\hat{k})}$, we have

$$\begin{aligned} \dot{y} &\geq y(-|g(t)| - M_y q(t)), \quad t \neq \tau_k, \\ y(t^+) &= (1 - u_k)y(t), \quad t = \tau_k, \quad k \in \mathbf{N}. \end{aligned} \quad (3.35)$$

Integrating the above inequality from $s_q^{(\hat{k})}$ to $t_q^{(\hat{k})}$, for any $q \geq K_1^{(\hat{k})}$, then we have

$$y(t_q^{(\hat{k})}) \geq y(s_q^{(\hat{k})}) \prod_{s_q^{(\hat{k})} < \tau_k < t_q^{(\hat{k})}} (1 - u_k) \exp\left(\int_{s_q^{(\hat{k})}}^{t_q^{(\hat{k})}} [-|g(t)| - M_y q(t)] dt\right). \quad (3.36)$$

Obviously, it follows from (3.33) that

$$\int_{s_q^{(\hat{k})}}^{t_q^{(\hat{k})}} [|g(t)| + M_y q(t)] dt > \ln(\hat{k}+1) \quad \text{for } q \geq K_1^{(\hat{k})}. \quad (3.37)$$

Hence, in view of the periodicity of $g(t)$ and $q(t)$, we get

$$t_q^{(\hat{k})} - s_q^{(\hat{k})} \rightarrow \infty \quad \text{as } \hat{k} \rightarrow \infty, \quad q \geq K_1^{(\hat{k})}. \quad (3.38)$$

By (3.14), (3.33), and (3.38), there are positive constants T and N_0 such that

$$y(s_q^{(\hat{k})}) = \prod_{0 < \tau_k \leq s_q^{(\hat{k})}} (1 - u_k) \frac{\mathcal{Q}y}{\hat{k}+1} < \varepsilon_0, \quad (3.39)$$

$$t_q^{(\hat{k})} - s_q^{(\hat{k})} > 2T, \quad (3.40)$$

$$\prod_{0 < \tau_k < \kappa} (1 - u_k) \exp\left(\int_0^\kappa \varphi_{\varepsilon_0}(t) dt\right) > 1 \quad (3.41)$$

for $\widehat{k} \geq N_0$, for $q \geq K_1^{(\widehat{k})}$, and for all $\kappa > T$. However, (3.39) implies that

$$y(t) < \varepsilon_0, \quad t \in [s_q^{(\widehat{k})}, t_q^{(\widehat{k})}] \quad (3.42)$$

for $\widehat{k} \geq N_0$, $q \geq K_1^{(\widehat{k})}$. Then, we get

$$\begin{aligned} \dot{x}_1 &\geq a(t)x_2 - b(t)x_1 - (d(t) + L\varepsilon_0 p_1(t))x_1^2, \\ \dot{x}_2 &\geq c(t)x_1 - (f(t) + L\varepsilon_0 p_2(t))x_2^2. \end{aligned} \quad (3.43)$$

Let $(x_{1\varepsilon_0}, x_{2\varepsilon_0})$ be the solution of (3.16) with $\mu = \varepsilon_0$ and $(x_{1\varepsilon_0}(T_3), x_{2\varepsilon_0}(T_3)) = (x_1(T_3), x_2(T_3))$, then by the vector comparison theorem, we obtain

$$x_i(t) \geq x_{i\varepsilon_0}(t), \quad i = 1, 2, \quad t \in [s_q^{(\widehat{k})}, t_q^{(\widehat{k})}]. \quad (3.44)$$

From $\lim_{q \rightarrow \infty} s_q^{(\widehat{k})} = \infty$ and Lemmas 3.2 and 3.3, we obtain that for any \widehat{k} , there is a $K_2^{(\widehat{k})} > K_1^{(\widehat{k})}$ such that for any $q \geq K_2^{(\widehat{k})}$,

$$\rho_x \leq x_i(s_q^{(\widehat{k})}) \leq M_x, \quad i = 1, 2. \quad (3.45)$$

For $\mu = \varepsilon_0$, (3.16) has a globally asymptotically stable positive ω -periodic solution $(x_{1\varepsilon_0}^*(t), x_{2\varepsilon_0}^*(t))$. By the global asymptotic stability of $(x_{1\varepsilon_0}^*(t), x_{2\varepsilon_0}^*(t))$, for the given $\varepsilon_0 > 0$, there exists $T_5 > T_4$, and T_5 is dependent of any \widehat{k} and q such that

$$x_{i\varepsilon_0}(t) > x_{i\varepsilon_0}^*(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_5 + s_q^{(\widehat{k})}, \quad i = 1, 2, \quad q \geq K_2^{(\widehat{k})}, \quad (3.46)$$

and hence, by (3.21), we get

$$x_{i\varepsilon_0}(t) > x_i^*(t) - \varepsilon_0, \quad t \geq T_5 + s_q^{(\widehat{k})}, \quad i = 1, 2, \quad q \geq K_2^{(\widehat{k})}. \quad (3.47)$$

By (3.44), there is an $N_1 \geq N_0$ such that $t_q^{(\widehat{k})} - s_q^{(\widehat{k})} \geq 2T$ for all $\widehat{k} \geq N_1$ and $q \geq K_2^{(\widehat{k})}$, where $T \geq T_5$. Hence, from (3.44) and (3.47), we obtain

$$x_i(t) \geq x_i^*(t) - \varepsilon_0, \quad i = 1, 2 \quad (3.48)$$

for all $t \in [T + s_q^{(\widehat{k})}, t_q^{(\widehat{k})}]$, $\widehat{k} \geq N_1$, and $q \geq K_2^{(\widehat{k})}$. Since, for any $t \in [T + s_q^{(\widehat{k})}, t_q^{(\widehat{k})}]$, $k \geq N_1$ and $q \geq K_2^{(\widehat{k})}$, by (1.4) and (3.42), we have

$$\begin{aligned} \dot{y} &\geq y \left[g(t) + \sum_{i=1}^2 h_i(t) \phi_i(t, (x_i^*(t) - \varepsilon_0)) (x_i^*(t) - \varepsilon_0)^2 - q(t) \varepsilon_0 \right], \quad t \neq \tau_k, \quad t \in [T + s_q^{(\widehat{k})}, t_q^{(\widehat{k})}], \\ y(t^+) &= (1 - u_k) y(t), \quad t = \tau_k, \quad k \in \mathbf{N}. \end{aligned} \quad (3.49)$$

Integrating from $T + s_q^{(\hat{k})}$ to $t_q^{(\hat{k})}$ for any $k \geq N_1$ and $q \geq K_2^{(\hat{k})}$, we obtain

$$y(t_q^{(\hat{k})}) \geq y(s_q^{(\hat{k})} + T) \prod_{s_q^{(\hat{k})} + T < \tau_k < t_q^{(\hat{k})} (1 - u_k) \exp \left(\int_{s_q^{(\hat{k})} + T}^{t_q^{(\hat{k})}} \psi_{\varepsilon_0}(t) dt \right). \quad (3.50)$$

Hence, by (3.34), (3.40), and (3.41), we finally derive that

$$\begin{aligned} \prod_{0 < \tau_k < t_q^{(\hat{k})} (1 - u_k) \frac{Qy}{(\hat{k} + 1)^2} &\geq \prod_{0 < \tau_k \leq t_q^{(\hat{k})} (1 - u_k) \frac{Qy}{(\hat{k} + 1)^2} \prod_{s_q^{(\hat{k})} + T < \tau_k < t_q^{(\hat{k})} (1 - u_k) \exp \left(\int_{s_q^{(\hat{k})} + T}^{t_q^{(\hat{k})}} \psi_{\varepsilon_0}(t) dt \right) \\ &> \prod_{0 < \tau_k < t_q^{(\hat{k})} (1 - u_k) \frac{Qy}{(\hat{k} + 1)^2}. \end{aligned} \quad (3.51)$$

This leads to a contradiction. This completes the proof. \square

According to Lemmas 3.2–3.5, we can directly prove the sufficient part of the theorem. Next, we are ready to show the necessity of this theorem.

Consider the case of

$$\mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) \leq 0, \quad (3.52)$$

it is then easy to derive that the predator population $y(t)$ must be extinct because of sterilization and impulsive harvesting. Suppose that

$$\begin{aligned} \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) &> 0, \\ \prod_{k=1}^q (1 - u_k) &\leq \exp \left(-\omega \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) \right). \end{aligned} \quad (3.53)$$

We will show that

$$\lim_{t \rightarrow \infty} y(t) = 0. \quad (3.54)$$

In fact, from (1.7) and (3.53), we know that for any given $0 < \varepsilon < 1$, there exists $\varepsilon_1 > 0$ such that

$$\begin{aligned} \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, (x_i^* + \varepsilon_1)) (x_i^* + \varepsilon_1)^2 - q\varepsilon \right) &> 0, \\ \prod_{k=1}^q (1 - u_k) &< \exp \left(-\omega \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, (x_i^* + \varepsilon_1)) (x_i^* + \varepsilon_1)^2 - q\varepsilon \right) \right). \end{aligned} \quad (3.55)$$

Note that $q(t) > 0$ for $t \geq 0$. Since

$$\begin{aligned}\dot{x}_1 &\leq a(t)x_2 - b(t)x_1 - d(t)x_1^2, \\ \dot{x}_2 &\leq c(t)x_1 - f(t)x_2^2,\end{aligned}\tag{3.56}$$

for the given $\epsilon_1 > 0$, it is easy to show that there exists $T^{(1)} > 0$ such that

$$x_i(t) \leq x_i^*(t) + \epsilon_1 \quad \text{for } t > T^{(1)}, \quad i = 1, 2,\tag{3.57}$$

we then have

$$\prod_{k=1}^q (1 - u_k) < \exp \left(-\omega \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i) (x_i)^2 - q\epsilon \right) \right).\tag{3.58}$$

We now show that there must exist $T^{(2)} (> T^{(1)})$ such that $y(T^{(2)}) < \epsilon$. Otherwise, by the last two equations in system (1.4), we have

$$\begin{aligned}\epsilon \leq y(t) &\leq y(T^{(1)}) \prod_{T^{(1)} < \tau_k < t} (1 - u_k) \exp \left(\int_{T^{(1)}}^t \left(g(s) + \sum_{i=1}^2 h_i(s) \phi_i(s, x_i(s)) x_i^2(s) - q(s)\epsilon \right) ds \right) \longrightarrow 0 \\ &\text{as } t \longrightarrow \infty.\end{aligned}\tag{3.59}$$

This implies that $\epsilon \leq 0$, which is a contradiction.

Let $M(\epsilon) = \max_{t \in [0, \omega]} \{g(t) + \sum_{i=1}^2 h_i(t) \phi_i(s, x_i(t)) x_i^2(t) + q(t)\epsilon\}$. Note that $M(\epsilon)$ is bounded. We then show that

$$y(t) \leq \epsilon \exp(M(\epsilon)\omega) \quad \text{for } t \geq T^{(2)}.\tag{3.60}$$

Otherwise, there exists $T^{(3)} > T^{(2)}$ such that $T^{(3)} \in (T^{(2)} + P_1\omega, T^{(2)} + (P_1 + 1)\omega]$, in which $P_1 \in \mathbf{Z}_+ = \{0, 1, 2, \dots\}$, and $y(T^{(3)}) > \epsilon \exp(M(\epsilon)\omega)$. By (3.58), we have

$$\begin{aligned}\epsilon \exp(M(\epsilon)\omega) &< y(T^{(3)}) \\ &= y(T^{(2)}) \prod_{T^{(2)} < \tau_k < T^{(3)}} (1 - u_k) \exp \left(\int_{T^{(2)}}^{T^{(3)}} \left(g(s) + \sum_{i=1}^2 h_i(s) \phi_i(s, x_i(s)) x_i^2(s) - q(s)\epsilon \right) ds \right) \\ &\leq y(T^{(2)}) \left[\prod_{k=1}^q (1 - u_k) \exp \left(\int_0^\omega \left(g(s) + \sum_{i=1}^2 h_i(s) \phi_i(s, x_i(s)) x_i^2(s) - q(s)\epsilon \right) ds \right) \right]^{P_1} \\ &\quad \times \prod_{T^{(2)} + P_1\omega < \tau_k < T^{(3)}} (1 - u_k) \exp \left(\int_0^\omega \left(g(s) + \sum_{i=1}^2 h_i(s) \phi_i(s, x_i(s)) x_i^2(s) - q(s)\epsilon \right) ds \right) \\ &< \epsilon \exp(M(\epsilon)\omega),\end{aligned}\tag{3.61}$$

which leads to a contradiction. This implies that (3.60) holds. In view of the arbitrariness of ϵ , we get that $y(t) \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof.

Remark 3.6. From the proof of Theorem 3.1 above, we note that the predator population y in system (1.4) will be extinct provided that

$$\mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) \leq 0, \quad (3.62)$$

or excessive harvesting, that is,

$$\begin{aligned} \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) &> 0, \\ \prod_{k=1}^q (1 - u_k) &\leq \exp \left(-\omega \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) \right). \end{aligned} \quad (3.63)$$

4. Discussion

Our result could provide a useful insight into the conservation of beneficial animals, especially rare animals. As an example, we depict the case of *Oreolalax omeimontis* tadpole, *Oreolalax omeimontis*, and red-eared slider (*T. scripta elegans*). *Oreolalax omeimontis* is a rare species of frog found near Mt. Omei in Sichuan (China). The red-eared slider is a native of the Mississippi Valley area of the United States [8, 20]. Since the 1970s, large numbers of red-eared sliders have been produced on turtle farms in the USA for the international pet trade. Red-eared Turtles are traded as pet animals and have been introduced to many countries. They are omnivorous and will eat insects, tadpoles, crayfish, shrimp, worms, snails, amphibians and small fish, as well as aquatic plants, and they hardly may be controlled by a natural enemy. In our model, the variables $x_1(t)$ and $x_2(t)$ represent the density of *Oreolalax omeimontis* tadpole and *Oreolalax omeimontis* at time t , respectively. The variable $y(t)$ describes the density of the red-eared slider at time t . It is well known that *Oreolalax omeimontis* is beneficial to humans because they eat so many insect pests. Ironically, although red-eared sliders have been widely introduced throughout the world, its detrimental effects have been reported by many researchers [21, 22]. The red-eared slider is one of main enemies of *Oreolalax omeimontis*. To protect these beneficial and rare toads, we must control the amount of red-eared sliders in the habitat of *Oreolalax omeimontis*. From a control point of view, our aim is to keep red-eared sliders at an acceptably low level with a minimum use of artificial control measures, not to eradicate all red-eared sliders. Hence, in the above example, the problem of nonextinction of populations becomes a description of reasonable harvesting rates u_k ($k = 1, 2, \dots, q$).

According to our main result, system (1.4) is permanent if and only if the growth of the predator y by foraging prey populations x_1 and x_2 plus its intrinsic rate of increase is positive on average during the period ω , and harvesting rates u_k ($k = 1, 2, \dots, q$) of red-eared sliders during the period ω are small enough to satisfy that

$$\prod_{k=1}^q (1 - u_k) > \exp \left(-\omega \mathbf{A}_\omega \left(g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} \right) \right). \quad (4.1)$$

These seem to be reasonable from a biological point of view; but it should be noted that condition (3.1) allows the predator to be $g + \sum_{i=1}^2 h_i \phi_i(t, x_i^*) x_i^{*2} < 0$ for some time intervals among $0 \leq t \leq \omega$. That is, under reasonable harvesting rates, the predator can survive together with prey populations even if the growth rate of the former is negative for some seasons during a period. We hope that our result can be used to help protect beneficial animals found in their habitats.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (10671001).

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