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Research Article

A General Discrete Time Model of Population Dynamics in the Presence of an Infection

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We present a set of difference equations which generalizes that proposed in the work of G. Izzo and A. Vecchio (2007) and represents the discrete counterpart of a larger class of continuous model concerning the dynamics of an infection in an organism or in a host population. The limiting behavior of this new discrete model is studied and a threshold parameter playing the role of the basic reproduction number is derived.

1. Introduction

Consider the following set of difference equations:

$$\begin{aligned} y(n+1) &= \alpha + (1-\beta)y(n) - \sum_{i=1}^{m} \psi_i(y(n+1))x_i(n+1), \quad n \geq 0, \\ x_1(n+1) &= (1-a_1)x_1(n) + \phi_1(y(n))x_L(n), \quad 1 \leq L \leq m, \\ x_i(n+1) &= (1-a_i)x_i(n) + \phi_i(y(n))x_{i-1}(n), \quad i = 2, 3, \dots, m, \end{aligned} \tag{1.1}$$

where $\psi_i(x), \phi_i(x) \in C^0(\mathbb{R})$, $1 \leq i \leq m$. Moreover ϕ_i, $1 \leq i \leq m$, and at least one of the ψ_j $(1 \leq j \leq L)$ is strictly monotone increasing functions and

$$y\psi_i(y) \geq 0, \quad \forall y \in \mathbb{R},\ i = 1, \dots, m. \tag{1.2}$$

This difference system contains a very large class of population dynamics models in the presence of an infection involving typically at least two populations: *susceptible* individuals and *infective* ones. The former is represented in (1.1) by the sequence y, while the latter is represented by one of the sequences x_i; we name it x_I where I is an integer between 1 and L.

The paper is organized as follows. In the next section we report various examples of continuous models which can be discretized by (1.1). The correspondence among the sequences appearing in (1.1) and the dependent variables of the continuous problem is indicated for each example. In Section 3 we prove some basic properties of the solution of the proposed scheme such as positivity and boundedness, which makes it meaningful in the applications. Our main result is proved in Section 4 where the question of the asymptotic behavior of the solution is investigated. We prove a necessary and sufficient condition for the vanishing of the sequences $\{x_i(n)\}$ and we derive the expression of the *basic reproduction number*, a threshold parameter which allows to predict whether the infection develops or not. Such a parameter permit to check that, in all the examples quoted in Section 2, the asymptotic behavior of the discrete and continuous problem coincides; therefore, our discrete system incorporates the dynamical characteristics (such as positivity and steady states) of the continuous-time models.

2. Continuous Models

In this section we report different classes of continuous models which can be discretized by means of (1.1). In order to avoid the introduction of many different symbols and for the sake of brevity we do not always use the symbolism found in the literature and we indicate the specific references for the explanation of their meaning.

Example 2.1 (see [1]). This continuous model represents the spread of HIV-1 infection inside the human organism. Here $S(t)$ represents the number of susceptible cells which are present at time t in a unit of plasma. The process of infection of a cell is divided into several sequential stages; therefore, $T_j(t)$ is the number of infected cells at time t at stage j. The variable $V(t)$ is the number of viruses at time t. The meaning of the rest of symbols can be found in [1]

$$\begin{gathered}
S'(t) = a - bS(t) - cS(t)V(t), \\
I_1'(t) = cS(t)V(t) - kI_1(t), \\
I_j'(t) = k\big(I_{j-1}(t) - I_j(t)\big), \quad j = 2, \dots, 5, \\
V'(t) = pI_5(t) - qV(t).
\end{gathered} \tag{2.1}$$

Rewrite (1.1) at a general time step t the length of which is h and put

$$\begin{gathered}
m = 6; \qquad I = L = 6; \qquad \psi_i(y) = 0, \quad i = 1, \dots, 5; \\
\psi_6(y) = \phi_1(y) = \gamma y; \qquad \phi_i(y) = kh, \quad i = 2, \dots, 5; \qquad \phi_6(y) = ph; \\
a_i = kh, \quad i = 1, \dots, 5; \qquad a_6 = qh; \qquad \alpha = ah; \qquad \beta = bh; \qquad \gamma = ch.
\end{gathered} \tag{2.2}$$

We obtain

$$\begin{aligned}
y(t+h)-y(t)&=ah-bhy(t)-chy(t+h)x_6(t+h),\\
x_1(t+h)-x(t)&=-khx_1(t)+chy(t)x_6(t),\\
x_i(t+h)-x(t)&=-khx_i(t)+khx_{i-1}(t),\quad i=2,\ldots,5,\\
x_6(t+h)-x(t)&=-qhx_6(t)+phx_5(t).
\end{aligned}\tag{2.3}$$

This can be easily seen (see, e.g., [2]) to be the discrete analogue of (2.1) by dividing each equation by h.

In conclusion, by assuming $h=1$, we have that (1.1) is the discrete counterpart of (2.1) provided that

$$\begin{gathered}
m=6;\qquad I=L=6;\qquad \phi_1(y)=cy;\qquad \phi_i(y)\equiv k,\quad i=2,\ldots,5,\qquad \phi_6(y)\equiv p;\\
a_i=k,\quad i=1,\ldots,5;\qquad a_6=q;\qquad \alpha=a;\qquad \beta=b.
\end{gathered}\tag{2.4}$$

The role of y, x_i, $i=1,\ldots,6$ is related to that of the variable of (2.1) according to the following scheme: $y\leftrightarrow S$, $x_1\leftrightarrow I_1,\ldots,x_5\leftrightarrow I_5$, $x_6\leftrightarrow V$.

Observe that $y(n)$ and $x_6(n)$ play the role of $S(t)$ and $V(t)$, respectively, and therefore they correspond to the susceptible and infective populations as we mentioned in the introduction.

Example 2.2 (see [3]). This represents the spread of HTLV-I infection in a human organism we refer to [3] for the meaning of the symbols

$$\begin{aligned}
T'(t)&=a-bT(t)-cT_A(t)T(t),\\
T_B'(t)&=cT_A(t)T(t)-d_BT_L(t),\\
T_A'(t)&=gT_B(t)-d_AT_A(t).
\end{aligned}\tag{2.5}$$

As in Example 2.1 we can see that (1.1) is the discrete counterpart of (2.5) provided that $m=2$; $I=L=2$; $\varphi_1(y)=0$; $\phi_1(y)=\varphi_2(y)=cy$; $\phi_2(y)\equiv g$; $a_1=d_B$, $a_2=d_A$; $\alpha=a$; $\beta=b$. The correspondence between the variables of (1.1) and (2.5) is summarized by $y\leftrightarrow T$, $x_1\leftrightarrow T_B$; $x_2\leftrightarrow T_A$.

Let us note that this model is mathematically equivalent to the classical SIR model [4, model (2.5)].

Example 2.3 (see [5]). This represents the spread of HIV-I infection in a human organism

$$\begin{aligned}
T'(t)&=a-bT(t)-cV_I(t)T(t),\\
V_I'(t)&=g_1T^*(t)-d_1V_I(t),\\
T^{*\prime}(t)&=cV_I(t)T(t)-d_2T^*(t),\\
V_{NI}'(t)&=g_2T^*(t)-d_1V_{NI}(t).
\end{aligned}\tag{2.6}$$

Here we have $m = 3$; $I = 1$; $L = 2$; $\phi_1(y) \equiv g_1$; $\phi_2(y) = \psi_1(y) = cy$; $\phi_3(y) \equiv g_2$; $\psi_2(y) = \psi_3(y) = 0$; $a_1 = d_1$, $a_2 = d_2$; $a_3 = d_1$; $\alpha = a$; $\beta = b$. $y \leftrightarrow T$, $x_1 \leftrightarrow V_I$; $x_2 \leftrightarrow T^*$; $x_3 \leftrightarrow V_{NI}$.

It is worth to note that all the continuous models just proposed can be discretized by means of the discrete model proposed in [6]. The following example shows instead a continuous model that has not this property but can be discretized by means of (1.1).

Example 2.4 (see [5]). This represents the spread of HIV-I infection in a human organism, too

$$\begin{aligned} T'(t) &= \lambda - dT(t) - (1-\varepsilon)kV(t)T(t) - qT^*(t)T(t), \\ T^{*\prime}(t) &= (1-\varepsilon)kV(t)T(t) - \delta T^*(t), \\ V'(t) &= N_T\delta T^*(t) - cV(t). \end{aligned} \tag{2.7}$$

This continuous model cannot be discretized by means of the discrete model proposed in [6] because of the presence of the two nonlinear terms ($kV(t)T(t)$ and $qT^*(t)T(t)$) in the first equation. Instead, that can be done by means of (1.1) and we have $m = 2$; $I = 1$; $L = 2$; $\alpha = \lambda$; $a_1 = \delta$; $a_2 = c$; $\beta = d$; $\psi_1(y) = qy$; $\psi_2(y) = \phi_1(y) = (1-\varepsilon)ky$; $\phi_2(y) = N_T a_1$; $T \leftrightarrow y$; $T^* \leftrightarrow x_1$; $V \leftrightarrow x_2$.

3. Basic Properties

Since functions y and x_i $(i = 1, 2, \ldots, m)$ represent populations, at first, we can prove in the following two theorems their positivity and boundedness by using very natural hypotheses.

Theorem 3.1. *Assume that*

(i) $\alpha > 0$;

(ii) $\beta < 1$;

(iii) $a_i < 1$, $i = 1, \ldots, m$;

(iv) $\phi_i(y)$ *is not decreasing and* $\phi_i(0) \geq 0$, $i = 1, \ldots, m$;

(v) $y\psi_i(y) \geq 0$, $\forall y \in \mathbb{R}$, $i = 1, \ldots, m$;

(vi) $\exists$ *s.t.* $1 \leq \hat{\imath} \leq L$ *and* $\psi_{\hat{\imath}}$ *is strictly increasing.*

Then $y(n) > 0$, $x_i(n) > 0$, $n \geq 0$, $i = 1, \ldots, m$.

Proof. From (iii), (iv) and the positivity of $x_i(0)$ we have $x_i(1) > 0$, $i = 1, \ldots, m$. Now assume

$$y(1) \leq 0. \tag{3.1}$$

From (v), (vi), we get $\sum_{i=1}^{m}\psi_i(y(1))x_i(1) \leq 0$ and from (i), (ii) and the first of (1.1) we obtain $y(1) > 0$ which contradicts (3.1). The rest of the theorem can be proved in the same way (by induction). □

Theorem 3.2. *Assume that*

(i) $\alpha > 0$;

(ii) $0 < \beta < 1$;

(iii) $0 < a_i < 1$, $i = 1, \ldots, m$;

(iv) $\phi_i(y)$ *is not decreasing and* $\phi_i(0) \geq 0$, $i = 1, \ldots, m$;

(v) $y\varphi_i(y) \geq 0$, $\forall y \in \mathbb{R}$, $i = 1, \ldots, m$;

(vi) $\exists\, \hat{\imath}$ *s.t.* $1 \leq \hat{\imath} \leq L$ *and* $\varphi_{\hat{\imath}}$ *is strictly increasing;*

(vii) $\exists\, \bar{i}$ *s.t.* $1 \leq \bar{i} \leq L$ *and*

$$\exists\, q > 0 : \begin{cases} \phi_1(y) \leq q\varphi_L(y),\ y \geq 0 & \text{if } \bar{i} = 1, \\ \phi_{\bar{i}}(y) \leq q\varphi_{\bar{i}-1}(y),\ y \geq 0 & \text{if } 2 \leq \bar{i} \leq L. \end{cases} \tag{3.2}$$

Then, the sequences $\{y(n)\}$, $\{x_i(n)\}$, $i = 1, \ldots, m$ *are bounded.*

Proof. In order to prove this theorem it is convenient to represent (1.1) in the form of the following system of Volterra difference equations (see, e.g., [7, 8]):

$$\begin{aligned} y(n+1) &= \frac{\alpha}{\beta}\left[1 - (1-\beta)^{n+1}\right] + (1-\beta)^{n+1} y(0) - \sum_{l=1}^{n+1} (1-\beta)^{n+1-l} \sum_{i=1}^{m} \varphi_i(y(l)) x_i(l), \\ x_1(n+1) &= (1-a_1)^{n+1} x_1(0) + \sum_{l=0}^{n} (1-a_1)^{n-l} \phi_1(y(l)) x_L(l), \quad 1 \leq L \leq m, \\ x_i(n+1) &= (1-a_i)^{n+1} x_i(0) + \sum_{l=0}^{n} (1-a_i)^{n-l} \phi_i(y(l)) x_{i-1}(l), \quad i = 2, \ldots, m. \end{aligned} \tag{3.3}$$

Since the hypotheses of the previous theorem hold, positivity of the sequences $\{y(n)\}$, $\{x_i(n)\}$ is assured. From the first of (3.3) we obtain

$$y(n+1) \leq \frac{\alpha}{\beta}\left[1 - (1-\beta)^{n+1}\right] + (1-\beta)^{n+1} y(0) \leq \max\left\{\frac{\alpha}{\beta}, y(0)\right\} = y_M \tag{3.4}$$

and so the boundedness of y. From the first of (1.1) and (3.4) we also obtain for all $j = 1, \ldots, m$

$$\begin{aligned} \varphi_j(y(l)) x_j(l) &\leq \sum_{i=1}^{m} \varphi_i(y(l)) x_i(l), \\ &= \alpha + (1-\beta) y(l-1) - y(l) \\ &\leq \alpha + (1-\beta) y_M. \end{aligned} \tag{3.5}$$

Assume that there exists $q > 0$ such that

$$\phi_1(y) \leq q\varphi_L(y), \qquad y \geq 0. \tag{3.6}$$

From the second of (3.3), (ii), (iii) and (3.5) we have

$$
\begin{aligned}
x_1(n+1) &= (1-a_1)^{n+1}x_1(0) + \sum_{l=0}^{n}(1-a_1)^{n-l}\phi_1(y(l))x_L(l) \\
&\le (1-a_1)^{n+1}x_1(0) + q\sum_{l=0}^{n}(1-a_1)^{n-l}\psi_L(y(l))x_L(l) \\
&\le x_1(0) + q\frac{\alpha+(1-\beta)y_M}{a_1} \\
&= x_{1,M}.
\end{aligned}
\tag{3.7}
$$

Let us consider the third of (3.3) for $i = 2$. From the boundedness of x_1 and (iv) we have

$$
\begin{aligned}
x_2(n+1) &\le x_2(0) + x_{1,M}\phi_2(y_M)\sum_{l=0}^{n}(1-a_2)^{n-l} \\
&\le x_2(0) + \frac{x_{1,M}\phi_2(y_M)}{a_2} \\
&= x_{2,M}.
\end{aligned}
\tag{3.8}
$$

The boundedness of the remaining sequences can be proved in the same way.

If (3.6) does not hold then, from (i) and (vii), there exists $q > 0$ and $\bar{i}$ such that $2 \le \bar{i} \le m$ and

$$
\phi_{\bar{i}}(y) \le q\psi_{\bar{i}-1}(y), \quad y \ge 0. \tag{3.9}
$$

Thus, by the third of (3.3), (ii), (iii), and (3.5), the boundedness of $x_{\bar{i}}$ (and then the x_j sequences, $j > \bar{i}$) can be proved with the same argumentation used before for x_1 and x_2. Since by $1 \le \bar{i} \le L$, x_i is bounded, we obtain the boundedness of the remaining sequences x_j, $1 \le j \le \bar{i}$. □

In order to simplify the theorems' proofs of the remaining section, let us set

$$
\underline{x}_0 = \underline{x}_L = \liminf_{n\to\infty} x_L(n), \qquad \overline{x}_0 = \overline{x}_L \limsup_{n\to\infty} x_L(n) \tag{3.10}
$$

and introduce the following basic lemma.

Lemma 3.3. *Let one assume that hypotheses of Theorem 3.2 hold. Then*

$$\frac{\alpha-\sum_{i=1}^{m}\psi_i(\overline{y})\overline{x}_i}{\beta}\le \underline{y}\le\overline{y}\le\frac{\alpha-\sum_{i=1}^{m}\psi_i\left(\underline{y}\right)\underline{x}_i}{\beta}\le\frac{\alpha}{\beta},$$
$$\frac{\phi_i\left(\underline{y}\right)}{a_i}\underline{x}_{i-1}\le\underline{x}_i\le\overline{x}_i\le\frac{\phi_i(\overline{y})}{a_i}\overline{x}_{i-1},\qquad i=1,2,\dots,m,$$
$$\underline{x}_i\ge\left(\prod_{j=1}^{L}\frac{\phi_j\left(\underline{y}\right)}{a_j}\right)\underline{x}_i,$$
$$\overline{x}_i\le\left(\prod_{j=1}^{L}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_i,\quad i=1,2,\dots,L. \tag{3.11}$$

Proof. From (1.1) and Theorems 3.1 and 3.2, we easily have that

$$\underline{y}\ge\alpha+(1-\beta)\underline{y}-\sum_{i=1}^{m}\psi_i(\overline{y})\overline{x}_i \tag{3.12}$$

and then

$$\underline{y}\ge\frac{\alpha-\sum_{i=1}^{m}\psi_i(\overline{y})\overline{x}_i}{\beta}. \tag{3.13}$$

In the same way it can be proved that

$$\overline{y}\le\frac{\alpha-\sum_{i=1}^{m}\psi_i\left(\underline{y}\right)\underline{x}_i}{\beta}\le\frac{\alpha}{\beta}. \tag{3.14}$$

Also, we easily have that

$$\frac{\phi_i\left(\underline{y}\right)}{a_i}\underline{x}_{i-1}\le\underline{x}_i\le\overline{x}_i\le\frac{\phi_i(\overline{y})}{a_i}\overline{x}_{i-1},\qquad i=1,2,\dots,m, \tag{3.15}$$

and we have that

$$\begin{aligned}
\overline{x}_L &\leq \left(\prod_{j=1}^{L}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_L,\\
\overline{x}_i \leq \frac{\phi_i(\overline{y})}{a_i}\overline{x}_{i-1} \leq \cdots &\leq \left(\prod_{j=2}^{i}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_1 \leq \left(\prod_{j=1}^{i}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_L\\
&\leq \left(\prod_{j=1}^{i}\frac{\phi_j(\overline{y})}{a_j}\right)\frac{\phi_L(\overline{y})}{a_L}\overline{x}_{L-1} \leq \cdots \leq \left(\prod_{j=1}^{i}\frac{\phi_j(\overline{y})}{a_j}\right)\left(\prod_{j=i+1}^{L}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_i\\
&= \left(\prod_{j=1}^{L}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_i, \quad 1 \leq i \leq L,
\end{aligned}\tag{3.16}$$

and then

$$\overline{x}_i \leq \left(\prod_{j=1}^{L}\frac{\phi_j(\overline{y})}{a_j}\right)\overline{x}_i, \quad 1 \leq i \leq L. \tag{3.17}$$

Similarly, we have that

$$\underline{x}_i \geq \left(\prod_{j=1}^{L}\frac{\phi_j(\underline{y})}{a_j}\right)\underline{x}_i, \quad 1 \leq i \leq L. \tag{3.18}$$

The remaining parts are obtained similarly. □

Note that if there exists an integer $i \in \{1,2,\ldots,m\}$ such that $\underline{x}_i > 0$, then by the last inequalities of (3.11) in Lemma 3.3, we have that

$$\prod_{j=1}^{L}\frac{\phi_j(\underline{y})}{a_j} \leq 1 \leq \prod_{j=1}^{L}\frac{\phi_j(\overline{y})}{a_j}. \tag{3.19}$$

4. Asymptotic Properties

We assume that hypotheses of Theorem 3.2 hold and

(viii) $P(\lambda) \equiv \prod_{j=1}^{L}(\phi_j(\lambda)/a_j)$ is a strictly increasing positive continuous function of λ on $(0,+\infty)$ and $0 \leq P(0) < 1$,

and put

$$R_0 = P\left(\frac{\alpha}{\beta}\right). \tag{4.1}$$

We have the following theorems.

Theorem 4.1. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold. Then $R_0 \le 1$ if and only if the desease free equilibrium point $E_0 = (\alpha/\beta, 0, \dots, 0)$ is global asymptotically stable. In this case*

$$\lim_{n\to\infty} y(n) = \frac{\alpha}{\beta}, \qquad \lim_{n\to\infty} x_i(n) = 0, \quad i = 1, 2, \dots, m. \tag{4.2}$$

Proof. Sufficient condition ("⇒") is proved by Lemmas 4.4 and 4.5. Necessary condition ("⇐") is proved by Lemma 4.7. □

Theorem 4.2. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold. Then $R_0 > 1$ if and only if*

$$\overline{y} < \frac{\alpha}{\beta}, \qquad \underline{x}_i > 0, \quad 1 \le i \le m. \tag{4.3}$$

In this case, it holds that for λ^* defined in Lemma 4.6,

$$\underline{y} \le \lambda^* \le \overline{y}, \qquad x_i \le \frac{\alpha - \beta\lambda^*}{\sum_{l=1}^{m} \varphi_l(\lambda^*) \prod_{j=i+1}^{l} (\phi_j(\lambda^*)/a_j)} \le \overline{x}_i, \quad i = 1, 2, \dots, m. \tag{4.4}$$

Proof. By Theorem 4.1 and (3.11) in Lemma 3.3, we obtain (4.3). Moreover, by Lemma 4.6 and (3.19), we have $\underline{y} \le \lambda^* \le \overline{y}$ and by (3.11), we have (4.4). □

Theorem 4.3. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold and let one suppose that there exists a globally asymptotically stable endemic equilibrium point $(y^*, x_1^*, \dots, x_m^*)$, then $R_0 > 1$ if and only if there exists a unique solution $0 < \lambda = \lambda^* < \alpha/\beta$ such that $P(\lambda) = 1$, and*

$$\begin{gathered} y^* = \lambda^* < \frac{\alpha}{\beta}, \\ x_i^* \le \frac{\alpha - \beta\lambda^*}{\sum_{l=1}^{m} \varphi_l(\lambda^*) \prod_{j=i+1}^{l} (\phi_j(\lambda^*)/a_j)}, \quad i = 1, 2, \dots, m. \end{gathered} \tag{4.5}$$

Proof. By Lemmas 4.5 and 3.3, we obtain the thesis. □

For the proofs of Theorems 4.1–4.3, we need the following lemmas.

Lemma 4.4. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold. Then, if $R_0 < 1$, then*

$$\lim_{n\to\infty} y(n) = \frac{\alpha}{\beta}, \qquad \lim_{n\to\infty} x_i(n) = 0, \quad i = 1, 2, \dots, m. \tag{4.6}$$

If $R_0 = 1$, then

$$\overline{y} = \frac{\alpha}{\beta}. \tag{4.7}$$

Proof. If $P(\overline{y}) = \prod_{j=1}^{L}(\phi_j(\overline{y})/a_j) < 1$, then by (3.17) and Lemma 3.3, we have that $\underline{x}_i = \overline{x}_i = 0$, $1 \leq i \leq m$ and $\underline{y} = \overline{y} = \alpha/\beta$ which implies (4.6). Therefore, if $R_0 < 1$, then $P(\overline{y}) \leq P(\alpha/\beta) = R_0 < 1$ and hence, (4.6) holds.

If $R_0 = 1$, then $\overline{y} = \alpha/\beta$, because if $\overline{y} < \alpha/\beta$ then by the fact that $P(\lambda)$ is a strictly increasing positive function of λ on $(0, +\infty)$, we have that $P(\overline{y}) < P(\alpha/\beta) = R_0 = 1$ and by the above discussion, we obtain (4.6), which is a contradiction. □

Lemma 4.5. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold. Then $R_0 = 1$ implies (4.6).*

Proof. First let us assume

$$y(0) \leq \frac{\alpha}{\beta}. \tag{4.8}$$

From Lemma 4.4 we have $\overline{y} = \alpha/\beta$, so there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} y(n_k) = \overline{y}$. Let us define the two sets:

$$\begin{aligned} E_{gt} &= \{n_k : y(n_k - 1) \geq y(n_k)\}, \\ E_{lt} &= \{n_k : y(n_k - 1) \leq y(n_k)\}. \end{aligned} \tag{4.9}$$

Let us consider two cases with respect to the cardinality of the set E_{gt}.

Case 1 ($E_{gt} = \infty$). Let us consider the subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ corresponding to indexes belonging to E_{gt}.

In this case we easily see (from Lemma 3.3 $\overline{y} \leq \alpha/\beta$) that $\lim_{j\to\infty} y(n_{k_j} - 1) = \overline{y}$. From the first of (1.1) computed in n_{k_j}, $j = 1, 2, \ldots$ we have

$$y\left(n_{k_j}\right) = \alpha + (1-\beta)y\left(n_{k_j} - 1\right) - \sum_{i=1}^{m} \varphi_i\left(y\left(n_{k_j}\right)\right)x_i\left(n_{k_j}\right), \tag{4.10}$$

and as j goes to infinity:

$$\sum_{i=1}^{m} \varphi_i(\overline{y}) \lim_{j\to\infty} x_i\left(n_{k_j}\right) = 0. \tag{4.11}$$

We know that there exists $\hat{\imath}$ such that $\varphi_{\hat{\imath}}$ is strictly increasing, so from positivity of $\varphi_{\hat{\imath}}(\overline{y})$, we obtain:

$$\lim_{j\to\infty} x_{\hat{\imath}}\left(n_{k_j}\right) = 0. \tag{4.12}$$

Case 2 ($E_{gt} < \infty$). In this case we have $|E_{lt}| = \infty$. Let us consider the subsequence corresponding to indexes belonging to E_{lt}, name it $\{n_{k_j}\}_{j=1}^{\infty}$ again, so we have

$$y\left(n_{k_j}\right) - y\left(n_{k_j} - 1\right) \geq 0 \quad \forall k \in \mathbb{N}. \tag{4.13}$$

From the first of (1.1) we have

$$\beta y\left(n_{k_j}\right) = \alpha + (1-\beta)y\left(n_{k_j}-1\right) - (1-\beta)y\left(n_{k_j}\right) - \sum_{i=1}^{m}\varphi_i\left(y\left(n_{k_j}\right)\right)x_i\left(n_{k_j}\right) \tag{4.14}$$

and then

$$y\left(n_{k_j}\right) = \frac{\alpha - (1-\beta)\left[y\left(n_{k_j}\right) - y\left(n_{k_j}-1\right)\right] - \sum_{i=1}^{m}\varphi_i\left(y\left(n_{k_j}\right)\right)x_i\left(n_{k_j}\right)}{\beta} \leq \frac{\alpha}{\beta}. \tag{4.15}$$

As j goes to infinity, we obtain

$$\lim_{j\to\infty}\frac{\alpha - (1-\beta)\left[y\left(n_{k_j}\right) - y\left(n_{k_j}-1\right)\right] - \sum_{i=1}^{m}\varphi_i\left(y\left(n_{k_j}\right)\right)x_i\left(n_{k_j}\right)}{\beta} = \frac{\alpha}{\beta} \tag{4.16}$$

and then

$$\lim_{j\to\infty}\left((1-\beta)\left[y\left(n_{k_j}\right) - y\left(n_{k_j}-1\right)\right] + \sum_{i=1}^{m}\varphi_i\left(y\left(n_{k_j}\right)\right)x_i\left(n_{k_j}\right)\right) = 0. \tag{4.17}$$

This leads (from (4.13) and positivity of $\sum_{i=1}^{m}\varphi_i(y(n_{k_j}))x_i(n_{k_j})$) to

$$\begin{gathered}\lim_{j\to\infty} y\left(n_{k_j}\right) = \lim_{j\to\infty} y\left(n_{k_j}-1\right) = \overline{y} = \frac{\alpha}{\beta},\\ \lim_{j\to\infty}\sum_{i=1}^{m}\varphi_i\left(y\left(n_{k_j}\right)\right)x_i\left(n_{k_j}\right) = 0.\end{gathered} \tag{4.18}$$

Once again from this last statement, from the strict monotonousness of $\varphi_{\hat{\imath}}$ and Theorem 3.1 we obtain

$$\lim_{j\to\infty} x_{\hat{\imath}}\left(n_{k_j}\right) = 0. \tag{4.19}$$

So we proved in both cases that exists a sequence $\{n_{k_j}\}_j$ such that

$$\begin{gathered}\lim_{j\to\infty} y\left(n_{k_j}\right) = \lim_{j\to\infty} y\left(n_{k_j}-1\right) = \overline{y} = \frac{\alpha}{\beta},\\ \lim_{j\to\infty} x_{\hat{\imath}}\left(n_{k_j}\right) = 0.\end{gathered} \tag{4.20}$$

In the same way, we can prove that there exists a subsequence of $\{n_{k_j}\}_j$, for the sake of simplicity we name it $\{n_l\}_l$, such that

$$\lim_{l\to\infty} y(n_l - L) = \lim_{l\to\infty} y(n_l - L - 1) = \overline{y} = \frac{\alpha}{\beta}, \tag{4.21}$$

$$\lim_{l\to\infty} x_j(n_l - L) = 0, \quad j = 1,\ldots,L. \tag{4.22}$$

Let us consider the positive and bounded sequence

$$A(x(n)) = x_L(n) + \sum_{i=1}^{L-1}\prod_{j=i}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}x_i(n). \tag{4.23}$$

Assume $L \neq 1$ and compute its first difference, there results

$$\begin{aligned} A(x(n+1)) - A(x(n)) &= x_L(n)\left[-a_L + \prod_{j=1}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\phi_1(y(n))\right] \\ &\quad + \sum_{i=2}^{L-1}\prod_{j=i}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\left[-a_i x_i(n) + \phi_i(y(n))x_{i-1}(n)\right] + \phi_L(y(n))x_{L-1}(n) \\ &\quad - \prod_{j=1}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}a_1 x_1(n) \\ &= x_L(n)\left[-a_L + \prod_{j=1}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\phi_1(y(n))\right] \\ &\quad + \sum_{i=1}^{L-1}\prod_{j=i}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\left[-a_i x_i(n)\right] + \sum_{i=2}^{L}\prod_{j=i}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\phi_i(y(n))x_{i-1}(n), \end{aligned} \tag{4.24}$$

where $\prod_{j=i}^{i-1} = 1$. Hence

$$\begin{aligned} A(x(n+1)) - A(x(n)) &= x_L(n)\left[-a_L + \prod_{j=1}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\phi_1(y(n))\right] \\ &\quad - \sum_{i=1}^{L-1}\prod_{j=i+1}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\phi_{i+1}\left(\frac{\alpha}{\beta}\right)x_i(n) \\ &\quad + \sum_{i=1}^{L-1}\prod_{j=i+1}^{L-1}\frac{\phi_{j+1}(\alpha/\beta)}{a_j}\phi_{i+1}(y(n))x_i(n). \end{aligned} \tag{4.25}$$

As it can be easily seen this equality also holds for $L = 1$. From the first of (3.3) and (4.8) it is

$$y(n) \leq \frac{\alpha}{\beta}, \tag{4.26}$$

then, by taking into account the fact that ϕ_i is nondecreasing, we have

$$A(x(n+1)) - A(x(n)) \leq a_L x_L(n)\left[-1 + \frac{\phi_1(y(n))}{\phi_1(\alpha/\beta)}\prod_{j=1}^{L}\frac{\phi_j(\alpha/\beta)}{a_j}\right], \quad 1 \leq L \leq m \tag{4.27}$$

and from definition of R_0

$$A(x(n+1)) - A(x(n)) \leq a_L x_L(n)\left[-1 + \frac{\phi_1(y(n))}{\phi_1(\alpha/\beta)}R_0\right], \quad 1 \leq L \leq m. \tag{4.28}$$

Once again by recalling that ϕ_1 is nondecreasing and $R_0 = 1$ we have

$$A(x(n+1)) - A(x(n)) \leq 0, \quad 1 \leq L \leq m. \tag{4.29}$$

This implies that the sequence $A(x(n))$ is convergent. Since, from (4.22), $\lim_{l\to\infty} A(x(n_l - L)) = 0$, we obtain $\lim_{n\to\infty} A(x(n)) = 0$, and considering that $x_L(n) \leq A(x(n))$, we obtain

$$\lim_{n\to\infty} x_L(n) = 0. \tag{4.30}$$

Equation (4.6) can be easily obtained from this last statement, from Lemma 3.3 and (1.1).

Now let us consider the case

$$y(i) > \frac{\alpha}{\beta} \quad \forall i \in \mathbb{N}. \tag{4.31}$$

From the first of (1.1), we obtain

$$y(n+1) - y(n) = \alpha - \beta y(n) - \sum_{i=1}^{m} \varphi_i(y(n+1))x_i(n+1) \leq 0, \tag{4.32}$$

so the bounded sequence $y(n)$ is monotonic, then it converges. From this and Lemma 3.3 we obtain:

$$\lim_{n\to\infty} y(n) = \frac{\alpha}{\beta}. \tag{4.33}$$

This implies that

$$\lim_{n\to\infty}\sum_{i=1}^{m}\psi_i(y(n))x_i(n)=0. \tag{4.34}$$

We know that exists $\hat{\imath}$ such that $\psi_{\hat{\imath}}$ is strictly increasing, and so we have

$$\lim_{n\to\infty}x_{\hat{\imath}}(n)=0. \tag{4.35}$$

From (1.1) we obtain $\lim_{n\to\infty}x_j(n)=0$ for $j<\hat{\imath}$ and $\lim_{n\to\infty}x_L(n)=0$. Moreover, from Lemma 3.3 we obtain $\lim_{n\to\infty}x_k(n)=0$ for $k>\hat{\imath}$.

Hence, we have:

$$\lim_{n\to\infty}x_i(n)=0,\quad i=1,\dots,m. \tag{4.36}$$

Otherwise, if (4.31) does not hold, there exists $r\in\mathbb{N}$ such that $y(r)\le\alpha/\beta$ then we can use $y(r)$ as starting value instead of $y(0)$. □

Lemma 4.6. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold. Then, if $R_0>1$, then there exist a unique solution $0<\lambda=\lambda^*<\alpha/\beta$ of $P(\lambda)=1$ and positive constant solutions of (1.1) such that for any $n=0,1,2,\dots$*

$$\begin{gathered}y(n)=\lambda^*<\frac{\alpha}{\beta},\\ x_0(n):=x_L(n)=\frac{\alpha-\beta\lambda^*}{\sum_{l=1}^{m}\psi_l(\lambda^*)\prod_{j=1}^{l}\left(\dfrac{\phi_j(\lambda^*)}{a_j}\right)}>0,\\ x_i(n)=\prod_{j=1}^{i}\frac{\phi_j(\lambda^*)}{a_j}x_0(n),\qquad i=1,2,\dots,m.\end{gathered} \tag{4.37}$$

Proof. If $R_0=P(\alpha/\beta)>1$, then by the fact that $P(\lambda)$ is a strictly increasing positive function of λ on $(0,+\infty)$ and $0\le P(0)<1$, we have that $P(\lambda)=1$ has a unique solutions $0<\lambda=\lambda^*<\alpha/\beta$. Then, by Lemma 3.3 and (3.19), we can easily see that there are positive constant solutions of (1.1) defined by (4.37). □

From Lemmas 4.4–4.6, we obtain the following.

Lemma 4.7. *Let one assume that hypotheses of Theorem 3.2 and (viii) hold. Then (4.6) implies $R_0\le 1$.*

Proof. If $R_0=P(\alpha/\beta)>1$, then by Lemma 4.6, we can see that there are positive constant solutions of (1.1). Hence the thesis holds. □

References

[1] Z. Grossman, M. Feinberg, V. Kuznetsov, D. Dimitrov, and W. Paul, "HIV infection: how effective is drug combination treatment?" *Immunology Today*, vol. 19, no. 11, pp. 528–532, 1998.

[2] A. Ramani, A. S. Carstea, R. Willox, and B. Grammaticos, "Oscillating epidemics: a discrete-time model," *Physica A*, vol. 333, no. 1–4, pp. 278–292, 2004.

[3] L. Wang, M. Y. Li, and D. Kirschner, "Mathematical analysis of the global dynamics of a model for HTLV-I infection and ATL progression," *Mathematical Biosciences*, vol. 179, no. 2, pp. 207–217, 2002.

[4] H. W. Hethcote, "The mathematics of infectious diseases," *SIAM Review*, vol. 42, no. 4, pp. 599–653, 2000.

[5] D. S. Callaway and A. S. Perelson, "HIV-1 infection and low steady state viral loads," *Bulletin of Mathematical Biology*, vol. 64, no. 1, pp. 29–64, 2002.

[6] G. Izzo and A. Vecchio, "A discrete time version for models of population dynamics in the presence of an infection," *Journal of Computational and Applied Mathematics*, vol. 210, no. 1-2, pp. 210–221, 2007.

[7] M. R. Crisci, V. B. Kolmanovskii, E. Russo, and A. Vecchio, "Boundedness of discrete Volterra equations," *Journal of Mathematical Analysis and Applications*, vol. 211, no. 1, pp. 106–130, 1997.

[8] S. N. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 1996.