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Research Article

A Predator-Prey Gompertz Model with Time Delay and Impulsive Perturbations on the Prey

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We introduce and study a Gompertz model with time delay and impulsive perturbations on the prey. By using the discrete dynamical system determined by the stroboscopic map, we obtain the sufficient conditions for the existence and global attractivity of the "predator-extinction" periodic solution. With the theory on the delay functional and impulsive differential equation, we obtain the appropriate condition for the permanence of the system.

1. Introduction

It is well known that Gompertz equation [1] describes the growth law for a single species. The model reads as

$$\frac{dx(t)}{dt} = rx(t)\ln\frac{K}{x(t)}, \tag{1.1}$$

where $x(t)$ is the density of the population, r is a positive constant called the intrinsic growth rate, the positive constant K is usually referred to as the environment carrying capacity or saturation level, and $r\ \ln(K/x)$ denotes relative growth rate.

Many systems in physics, chemistry, biology, and information science have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes. This dynamical behavior can be modeled by impulsive differential equations. The theory of impulsive differential systems has been developed by numerous mathematicians [2–4]. But more realistic models should include some of the past states of these systems, that is, ideally, a real system should be modeled by differential equations with time delays. Recently, the investigation of impulsive delay differential equations is beginning [5–7].

In this paper, we need to consider a function

$$\phi(x) = \frac{x^2(t)}{1 + mx^2(t)}, \tag{1.2}$$

which is called Holling III functional response. The coefficient m is a positive constant and its biological meaning can be found in [8].

We introduce the model as follows:

$$\begin{aligned} &\frac{dx(t)}{dt} = rx(t)\ln\frac{K}{x(t)} - \frac{x^2(t)}{1+mx^2(t)}y(t), \\ &\frac{dy(t)}{dt} = \beta\frac{x^2(t-\tau)}{1+mx^2(t-\tau)}y(t-\tau) - dy(t), \end{aligned} \quad t \neq nT,\ n \in N,$$
$$\begin{aligned} &\Delta x(t) = -px(t), \\ &\Delta y(t) = 0, \end{aligned} \quad t = nT,\ n \in N, \tag{1.3}$$

where $x(t)$ and $y(t)$ are densities of the prey and the predator, respectively, r is the Gomportz intrinsic growth rate of the prey in the absence of the predator, β is the conversion rate, d is the death rate of the predator, τ is mean length of the digest period, and p $(0 < p < 1)$ represents impulsive harvest to preys by catching or pesticides at $t = nT$, $n \in N$.

The initial conditions for system (1.3) are

$$(\phi_1(s), \phi_2(s)) \in C_+ = C\left([-\tau, 0], R_+^2\right), \quad \phi_i(0) > 0 \ (i = 1, 2). \tag{1.4}$$

From the biological point of view, we only consider system (1.3) in the biological meaning region $D = \{(x, y) \mid x, y \geq 0\}$.

All outline of this paper is as follows. We give some basic knowledge in Section 2. In Section 3, using discrete dynamical system determined by the stroboscopic map, we obtain the existence and global attractivity of the "predator-extinction" periodic solution. In Section 4, with the theory of delay and impulsive different equations, we obtain the sufficient condition for the permanence of the system. In the last section, we give the numerical simulation and discussion.

2. Basic Knowledge

Let $R_+ = [0, \infty)$, $R_+^2 = \{x \in R^2 | x \geq 0\}$, $\Omega = \operatorname{int} R_+^2$, and N be the set of all nonnegative integers. Denote that $f = (f_1, f_2)^T$ is the map defined by the right-hand side of the first two equations of system (1.3). Let $V_0 = \{V : R_+ \times R_+^2 \to R_+$, continuous on $(nT, (n+1)T] \times R_+^2$, and $\lim_{(t,z)\to(nT^+,x)} V(t, z) = V(nT^+, x)$ exist$\}$.

Definition 2.1. Let $V \in V_0$, then for $(t, X) \in (nT, (n+1)T] \times R_+^2$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential system (1.3) is defined as

$$D^+V(t, X) = \limsup_{h \to 0^+} \frac{1}{h}\left[V(t+h, X + hf(t, x)) - V(t, X)\right]. \tag{2.1}$$

The solution of system (1.3) is a piecewise continuous function. The smoothness properties of f guarantee the global existence and uniqueness of the solution of system (1.3) [9].

Lemma 2.2 (see [9]). *Considering the following impulsive differential inequalities:*

$$\begin{aligned}\frac{dm(t)}{dt} &\le p(t)m(t)+q(t), \quad t\neq t_k,\ k=1,2\ldots,\\ m(w_k^+) &\le d_k m(t_k)+b_k, \quad t=t_k,\ t\ge t_0,\end{aligned}\tag{2.2}$$

where $p,q\in PC[R^+,R]$ and $d_k\ge 0$, b_k are constants, then

$$\begin{aligned}m(t)\le{}& m(t_0)\prod_{t_0<t_k<t} d_k\exp\left(\int_{t_0}^{t}p(s)ds\right)+\sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}d_j\exp\left(\int_{t_0}^{t}p(s)ds\right)\right)b_k\\ &+\int_{t_0}^{t}\prod_{s<t_k<t}d_k\exp\left(\int_{s}^{t}p(\sigma)d\sigma\right)q(s)ds, \quad t\ge t_0.\end{aligned}\tag{2.3}$$

Lemma 2.3. *There exists a constant $L=\beta rK(Ke^{d/r}-1)/d>0$ such that $x(t)\le K$ and $y(t)\le L$ for each solution of system* (1.3) *with t large enough.*

Proof. Define

$$V(t)=\beta x(t)+y(t+\tau),\tag{2.4}$$

then $V\in V_0$. Since $dx/dt\le rx(t)\ \ln(K/x(t))$, $dx/dt|_{x=K}=0$; in addition, $x(nT^+)\le x(nT)$. Thus $x(t)\le K$ for t large enough.

We calculate the upper right derivative of $V(t)$ along a solution of system (1.3) as

$$\begin{aligned}D^+V(t)+dV(t)&\le\beta x(t)r\ \ln\frac{Ke^{d/r}}{x(t)}\\ &=\beta x(t)r\ \ln Ke^{d/r}\\ &\le\beta rK\left(Ke^{d/r}-1\right).\end{aligned}\tag{2.5}$$

So we have that

$$\begin{aligned}D^+V(t)&\le -dV(t)+M, \quad t\neq nT,\ n\in N,\\ V(t^+)&\le V(t), \quad t=nT,\ n\in N.\end{aligned}\tag{2.6}$$

By Lemma 2.2, for $t \in (nT, (n+1)T]$, we have that

$$V(t) = \left(V(0^+) - \frac{\beta r K(Ke^{d/r} - 1)}{d}\right)\exp(-dt) + \frac{\beta r K(Ke^{d/r} - 1)}{d},$$
$$\lim_{t\to\infty} V(t) \le \frac{\beta r K(Ke^{d/r} - 1)}{d}. \tag{2.7}$$

So, $V(t)$ is ultimately bounded. Therefore by the definition of $V(t)$, there exists a constant $L = \beta r K(Ke^{d/r} - 1)/d > 0$ such that $x(t) \le K$ and $y(t) \le L$ for each solution of system (1.3) with t large enough. This completes the proof. □

Lemma 2.4 (see [10]). *Consider a delay equation*

$$\dot{x}(t) = ax(t-\tau) - bx(t), \tag{2.8}$$

where a, b, τ are all positive constants and $x(t) > 0$ for $-\tau \le t \le 0$. Thus one has the following.

(1) *If $a > b$, then* $\lim_{t\to\infty} x(t) = +\infty$.

(2) *If $a < b$, then* $\lim_{t\to\infty} x(t) = 0$.

3. "Predator-Extinction" Periodic Solution

3.1. Existence of a "Predator-Extinction" Periodic Solution

First, we begin analyzing the existence of a "predator-extinction" solution, in which predator is absent from the system, that is,

$$y(t) = 0, \quad t \ge 0. \tag{3.1}$$

In this condition, we know the growth of the prey in the time-interval $nT \le t \le (n+1)T$ and give some basic properties of the following subsystem of (1.3):

$$\frac{dx(t)}{dt} = rx(t)\ln\frac{K}{x(t)}, \quad t \ne nT,\ n \in N,$$
$$\Delta x(t) = -px(t), \quad t = nT,\ n \in N. \tag{3.2}$$

For system (3.2), we let $\varpi(t) = \ln x(t)$ and obtain a linear nonhomogeneous impulsive equation

$$\frac{d\varpi(t)}{dt} = -r\varpi(t) + r\ \ln K, \quad t \ne nT,\ n \in N,$$
$$\Delta\varpi(t) = \ln(1-p), \quad t = nT,\ n \in N. \tag{3.3}$$

Solving the first equation of system (3.3) between pulses yields

$$\varpi(t) = \ln K + (\varpi((n-1)T) - \ln K)e^{-r(t-(n-1)T)}, \quad (n-1)T < t \leq nT. \tag{3.4}$$

For the second equation of system (3.3), using discrete dynamical system determined by the stroboscopic map yields

$$\varpi(nT) = \ln K + (\varpi((n-1)T) - \ln K)e^{-rT} + \ln(1-p) =: f(\varpi(n-1)T), \tag{3.5}$$

where $f(\varpi) = \ln K + (\varpi - \ln K)e^{-rT} + \ln(1-p)$. From (3.5), we can see that this difference system has an equilibrium $\varpi_0^* = \ln K + \ln(1-p)/(1-e^{-rT})$, which implies that system (3.3) has a unique T-periodic solution

$$\varpi^*(t) = \ln K + \left(\varpi_0^* - \ln K\right)e^{-r(t-(n-1)T)}, \quad (n-1)T < t \leq nT. \tag{3.6}$$

Since $x_0^* = e^{\varpi_0^*}$, then $x^*(t) = e^{\varpi^*(t)}$ is the unique positive T-periodic solution of (3.2).

In the following, we will prove that $x^*(t)$ is globally asymptotically stable. By Lemma 2.3, we find that any solution of (3.2) is ultimately upper bounded, so we need only to prove that

$$\lim_{t\to\infty} |\varpi(t) - \varpi^*(t)| = 0, \tag{3.7}$$

where $\varpi^*(t)$ is the periodic solution of system (3.3) Suppose that with $\ln x^*(0) = \ln x_0^* = \varpi^*(0) = \varpi_0^*$.

$$|\varpi(t) - \varpi^*(t)| = \left|\varpi(0) - \varpi_0^*\right|e^{-rt} \leq e^{-(n-1)rT}, \tag{3.8}$$

since $\lim_{t\to\infty} e^{-(n-1)rT} = 0$; thus, $\lim_{t\to\infty}|\varpi(t) - \varpi^*(t)| = 0$.

Theorem 3.1. *System* (3.2) *has unique a positive periodic solution* $x^*(t)$ *which is globally asymptotically stable. That is, system* (1.3) *has a "predator-extinction" periodic solution* $(x^*(t), 0)$ *for* $t \in (nT, (n+1)T]$, $n \in N$.

3.2. Global Attractivity of the "Predator-Extinction" Periodic Solution

Denote that

$$\mathfrak{R}_1 = \frac{\beta K^2}{d\left[mK^2 + \exp(2\ln(1/1-p))/(\exp(rT)-1)\right]}. \tag{3.9}$$

Theorem 3.2. *If* $\mathfrak{R}_1 < 1$, *then the "predator-extinction" periodic solution* $(x^*(t), 0)$ *of system* (1.3) *is globally attractive.*

Proof. We denote that $(x(t), y(t))$ be any solution of system (1.3) with initial condition (1.4). From the second equation of system (1.3), we have that

$$\frac{dy(t)}{dt} \leq \frac{\beta}{m} y(t-\tau) - dy(t). \tag{3.10}$$

We consider the following comparison equation:

$$\frac{dz(t)}{dt} = \frac{\beta}{m} z(t-\tau) - dz(t). \tag{3.11}$$

If $\beta/m - d < 0$, then $\Re_1 < 1$; according to Lemma 2.4, we obtain $\lim_{t\to\infty} z(t) = 0$.

Since $y(s) = z(s) = \phi_2(s) > 0$ for all $s \in [-\tau, 0]$, by the comparison theorem in differential equation and the positivity of the solution, we have that $y(t) \to 0$ as $t \to \infty$.

In the following, we suppose that $\beta/m \geq d$. Since $\Re_1 < 1$, we have that

$$\frac{\beta K^2 \exp(2\ln(1-p)/(\exp(rT)-1))}{1 + mK^2 \exp(2\ln(1-p)/(\exp(rT)-1))} - d < 0. \tag{3.12}$$

Because the function $x^2/(1+mx^2)$ is monotonically increasing with respect to x, we choose a sufficiently small positive constant ϵ such that

$$\frac{\beta\left[K \exp(\ln(1-p)/(\exp(rT)-1)) + \epsilon\right]^2}{1 + m\left[K \exp(\ln(1-p)/(\exp(rT)-1)) + \epsilon\right]^2} - d < 0. \tag{3.13}$$

Noting that $dx(t)/dt \leq rx(t)\ln(K/x(t))$, $\Delta x(t) = -px(t)$ for $nT < t \leq (n+1)T$, then we consider the following comparison system:

$$\begin{gathered} \frac{dz(t)}{dt} = rz(t) \ln \frac{K}{z(t)}, \quad t \neq nT,\ n \in N, \\ z(t^+) = (1-p)z(t), \quad t = nT,\ n \in N, \\ z(0^+) = x(0^+). \end{gathered} \tag{3.14}$$

From Section 3.1, we see that

$$z^*(t) = \exp\left\{\ln K + \frac{\ln(1-p)}{1-e^{-rT}} e^{-r(t-(n-1)T)}\right\} \quad (n-1)T < t \leq nT, \tag{3.15}$$

which is unique a globally asymptotically stable positive T-periodic solution of system (3.14).

There exist a positive integer n_1 and an arbitrarily small positive constant ϵ for all $t \geq n_1 T$ such that

$$x(t) \leq z^*(t) + \epsilon \leq \exp\left\{\ln K + \frac{\ln(1-p)}{1-e^{-rT}} e^{-rT}\right\} + \epsilon =: \sigma. \tag{3.16}$$

From (3.16) and the second equation of system (1.3), for $t > n_1T + \tau$ we have

$$\frac{dy(t)}{dt} \leq \frac{\beta\sigma^2}{1+m\sigma^2} y(t-\tau) - dy(t). \tag{3.17}$$

Considering the following comparison equation:

$$\frac{dz(t)}{dt} = \frac{\beta\sigma^2}{1+m\sigma^2} z(t-\tau) - dz(t), \tag{3.18}$$

we can see (3.13), that is, $\beta\sigma^2/(1+m\sigma^2) < d$; according to Lemma 2.4, we obtain $\lim_{t\to\infty} z(t) = 0$.

Since $y(s) = z(s) = \phi_2(s) > 0$ for all $s \in [-\tau, 0]$, by the comparison theorem in differential equation and the positivity of the solution, we have that $y(t) \to 0$ as $t \to \infty$.

Next, we will prove that $x(t) \to x^*(t)$ as $t \to \infty$. Without loss of generality, we may assume that $0 < y(t) \leq \epsilon$ for all $t \geq 0$. By the first equation of system (1.3), we have that

$$\begin{aligned} \frac{dx(t)}{dt} &\geq rx(t)\ln\frac{K}{x(t)} - x^2(t)y(t) \\ &\geq rx(t)\ln\frac{K}{x(t)} - Kx(t)\epsilon \\ &\geq rx(t)\ln\frac{Ke^{-K\epsilon/r}}{x(t)}. \end{aligned} \tag{3.19}$$

Then we have that $\tilde{z}_1(t) \to x^*(t)$ as $\epsilon \to 0$ as $t \to \infty$, where $\tilde{z}_1(t)$ is unique a positive solution of

$$\begin{gathered} \frac{dz_1(t)}{dt} = rz_1(t)\ln\frac{Ke^{-K\epsilon/r}}{z_1(t)}, \quad t \neq nT,\ n \in N, \\ z_1(t^+) = (1-p)z_1(t), \quad t = nT,\ n \in N, \\ z_1(0^+) = x(0^+). \end{gathered} \tag{3.20}$$

From Section 3.1, we obtain

$$\tilde{z}_1(t) = \exp\left\{\ln\left(Ke^{-K\epsilon/r}\right) + \frac{\ln(1-p)}{1-e^{-rT}} e^{-r(t-nT)}\right\}, \quad nT < t \leq (n+1)T. \tag{3.21}$$

By using comparison theorem of impulsive differential equation [11], for any $\epsilon_1 > 0$, there exists $T_1 > 0$ and such that $t > T_1$

$$x(t) > \tilde{z}_1(t) - \epsilon_1. \tag{3.22}$$

On the other hand, from the first equation of (1.3), we have that $dx(t)/dt \leq rx(t)\ln(K/x(t))$.

Table 1: Critical values of some parameters of system (1.3) ($R_1 < 1$ must be satisfied).

The conditions for global attractivity of $(x^*(t), 0)$	
$p > p^*$	$p^* = 1 - 1/\sqrt{\left(\dfrac{\beta K^2}{d} - mK^2\right)^{e^{rT-1}}}$
$T < T^*$	$T^* = \dfrac{1}{r}\ln\left[1 - \dfrac{\ln(1-p)^2}{\ln(\beta K^2/d - mK^2)}\right]$
$m > m^*$	$m^* = \dfrac{\beta}{d} - \dfrac{1}{K^2}\exp\left(\dfrac{-\ln(1-p)^2}{e^{rT}-1}\right)$

Considering the following comparison system:

$$
\begin{aligned}
\frac{dz_2(t)}{dt} &= rz_2(t)\ln\frac{K}{z_2(t)}, \quad t \neq nT,\\
z_2(t^+) &= (1-p)z_2(t), \quad t = nT,\\
z_2(0^+) &= x(0^+),
\end{aligned}
\tag{3.23}
$$

we have that

$$
\widetilde{z}_2(t) = \exp\left\{\ln K + \frac{\ln(1-p)}{1-e^{-rT}}e^{-r(t-nT)}\right\}, \quad nT < t \leq (n+1)T, \tag{3.24}
$$

$$
x(t) < \widetilde{z}_2(t) + \epsilon_1, \tag{3.25}
$$

as $t \to \infty$ and $\widetilde{z}_2(t) = x^*(t)$. Let $\epsilon_1 \to 0$, then it follows from (3.22) and (3.25), that we can see $x^*(t) - \epsilon_1 < x(t) < x^*(t) + \epsilon_1$ for t large enough, which implies $x(t) \to x^*(t)$ as $t \to \infty$. The proof is completed. □

Now, we give the following theorem.

Theorem 3.3. (1) *If $\beta/m < d$, then the "predator-extinction" periodic solution $(x^*(t), 0)$ is globally attractive.*

(2) *If $\beta/m \geq d$ and $p > p^*$, $T < T^*$, $m > m^*$, then the "predator-extinction" periodic solution $(x^*(t), 0)$ is globally attractive, where the critical values p^*, T^*, m^* are listed in Table 1.*

4. Permanence

In the above section, we have proved that, when $p > p^*$, $T < T^*$, or $m > m^*$, the "predator-extinction" periodic solution $(x^*(t), 0)$ is globally attractive. But in natural world, the predator cannot be eradicated totally. In order to save resources, we need to keep the prey and predator coexisting when the prey does not bring about immense economic losses. Next, we will discuss the permanence of system (1.3).

Definition 4.1. System (1.3) is said to be uniformly persistent if there are positive constants m_i $(i = 1, 2)$ and a finite time T_0 such that for all solutions $(x(t), y(t))$ with initial values $x(0^+) > 0$, $y(0^+) > 0$, $x(t) \geq m_1$, and $y(t) \geq m_2$ hold for all $t > T_0$.

Definition 4.2. System (1.3) is said to be permanent if there exists a compact region $D \subset \operatorname{int}\Omega$, such that every solution of system (1.3) with initial condition (1.4) will eventually enter and remain in region D.

Denote that

$$\Re_2 = \frac{\beta K^2}{d\left[mK^2 + \exp(2\ln(1/1-p)/(1-\exp(-rT)))\right]}. \tag{4.1}$$

Theorem 4.3. *If* $\Re_2 > 1$, *then there exist two positive constants* m_1 *and* m_2 *such that* $x(t) \geq m_1$, $y(t) \geq m_2$ *for t large enough, that is, system* (1.3) *is uniformly persistent.*

Proof. Suppose that $(x(t), y(t))$ is any positive solution of system (1.3) with initial condition (1.4). From Lemma 2.3, we know that $x(t) \leq K$, $y(t) \leq L$.

Firstly, from the first equation of system (1.3), we know that $dx(t)/dt \geq rx(t)\ln(Ke^{-KL/r}/x(t))$. Now, we consider the following equation:

$$\begin{gathered}
\frac{dz_3(t)}{dt} = rz_3(t)\ln\frac{Ke^{-KL/r}}{z_3(t)}, \quad t \neq nT,\ n \in N, \\
z_3(t^+) = (1-p)z_3(t), \quad t = nT,\ n \in N, \\
z_3(0^+) = x(0^+).
\end{gathered} \tag{4.2}$$

therefore we have that

$$\tilde{z}_3(t) = \exp\left\{\ln\ Ke^{-KL/r} + \frac{\ln(1-p)}{1-e^{-rT}}e^{-r(t-nT)}\right\} \quad nT < t \leq (n+1)T. \tag{4.3}$$

There exists a $\epsilon > 0$ small enough such that, for sufficiently large t,

$$x(t) > \tilde{z}_3(t) - \epsilon \geq \exp\left\{\ln\left(Ke^{-KL/r}\right) + \frac{\ln(1-p)}{1-e^{-rT}}\right\} - \epsilon =: m_1. \tag{4.4}$$

Secondly, we will find m_2 such that $y(t) \geq m_2$.

The second equation of system (1.3) may be rewritten as follows:

$$\frac{dy(t)}{dt} = \left(\frac{\beta x^2(t)}{1+mx^2(t)} - d\right)y(t) - \frac{d}{dt}\int_{t-\tau}^{t}\frac{\beta x^2(\theta)}{1+mx^2(\theta)}y(\theta). \tag{4.5}$$

Define

$$V(t) = y(t) + \int_{t-\tau}^{t} \frac{\beta x^2(\theta)}{1 + m x^2(\theta)} y(\theta). \tag{4.6}$$

Calculating the derivative of $V(t)$ along the solution of (1.3), it follows from (4.5) that

$$\begin{aligned} \frac{dV(t)}{dt} &= \beta \frac{x^2(t-\tau)}{1 + m x^2(t-\tau)} y(t-\tau) - d y(t) + \frac{\beta x^2(t)}{1 + m x^2(t)} y(t) - \frac{\beta x^2(t-\tau)}{1 + m x^2(t-\tau)} y(t-\tau) \\ &= d\left(\frac{\beta}{d} \frac{x^2(t)}{1 + m x^2(t)} - 1 \right) y(t). \end{aligned} \tag{4.7}$$

Since $\Re_2 > 1$, then we can deduce that

$$\frac{r}{2K} \left[\frac{2\ln(1-p)}{1 - e^{-rT}} + \ln\left(K^2 \left(\frac{\beta}{d} - m \right) \right) \right] > 0. \tag{4.8}$$

Hence, we choose two positive constants m_2^* and ϵ_1 small enough such that

$$\frac{\beta}{d} \frac{\zeta^2}{1 + m\zeta^2} > 1, \tag{4.9}$$

where

$$\begin{aligned} \zeta &= K \exp\left\{ -\frac{K m_2^*}{r} \right\} \cdot \exp\left\{ \frac{\ln(1-p)}{1 - e^{-rT}} \right\} - \epsilon_1 > 0, \\ 0 &< m_2^* < \frac{r}{2K} \left[\frac{2\ln(1-p)}{1 - e^{-rT}} + \ln\left(K^2 \left(\frac{\beta}{d} - m \right) \right) \right]. \end{aligned} \tag{4.10}$$

In the following, we will find m_2 such that $y(t) \geq m_2$. There are two cases.

(1) We claim that the inequality $y(t) < m_2^*$ cannot hold for all $t > t_1$ $(t_1 > 0)$; otherwise, there is a positive constant t_1 such that $y(t) < m_2^*$ for all $t > t_1$.

From the first equation of system (1.3), we have that

$$\frac{dx(t)}{dt} \geq r x(t) \ln \frac{K e^{-K m_2^*/r}}{x(t)}. \tag{4.11}$$

We have that $x(t) \geq \widetilde{z}_4(t)$, where $\widetilde{z}_4(t)$ is unique a positive solution of

$$\begin{gathered}
\frac{dz_4(t)}{dt} = rz_4(t)\ln\frac{Ke^{-Km_2^*/r}}{z_4(t)}, \quad t \neq nT,\ n \in N, \\
z_4(t^+) = (1-p)z_4(t), \quad t = nT,\ n \in N, \\
z_4(0^+) = x(0^+).
\end{gathered} \tag{4.12}$$

From Section 3.1, we have that

$$\widetilde{z}_4(t) = \exp\left\{\ln\, Ke^{-Km_2^*/r} + \frac{\ln(1-p)}{1-e^{-rT}}e^{-r(t-nT)}\right\} \quad nT < t \leq (n+1)T. \tag{4.13}$$

By comparison theory [11], for any $\epsilon_1 > 0$, there exists a $T_1 > 0$, for $t > T_1$, such that

$$x(t) > \widetilde{z}_4(t) - \epsilon_1 \geq \exp\left\{\ln\, Ke^{-Km_2^*/r} + \frac{\ln(1-p)}{1-e^{-rT}}\right\} - \epsilon_1 =: \zeta. \tag{4.14}$$

From (4.7) and (4.14), we have that

$$\frac{dV(t)}{dt} \geq d\left(\frac{\beta}{d}\frac{\zeta^2(t)}{1+m\zeta^2(t)} - 1\right)y(t), \quad t \geq T_1. \tag{4.15}$$

Denote that

$$y_l = \min_{t\in[T_1,T_1+\tau]} y(t). \tag{4.16}$$

We show that $y(t) \geq y_l$ for all $t \geq T_1$. Otherwise, there exists a nonnegative constant T_2 such that $y(t) \geq y_l$, for $t \in [T_1, T_1+\tau+T_2)$, $y(T_1+\tau+T_2) = y_l$ and $\dot{y}(T_1+\tau+T_2) \leq 0$.

Thus, from the second equation of (1.3) and (4.12), (4.14), we can see that

$$\frac{dy(T_1+\tau+T_2)}{dt} > \left(\beta\frac{\zeta^2}{1+m\zeta^2} - d\right)y_l = d\left(\frac{\beta}{d}\frac{\zeta^2}{1+m\zeta^2} - 1\right)y_l > 0, \tag{4.17}$$

which is a contradiction to $y(t) \leq L$. Hence we get that $y(t) \geq y_l > 0$ for all $t \geq T_1$. Meanwhile, we can see that $dV(t)/dt > 0$, which implies that $V(t) \to \infty$, $t \to \infty$.

This is a contradiction to $V(t) \leq (1+\beta\tau K)L$. Thus, for any positive constant t_1, the inequality $y(t) \leq m_2^*$ cannot hold for all $t \geq t_1$.

(2) If $y(t) \geq m_2^*$ holds for all t large enough, then our aim is obtained. Otherwise, $y(t)$ is oscillatory about m_2^*. Let $m_2 = \min\{m_2^*/2, m_2^*e^{-d\tau}\}$.

We will prove that $y(t) \geq m_2$. There exist two positive constants $\bar{t}$, χ such that $y(\bar{t}) = y(\bar{t}+\chi) = m_2^*$ and $y(t) < m_2^*$ for $t \in [\bar{t}, \bar{t}+\chi]$.

When $\bar{t}$ is large enough, the inequality $x(t) > \sigma$ holds true for $t \in [\bar{t}, \bar{t}+\chi]$.

Table 2: Critical values of some parameters of system (1.3) ($R_2 > 1$ must be satisfied).

The conditions for the permanence of system (1.3)	
$p < p_*$	$p^* = 1 - 1/\sqrt{\left(\frac{\beta K^2}{d} - mK^2\right)^{1-e^{-rT}}}$
$T > T_*$	$T^* = -\frac{1}{r}\ln\left[1 + \frac{\ln(1-p)^2}{\ln(\beta K^2/d - mK^2)}\right]$
$m < m_*$	$m^* = \frac{\beta}{d} - \frac{1}{K^2}\exp\left\{\frac{-\ln(1-p)^2}{1-e^{-rT}}\right\}$

Since $y(t)$ is continuous and bounded and does not have impulsive effort, we conclude that $y(t)$ is uniformly continuous. There exist a constant T_3 ($0 < T_3 < \tau$ and T_3 is independent to $\bar{t}$) such that $y(t) > m_2^*/2$ for all $t \in [\bar{t}, \bar{t} + T_3]$.

If $\chi < T_3$, then our aim is obtained.

If $T_3 < \chi \leq \tau$, from the second equation of (1.3), we have that $dy(t)/dt \geq -dy(t)$ for $t \in [\bar{t}, \bar{t} + \chi]$, and then we have that $y(t) \geq m_2^* e^{-d\tau}$ for $\bar{t} < t \leq \bar{t} + \chi \leq \bar{t} + \tau$. Therefore, $y(t) \geq m_2$ for $t \in [\bar{t}, \bar{t} + \tau]$.

If $\chi > \tau$, by the second equation of system (1.3), we have that $y(t) \geq m_2$ for $t \in [\bar{t}, \bar{t} + \tau]$, the same as above claim; we can obtain $y(t) \geq m_2$ for $t \in [\bar{t} + \tau, \bar{t} + \chi]$.

Since the interval $[\bar{t}, \bar{t} + \chi]$ is arbitrarily chosen, we get that $y(t) \geq m_2$ for t large enough.

This proof is complete. □

Theorem 4.4. *If* $\mathfrak{R}_2 > 1$, *then system* (1.3) *is permanent.*

Proof. Suppose that $(x(t), y(t))$ is any solution of system (1.3) with initial condition (1.4). By Theorem 4.3, there exist positive constants m_1, m_2, and T' such that $x(t) \geq m_1$, $y(t) \geq m_2$ for $t \geq T'$. Set

$$D = \left\{(x, y) \in R_+^2 \mid m_1 \leq x(t) \leq K, m_2 \leq y(t) \leq L\right\}. \tag{4.18}$$

Then D is a bounded compact region and $D \subset \operatorname{int}\Omega$. By Theorem 4.3, every solution of system (1.3) with initial condition (1.4) eventually enters and remains in region D.

The proof is complete. □

Theorem 4.5. *If* $p < p_*$, $T > T_*$, *or* $m < m_*$, *then system* (1.3) *is permanent, where the critical values* p_*, T_*, m_* *are listed in Table 2.*

5. Numerical Simulation and Discussion

In this paper, we introduce and discuss a predator-prey system model with Holling III response functional under time delay on the predator and impulsive perturbations on the prey. From Section 3, there exists a predator-extinction periodic solution of system (1.3); when

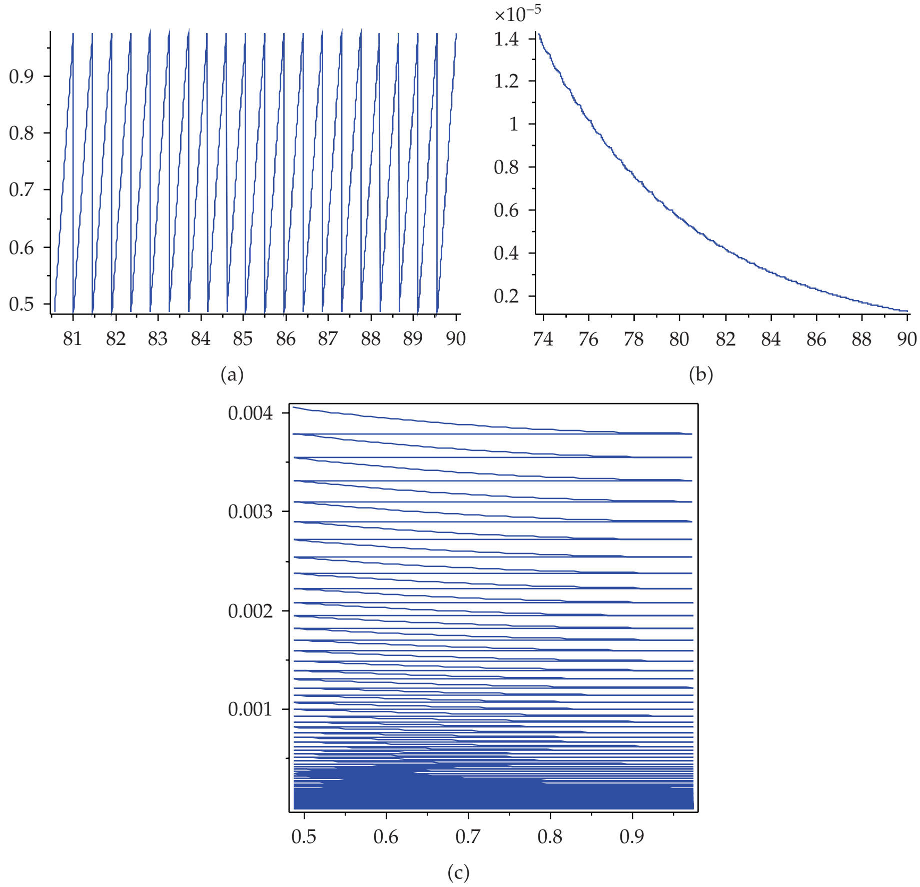

Figure 1

$\mathfrak{R}_1 < 1$, the predator eradication periodic solution is globally attractive. From Section 4, when $\mathfrak{R}_2 > 1$, system (1.3) is permanent.

In the following, we will analyze the influence of them on the dynamics of system (1.3). We consider the hypothetical set of parameter values as $r = 1.5$, $K = 2$, $\beta = 0.6$, $d = 0.4$, and $\tau = 0$.

Figure 1 is the dynamical behavior of system (1.3) with $r = 1.5$, $K = 2$, $\beta = 0.6$, $d = 0.4$, $\tau = 0$, $p = 0.5$, $T = 0.45$, and $\mathfrak{R}_1 = 0.9658214856 < 1$. (a) is the timeseries of prey population (x) for periodic oscillation; (b) is the timeseries of population (y) for extinction; (c) is the phase portrait of the prey and the predator population for global attractivity of the "predator-eradication" periodic solution.

Figure 2 is the dynamical behavior of system (1.3) with $r = 1.5$, $K = 2$, $\beta = 0.6$, $d = 0.4$, $\tau = 0$, $p = 0.2$, $T = 1$, and $\mathfrak{R}_2 = 2.083554740 > 1$. (a) is the timeseries of prey population (x) for permanence; (b) is the timeseries of population (y) for permanence; (c) is the phase portrait (T-periodic solution) of the prey and the predator population of system (1.3).

By Theorems 3.2 and 4.3, we know that, when $\mathfrak{R}_1 = 0.9658214856 < 1$, the "predator-eradication" periodic solution is globally attractive (Figure 1); when $\mathfrak{R}_2 = 2.083554740 > 1$, system (1.3) is permanence (Figures 2(a) and 2(b)).

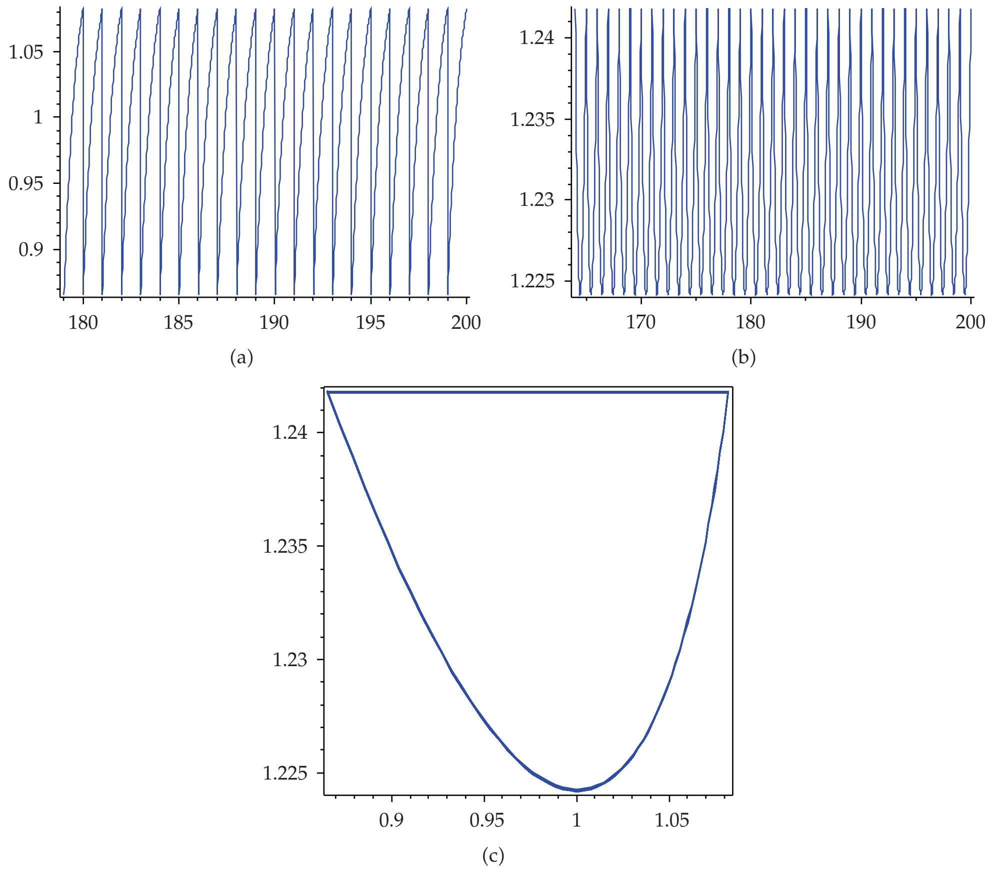

Figure 2

In Figure 1(b), we know that the predator population dies ultimately; we know that, although impulsive catching is larger or the period of pulsing is shorter, we kill the prey largely, and the predator population will decrease largely. It is very difficult for it to prey on prey; the predator can die out earlier than the prey. From Figure 2, we suppose that a smaller impulsive catching rate or a longer period can cause the prey and predator populations to coexist.

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