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*Research Article*

# Existence of Solutions to Boundary Value Problems for the Discrete Generalized Emden-Fowler Equation

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We consider the existence of solutions to boundary value problems for the discrete generalized Emden-Fowler equation. By means of the minimax methods in the critical point theory, some new results are obtained. Two examples are also given to illustrate the main results.



## 1. Introduction and Statements of Main Results

Let $\mathbf{Z}$ and $\mathbf{R}$ be the sets of all integers and real numbers, respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}[a] = \{a, a+1, \ldots\}$, $\mathbf{Z}[a,b] = \{a, a+1, \ldots, b\}$ when $a \leq b$.

In this paper, we consider the following boundary value problem (BVP for short) consisting of the discrete generalized Emden-Fowler equation:

$$\Delta\left[p(t)\Delta u(t-1)\right] + q(t)u(t) = f(t, u(t)), \quad t \in \mathbf{Z}[1,k], \tag{1.1}$$

and the boundary value conditions:

$$u(0) + \alpha u(1) = A, \qquad u(k+1) + \beta u(k) = B, \tag{1.2}$$

where $k$ is a positive integer, $\alpha$, $\beta$, $A$, and $B$ are constants, and $\Delta u(t) = u(t+1) - u(t)$ is the forward difference operator. We assume that $p(t)$ is nonzero and realvalued for each $t \in \mathbf{Z}[1, k+1]$. We also assume that $q(t)$ is realvalued for each $t \in \mathbf{Z}[1,k]$, $f(t,x)$ is realvalued for each $(t,x) \in \mathbf{Z}[1,k] \times \mathbf{R}$, and $f(t,x)$ is continuous in the second variable $x$.

Equation (1.1) has been extensively studied by many authors; for example, see [1–9] concerning its disconjugacy, disfocality, oscillation, asymptotic behaviour, existence of periodic solutions, and solutions to boundary value problem.

Recently, Yu and Guo in [10] employed the critical point theory to obtain the existence of solutions to the BVP (1.1)-(1.2). Motivated by this and the results in [11], the main purpose of this paper is to give some new sufficient conditions for the existence of solutions to the BVP (1.1)-(1.2) by applying the Saddle Point Theorem and the Least Action Principle.

Before giving the main results, we first set

$$F(t,x)=\int_0^x f(t,s)ds, \qquad c(t)=q(t)-p(t)-p(t+1),$$
$$M=\begin{pmatrix} c(1)-\alpha p(1) & p(2) & 0 & \cdots & 0 & 0 \\ p(2) & c(2) & p(3) & \cdots & 0 & 0 \\ 0 & p(3) & c(3) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c(k-1) & p(k) \\ 0 & 0 & 0 & \cdots & p(k) & c(k)-\beta p(k+1) \end{pmatrix},$$
$$\eta=(\eta(1),\eta(2),\ldots,\eta(k))^{\tau}=\begin{pmatrix} p(1)A \\ 0 \\ \vdots \\ 0 \\ p(k+1)B \end{pmatrix}. \tag{1.3}$$

The main results are as follows.

**Theorem 1.1.** *Suppose that $f$, $M$ satisfy the following assumptions.*

($F_1$) *There are constants $C_1>0$, $C_2>0$, $1/2\le\theta<1$ such that for all $(t,x)\in\mathbf{Z}[1,k]\times\mathbf{R}$,*

$$|f(t,x)|\le C_1|x|^{\theta}+C_2. \tag{1.4}$$

($F_2$) *One has*

(i) *either $\|u\|^{-2\theta}\sum_{t=1}^{k}F(t,u(t))\to-\infty$ as $\|u\|\to\infty$, or*
(ii) $\|u\|^{-2\theta}\sum_{t=1}^{k}F(t,u(t))\to+\infty$ *as $\|u\|\to\infty$,*

*where for all $u=(u(1),u(2),\ldots,u(k))^{\tau}\in\mathbf{R}^k$, $\|u\|=(\sum_{t=1}^{k}|u(t)|^2)^{1/2}$.*

($P_1$) *The matrix $M$ is singular.*

*Then the BVP* (1.1)-(1.2) *has at least one solution.*

*Remark 1.2.* There are functions $p(t)$, $q(t)$, and $f(t,x)$ satisfying our Theorem 1.1 and not satisfying the results in [10]. For example, let $A$, $B$ be arbitrary constants, $k = 4$, $\alpha = \beta = -1$, $\theta = 1/2$ and

$$p(t) \equiv 1, \qquad q(t) \equiv 0, \qquad f(t,x) = g(t)\frac{x}{\sqrt[4]{1+x^2}} + 1, \tag{1.5}$$

where $g(t) < 0$ for every $t \in \mathbf{Z}[1,4]$. It is easy to verify that $(F_1)$, $(F_2)$(i), and $(P_1)$ are satisfied. Then the BVP (1.1)-(1.2) has at least one solution. And it is easy to see that this solution is a nonzero solution since $f(t,0) \neq 0$.

**Theorem 1.3.** *Suppose that $f$, $M$ satisfy the following assumptions.*

$(F_3)$ *For any $t \in \mathbf{Z}[1,k]$, $\lim_{|x|\to\infty}(f(t,x)/x) = 0$.*

$(P_2)$ *The matrix $M$ is nonsingular.*

*Then the BVP* (1.1)-(1.2) *has at least one solution.*

**Corollary 1.4.** *Suppose that $f$, $M$ satisfy $(F_1)$ and $(P_2)$. Then the BVP* (1.1)-(1.2) *has at least one solution.*

*Remark 1.5.* Since for all $(t,x) \in \mathbf{Z}[1,k] \times \mathbf{R}$, $|f(t,x)| \leq C_2$ implies $|f(t,x)| \leq C_1|x|^{\theta} + C_2$, our Corollary 1.4 extends Theorem 3.2 in [10].

**Theorem 1.6.** *Suppose that $f$, $M$ satisfy the following assumptions.*

$(F_4)$ *For any $t \in \mathbf{Z}[1,k]$, $\lim_{|x|\to\infty}(F(t,x)/x^2) = 0$.*

$(F_5)$ *One has*

(i) *either $2F(t,x) - (f(t,x) + \eta(t))x \to -\infty$ as $|x| \to \infty$ for all $t \in \mathbf{Z}[1,k]$, or*

(ii) *$2F(t,x) - (f(t,x) + \eta(t))x \to +\infty$ as $|x| \to \infty$ for all $t \in \mathbf{Z}[1,k]$.*

$(P_3)$ *The matrix $M$ is singular and indefinite.*

*Then the BVP* (1.1)-(1.2) *has at least one solution.*

*Remark 1.7.* There are functions $p(t)$, $q(t)$, and $f(t,x)$ satisfying our Theorem 1.6 and not satisfying the results in [10]. For example, let $A$, $B$ be arbitrary constants, $k = 4$, $\alpha = 1$, $\beta = 0$ and

$$\begin{gathered} p(1) = p(3) = -1, \qquad p(2) = p(4) = 1, \qquad p(5) = -2, \qquad q(t) \equiv 0, \\ f(t,x) = g(t)\left(\frac{x}{\sqrt{1+x^2}}\ln\left(1+x^2\right) + \sqrt{1+x^2}\frac{2x}{1+x^2} + 1\right), \end{gathered} \tag{1.6}$$

where $g(t) < 0$ for every $t \in \mathbf{Z}[1,4]$. It is easy to verify that $(F_4)$, $(F_5)$(i) and $(P_3)$ are satisfied. Then the BVP (1.1)-(1.2) has at least one solution. And it is easy to see that this solution is a nonzero solution since $f(t,0) \neq 0$.

**Corollary 1.8.** *Assume that $(F_4)$ holds. If one of the following conditions is satisfied: $(H_1)$ the matrix $M$ is negative semi-definite, and $(F_5)$(i) holds, $(H_2)$ the matrix $M$ is positive semi-definite, and $(F_5)$(ii) holds, then, the BVP* (1.1)-(1.2) *has at least one solution.*

## 2. Variational Structure and Two Basic Lemmas

Let $\mathbf{R}^k$ be the real Euclidean space with dimension $k$. For any $u, v \in \mathbf{R}^k$, $\|u\|$ and $(u, v)$ denote the usual norm and inner product in $\mathbf{R}^k$, respectively.

Define the functional $J$ on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2}(Mu, u) + (\eta, u) - \sum_{t=1}^{k} F(t, u(t)), \quad \forall u = (u(1), u(2), \ldots, u(k))^\tau \in \mathbf{R}^k. \tag{2.1}$$

It is well known that $u = (u(1), u(2), \ldots, u(k))^\tau \in \mathbf{R}^k$ is a critical point of $J$ if and only if $\{u(t)\}_{t=0}^{k+1} = (u(0), u(1), u(2), \ldots, u(k), u(k+1))^\tau$ is a solution of the BVP (1.1)-(1.2), where $u(0) = A - \alpha u(1)$ and $u(k+1) = B - \beta u(k)$. For details, see [10]. It follows from the continuity of $f$ that $J$ is continuously differentiable on $\mathbf{R}^k$. Moreover, one has

$$(J'(u), h) = (Mu, h) + (\eta, h) - \sum_{t=1}^{k} f(t, u(t)) \cdot h(t), \quad \forall u, h \in \mathbf{R}^k. \tag{2.2}$$

When the matrix $M$ is singular and indefinite, we suppose that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_l$ and $0 > -\mu_1 \ge -\mu_2 \ge \cdots \ge -\mu_s$ are the positive and negative eigenvalues of $M$, respectively, and $l + s = k - 1$. We also suppose that $\varsigma_i$, $1 \le i \le l$ and $\xi_j$, $1 \le j \le s$ are the eigenvectors of $M$ corresponding to eigenvalues $\lambda_i$, $1 \le i \le l$ and $-\mu_j$, $1 \le j \le s$ satisfying

$$(\varsigma_{i_1}, \varsigma_{i_2}) = \begin{cases} 0, & i_1 \neq i_2, \\ 1, & i_1 = i_2, \end{cases} \qquad (\varsigma_i, \xi_j) = 0, \qquad (\xi_{j_1}, \xi_{j_2}) = \begin{cases} 0, & j_1 \neq j_2, \\ 1, & j_1 = j_2, \end{cases} \tag{2.3}$$

where $i, i_1, i_2 \in \mathbf{Z}[1, l]$, $j, j_1, j_2 \in \mathbf{Z}[1, s]$. We denote

$$R^+ = \operatorname{span}\{\varsigma_i \mid i \in \mathbf{Z}[1, l]\}, \qquad R^- = \operatorname{span}\{\xi_j \mid j \in \mathbf{Z}[1, s]\}, \tag{2.4}$$

$$R^0 = \left(R^+ \oplus R^-\right)^\perp. \tag{2.5}$$

Then $\mathbf{R}^k$ has the direct sum decomposition $\mathbf{R}^k = R^+ \oplus R^0 \oplus R^-$. So, for each $u \in \mathbf{R}^k$, $u$ can be expressed by

$$u = u^+ + u^0 + u^-, \tag{2.6}$$

where $u^* \in R^*$, $*$ is $+, 0$, or $-$, respectively. Furthermore, we have the following estimates:

$$\begin{aligned} \lambda_1 \|u^+\|^2 &\le (Mu^+, u^+) \le \lambda_l \|u^+\|^2, \quad u^+ \in R^+, \\ -\mu_s \left\|u^-\right\|^2 &\le \left(Mu^-, u^-\right) \le -\mu_1 \left\|u^-\right\|^2, \quad u^- \in R^-. \end{aligned} \tag{2.7}$$

Set

$$\|u\|_\infty = \max_{t\in\mathbf{Z}[1,k]}|u(t)|, \qquad \|u\|_p = \left(\sum_{t=1}^{k}|u(t)|^p\right)^{1/p}, \tag{2.8}$$

where $u = (u(1), u(2), \dots, u(k))^\tau \in \mathbf{R}^k$ and $\infty > p \ge 1$. Then for any $u \in \mathbf{R}^k$,

$$\frac{1}{\sqrt{k}}\|u\| \le \|u\|_\infty \le \|u\|, \qquad \frac{1}{\sqrt{k}}\|u\| \le \|u\|_p \le k^{1/p}\|u\|. \tag{2.9}$$

We will make use of the least action principle and saddle point theorem to obtain the critical points of $J$. Let us first recall these theorems.

**Lemma 2.1** (the least action principle, see [12]). *Let $X$ be a real Banach space, and assume that $J \in C^1(X,\mathbf{R})$ is bounded from below in $X$ and satisfies the Palais-Smale condition ((PS) condition for short). Then $c = \inf_{u\in X} J(u)$ is a critical value of $J$.*

**Lemma 2.2** (saddle point theorem, see [13]). *Let $X$ be a real Hilbert space, $X = X_1 \oplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(X,\mathbf{R})$ satisfies the (PS) condition and*

- ($\varphi_1$) *there exist constants $\sigma, \rho > 0$ such that $J|_{\partial B_\rho \cap X_1} \le \sigma$, where $B_\rho = \{u \in X \mid \|u\| < \rho\}$, $\partial B_\rho$ denotes the boundary of $B_\rho$;*
- ($\varphi_2$) *there exist $e \in B_\rho \cap X_1$ and a constant $\omega > \sigma$ such that $J|_{e+X_2} \ge \omega$.*

*Then $J$ possesses a critical value $c \ge \omega$ and*

$$c = \inf_{h\in\Phi}\max_{u\in B_\rho\cap X_1} J(h(u)), \tag{2.10}$$

*where $\Phi = \{h \in C(\overline{B}_\rho \cap X_1, X) \mid h|_{\partial B_\rho \cap X_1} = id\}$.*

*Remark 2.3.* As shown in [14], a deformation lemma can be proved with the weaker condition ($C$) replacing the usual (PS) condition, and it turns out that the saddle point theorem holds under condition ($C$).

## 3. Proofs of the Main Results

In order to prove Theorem 1.1, we need to prove the following lemma.

**Lemma 3.1.** *Assume that conditions ($F_1$), ($F_2$), and ($P_3$) hold. Then the functional $J$ (see (2.1)) satisfies the (PS) condition; that is, for any sequence $\{u_m\}$ such that $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$, there exists a subsequence of $\{u_m\}$ which is convergent in $\mathbf{R}^k$.*

*Proof.* First suppose that ($F_1$), ($F_2$)(i), and ($P_3$) hold. Recall that $\mathbf{R}^k$ is a finite dimensional Hilbert space. Consequently, in order to prove that $J$ satisfies the (PS) condition, we only need to prove that $\{u_m\}$ is bounded. Let $\{u_m\}$ be a sequence in $\mathbf{R}^k$ such that $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$. Then there exist $C_3 > 0$ and $m_0 \in \mathbf{Z}[1]$ such that

$$|J(u_m)| \le C_3, \qquad \left|\left(J'(u_m), h\right)\right| \le \|h\| \tag{3.1}$$

for all $m > m_0$, $h \in \mathbf{R}^k$.

Since $M$ is singular and indefinite, we write $u_m = u_m^+ + u_m^0 + u_m^-$ with $u_m^* \in R^*$, where $* = +, 0, -$, respectively. By ($F_1$), (2.9), and Hölder$'s$ inequality ($p = 1/\theta, q = 1/(1-\theta)$), we have

$$\begin{aligned}
&\left|\sum_{t=1}^{k} f(t, u_m(t)) \cdot \left(u_m^-(t) - u_m^+(t)\right)\right| \\
&\quad\le \sum_{t=1}^{k} \left(C_1 \left|u_m^0(t) + \left(u_m^+(t) + u_m^-(t)\right)\right|^{\theta} + C_2\right) \cdot \left|u_m^-(t) - u_m^+(t)\right| \\
&\quad\le 2C_1 \left\|u_m^0\right\|_{\infty}^{\theta} \sum_{t=1}^{k} \left|u_m^-(t) - u_m^+(t)\right| + 2C_1 \sum_{t=1}^{k} \left|u_m^+(t) + u_m^-(t)\right|^{\theta} \left|u_m^-(t) - u_m^+(t)\right| \\
&\qquad + C_2 \sum_{t=1}^{k} \left|u_m^-(t) - u_m^+(t)\right| \le \frac{2C_1^2 k}{\tau_1} \left\|u_m^0\right\|_{\infty}^{2\theta} + \frac{\tau_1}{2} \left\|u_m^- - u_m^+\right\|^2 \\
&\qquad + 2C_1 \left\|u_m^+ + u_m^-\right\|_1^{\theta} \left\|u_m^- - u_m^+\right\|_q + C_2 \sqrt{k} \left\|u_m^- - u_m^+\right\| \\
&\quad\le \frac{2C_1^2 k}{\tau_1} \left\|u_m^0\right\|^{2\theta} + \frac{\tau_1}{2} \left\|u_m^- - u_m^+\right\|^2 + 2C_1 k \left\|u_m^+ + u_m^-\right\|^{\theta} \left\|u_m^- - u_m^+\right\| \\
&\qquad + C_2 \sqrt{k} \left\|u_m^- - u_m^+\right\|,
\end{aligned} \tag{3.2}$$

where $\tau_1 = \min\{\lambda_1, \mu_1\}$. On the other hand, by the fact that $u_m^+$ and $u_m^-$ are mutually orthogonal, one has $\|u_m^- - u_m^+\| = \|u_m^+ + u_m^-\|$. Hence we have

$$\begin{aligned}
\left|\sum_{t=1}^{k} f(t, u_m(t)) \cdot \left(u_m^-(t) - u_m^+(t)\right)\right| &\le \frac{2C_1^2 k}{\tau_1} \left\|u_m^0\right\|^{2\theta} + \frac{\tau_1}{2} \left\|u_m^+ + u_m^-\right\|^2 \\
&\quad + 2C_1 k \left\|u_m^+ + u_m^-\right\|^{\theta+1} + C_2 \sqrt{k} \left\|u_m^+ + u_m^-\right\|.
\end{aligned} \tag{3.3}$$

Similarly to (3.3), we have

$$\left|\sum_{t=1}^{k}\left[F(t,u_m(t))-F\left(t,u_m^0(t)\right)\right]\right| \leq \frac{2C_1^2k}{\mu_s}\left\|u_m^0\right\|^{2\theta}+\frac{\mu_s}{2}\|u_m^+ + u_m^-\|^2 + 2C_1k\|u_m^+ + u_m^-\|^{\theta+1}+C_2\sqrt{k}\|u_m^+ + u_m^-\|. \tag{3.4}$$

Take $h = u_m^- - u_m^+$ in (2.2). Then

$$-(Mu_m, u_m^- - u_m^+) = -(J'(u_m), u_m^- - u_m^+) + (\eta, u_m^- - u_m^+) - \sum_{t=1}^{k} f(t,u_m(t))(u_m^-(t) - u_m^+(t)). \tag{3.5}$$

We know

$$-(Mu_m, u_m^- - u_m^+) \geq \lambda_1\|u_m^+\|^2 + \mu_1\|u_m^-\|^2 \geq \tau_1\|u_m^+ + u_m^-\|^2. \tag{3.6}$$

Thus, by (3.1) and (3.3), we have

$$\tau_1\|u_m^+ + u_m^-\|^2 \leq (1+\|\eta\|)\|u_m^+ + u_m^-\| + \frac{2C_1^2k}{\tau_1}\left\|u_m^0\right\|^{2\theta}+\frac{\tau_1}{2}\|u_m^+ + u_m^-\|^2 + 2C_1k\|u_m^+ + u_m^-\|^{\theta+1}+C_2\sqrt{k}\|u_m^+ + u_m^-\|, \tag{3.7}$$

that is,

$$\frac{\tau_1}{2}\|u_m^+ + u_m^-\|^2 - 2C_1k\|u_m^+ + u_m^-\|^{\theta+1} - \left(1+\|\eta\|+C_2\sqrt{k}\right)\|u_m^+ + u_m^-\| \leq \frac{2C_1^2k}{\tau_1}\left\|u_m^0\right\|^{2\theta}. \tag{3.8}$$

It follows from (3.8) and $1/2 \leq \theta < 1$ that

$$\|u_m^+ + u_m^-\| \leq C_4\left\|u_m^0\right\|^{\theta} + C_5 \tag{3.9}$$

for all $m > m_0$ and some positive constants $C_4, C_5$.

By (3.1), (2.9), (3.4), and (3.9), we have

$$
\begin{aligned}
C_3 \geq J(u_m) &= \frac{1}{2}(Mu_m, u_m) + (\eta, u_m) - \sum_{t=1}^{k} F(t, u_m(t)) \\
&\geq \frac{1}{2}\left(\lambda_1 \|u_m^+\|^2 - \mu_s \|u_m^-\|^2\right) - \|\eta\|\left(\left\|u_m^0\right\| + \|u_m^+ + u_m^-\|\right) \\
&\quad - \sum_{t=1}^{k}\left(F(t, u_m(t)) - F\left(t, u_m^0(t)\right)\right) - \sum_{t=1}^{k} F\left(t, u_m^0(t)\right) \\
&\geq -\frac{\mu_s}{2}\|u_m^+ + u_m^-\|^2 - \|\eta\|\left\|u_m^0\right\| - \frac{2C_1^2 k}{\mu_s}\left\|u_m^0\right\|^{2\theta} - \frac{\mu_s}{2}\|u_m^+ + u_m^-\|^2 \\
&\quad - 2C_1 k \|u_m^+ + u_m^-\|^{\theta+1} - \left(\|\eta\| + C_2\sqrt{k}\right)\|u_m^+ + u_m^-\| - \sum_{t=1}^{k} F\left(t, u_m^0(t)\right) \\
&\geq -\mu_s\left(C_4^2\left\|u_m^0\right\|^{2\theta} + 2C_4C_5\left\|u_m^0\right\|^{\theta} + C_5^2\right) - \|\eta\|\left\|u_m^0\right\| - \frac{2C_1^2 k}{\mu_s}\left\|u_m^0\right\|^{2\theta} \\
&\quad - 2C_1 k\left(C_4\left\|u_m^0\right\|^{\theta} + C_5\right)^{\theta+1} - \left(\|\eta\| + C_2\sqrt{k}\right)\left(C_4\left\|u_m^0\right\|^{\theta} + C_5\right) - \sum_{t=1}^{k} F\left(t, u_m^0(t)\right) \\
&= -\left(\mu_s C_4^2 + \frac{2C_1^2 k}{\mu_s}\right)\left\|u_m^0\right\|^{2\theta} - \|\eta\|\left\|u_m^0\right\| - 2C_1 k\left(C_4\left\|u_m^0\right\|^{\theta} + C_5\right)^{\theta+1} \\
&\quad - 2\mu_s C_4 C_5\left\|u_m^0\right\|^{\theta} - \left(\|\eta\| + C_2\sqrt{k}\right)\left(C_4\left\|u_m^0\right\|^{\theta} + C_5\right) - \mu_s C_5^2 - \sum_{t=1}^{k} F\left(t, u_m^0(t)\right).
\end{aligned}
\tag{3.10}
$$

Since $1/2 \leq \theta < 1$, we deduce

$$
-C_3 \leq \left\|u_m^0\right\|^{2\theta}\left(\left\|u_m^0\right\|^{-2\theta}\sum_{t=1}^{T} F\left(t, u_m^0(t)\right) + C_6\right) + C_7 \tag{3.11}
$$

for all $m > m_0$ and some positive constants $C_6, C_7$. The above inequality and $(F_2)$(i) imply that $\{u_m^0(t)\}$ is bounded. Then it follows from (3.9) that $\{u_m^+ + u_m^-\}$ is bounded. Thus we conclude that $\{u_m\}$ is bounded, and the (PS) condition is verified.

Now, suppose that $(F_1)$, $(F_2)$(ii), and $(P_3)$ hold. By a similar argument as above, we know also that $J$ satisfies the (PS) condition. The proof is complete. □

*Proof of Theorem 1.1.* Assume that $(F_1)$, $(F_2)$(i), and $(P_1)$ hold. The proof for the case when $(F_1)$, $(F_2)$(ii), and $(P_1)$ hold is similar and will be omitted here. Since $p(t)$ is nonzero for each $t \in \mathbf{Z}$, the singular symmetric matrix $M$ has at least one nonzero eigenvalue and we will give the proof in three cases.

(i) Suppose that the matrix $M$ is singular and indefinite. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^0 \oplus R^-$. In view of Lemma 3.1, we only to check that conditions $(\varphi_1)$ and $(\varphi_2)$ in the saddle point theorem hold. To this end, let $X_1 =$

$R^+ \oplus R^0$, $X_2 = R^-$. For any $u = u^+ + u^0 \in X_1$, by $(F_1)$, (2.9), and the mean value theorem, we have

$$\begin{aligned}
J(u) &\geq \frac{\lambda_1}{2}\|u^+\|^2 - \|\eta\|\left(\left\|u^0\right\| + \|u^+\|\right) - \sum_{t=1}^{k} f\left(t, u^0(t) + \theta_1(t)u^+(t)\right) \cdot u^+(t) - \sum_{t=1}^{k} F\left(t, u^0(t)\right) \\
&\geq \frac{\lambda_1}{2}\|u^+\|^2 - \|\eta\|\left(\left\|u^0\right\| + \|u^+\|\right) - 2C_1\sqrt{k}\left\|u^0\right\|^{\theta}\|u^+\| \\
&\quad - 2C_1 k\|u^+\|^{\theta+1} - C_2\sqrt{k}\|u^+\| - \sum_{t=1}^{k} F\left(t, u^0(t)\right) \\
&\geq \frac{\lambda_1}{2}\|u^+\|^2 - \|\eta\|\left(\left\|u^0\right\| + \|u^+\|\right) - \frac{4C_1^2 k}{\lambda_1}\left\|u^0\right\|^{2\theta} - \frac{\lambda_1}{4}\|u^+\|^2 \\
&\quad - 2C_1 k\|u^+\|^{\theta+1} - C_2\sqrt{k}\|u^+\| - \sum_{t=1}^{k} F\left(t, u^0(t)\right) \\
&= \frac{\lambda_1}{4}\|u^+\|^2 - 2C_1 k\|u^+\|^{\theta+1} - \left(\|\eta\| + C_2\sqrt{k}\right)\|u^+\| \\
&\quad - \left\|u^0\right\|^{2\theta}\left(\left\|u^0\right\|^{-2\theta}\sum_{t=1}^{k} F\left(t, u^0(t)\right) + \frac{4C_1^2 k}{\lambda_1} + \|\eta\|\left\|u^0\right\|^{-2\theta+1}\right),
\end{aligned} \tag{3.12}$$

where $0 < \theta_1(t) < 1$, $t = 1, 2, \ldots, k$. Since $1/2 \leq \theta < 1$ and $(F_2)$(i), we have

$$J(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } X_1. \tag{3.13}$$

On the other hand, for any $u = u^- \in X_2$, by $(F_1)$, (2.9), and the mean value theorem, we have

$$\begin{aligned}
J(u) &\leq -\frac{1}{2}\mu_1\|u\|^2 + \|\eta\|\|u\| - \sum_{t=1}^{k} f(t, \theta_2(t)u(t)) \cdot u(t) - \sum_{t=1}^{k} F(t, 0) \\
&\leq -\frac{1}{2}\mu_1\|u\|^2 + \|\eta\|\|u\| + \sum_{t=1}^{k} C_1|u(t)|^{\theta+1} + C_2\sqrt{k}\|u\| \\
&\leq -\frac{1}{2}\mu_1\|u\|^2 + C_1 k\|u\|^{\theta+1} + \left(\|\eta\| + C_2\sqrt{k}\right)\|u\|,
\end{aligned} \tag{3.14}$$

where $0 < \theta_2(t) < 1$, $t = 1, 2, \ldots, k$. Since $1/2 \leq \theta < 1$, we can obtain

$$J(u) \longrightarrow -\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } X_2. \tag{3.15}$$

Let $e = 0$, then it follows from (3.13) and (3.15) that $(\varphi_1)$ and $(\varphi_2)$ are satisfied. By the saddle point theorem, $J$ has at least one critical point.

(ii) Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ are the positive eigenvalues of $M$ and $l = k-1$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^0$, where $R^+$ and $R^0$ are defined as in (2.4) and (2.5), respectively. By a similar argument as in the proof of Lemma 3.1, we see that $J$ satisfies the (PS) condition. By (3.13), $J$ is bounded from below. Then, by the least action principle, $c_1 = \inf_{u\in\mathbf{R}^k} J(u)$ is a critical value of $J$.

(iii) Suppose that $0 > -\mu_1 \geq -\mu_2 \geq \cdots \geq -\mu_s$ are the negative eigenvalues of $M$ and $s = k-1$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^0 \oplus R^-$. Following almost the same procedure as the proof of Lemma 3.1, we know also that $J$ satisfies the (PS) condition in this case.

For any $u = u^- \in R^-$, (3.15) holds. For any $u = u^0 \in R^0$, since $Mu = 0$, we have

$$J(u) \geq -\|\eta\|\left\|u^0\right\| - \sum_{t=1}^{k} F\left(t, u^0(t)\right) = -\left\|u^0\right\|^{2\theta}\left(\left\|u^0\right\|^{-2\theta}\sum_{t=1}^{k} F\left(t, u^0(t)\right) + \|\eta\|\left\|u^0\right\|^{-2\theta+1}\right). \tag{3.16}$$

Due to $(\mathrm{F}_2)$(i) and $1/2 \leq \theta < 1$,

$$J(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } R^0. \tag{3.17}$$

By the saddle point theorem, there exists at least one critical point of $J$.

Since $J$ has at least one critical point in all three cases, the BVP (1.1)-(1.2) has at least one solution. □

*Proof of Theorem 1.3.* Since the matrix $M$ is nonsingular, we will give the proof in three cases.

(i) Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ and $0 > -\mu_1 \geq -\mu_2 \geq \cdots \geq -\mu_s$ are the positive and negative eigenvalues of $M$, respectively, and $l + s = k$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^-$. Denote $\tau_1 = \min\{\lambda_1, \mu_1\}$. It follows from $(\mathrm{F}_3)$ that there exists a positive constant $a_1$ such that

$$\left|f(t,x)\right| \leq \frac{1}{4}\tau_1|x| + a_1 \tag{3.18}$$

for any $(t, x) \in \mathbf{Z}[1,k] \times \mathbf{R}$.

We now prove that the functional $J$ satisfies the (PS) condition. Let $\{u_m\}$ be a sequence in $\mathbf{R}^k$ such that $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$. Write $u_m = u_m^+ + u_m^-$, where $u_m^+ \in R^+$, $u_m^- \in R^-$. Similarly to (3.8), we have, by (3.18),

$$\frac{3\tau_1}{4}\left\|u_m^+ + u_m^-\right\|^2 \leq \left(1 + \|\eta\| + a_1\sqrt{k}\right)\left\|u_m^+ + u_m^-\right\|. \tag{3.19}$$

Thus $\{u_m\}$ is bounded, and the (PS) condition is verified.

For any $u = u^+ \in R^+$, by (3.18) and the mean value theorem, we have

$$J(u) \geq \frac{1}{2}\lambda_1\|u\|^2 - \|\eta\|\|u\| - \sum_{t=1}^{k} f(t, \theta_3(t)u(t)) \cdot u(t) - \sum_{t=1}^{k} F(t,0)$$
$$\geq \left(\frac{\lambda_1}{2} - \frac{\tau_1}{4}\right)\|u\|^2 - \|\eta\|\|u\| - a_1\sqrt{k}\|u\|, \tag{3.20}$$

where $0 < \theta_3(t) < 1$, $t = 1, 2, \ldots, k$. For any $u = u^- \in R^-$, Similarly to (3.20), we have

$$J(u) \leq -\left(\frac{\mu_1}{2} - \frac{\tau_1}{4}\right)\|u\|^2 + \|\eta\|\|u\| + a_1\sqrt{k}\|u\|. \tag{3.21}$$

Let $e = 0$, then it follows from (3.20) and (3.21) that $(\varphi_1)$ and $(\varphi_2)$ are satisfied. By the saddle point theorem, $J$ has at least one critical point.

(ii) Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ are the positive eigenvalues of $M$ and $l = k$. Then $E$ has the direct sum decomposition: $\mathbf{R}^k = R^+$. It follows from (3.20) that $J$ satisfies the (PS) condition and is bounded from below. Then, by the least action principle, $c_2 = \inf_{u\in\mathbf{R}^k} J(u)$ is a critical value of $J$.

(iii) Suppose that $0 > -\mu_1 \geq -\mu_2 \geq \cdots \geq -\mu_s$ are the negative eigenvalues of $M$ and $s = k$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^-$. It follows from (3.21) that $J$ satisfies the (PS) condition and is bounded from above. Then, by the least action principle, $c_3 = -\inf_{u\in\mathbf{R}^k} J(u)$ is a critical value of $-J$.

Since $J$ has a critical point in all three cases, the BVP (1.1)-(1.2) has at least one solution. □

*Proof of Corollary 1.4.* This is immediate from Theorem 1.3. □

The following lemma is useful for proving Theorem 1.6 and Corollary 1.8.

**Lemma 3.2.** *Under the condition ($F_5$), the functional $J$ satisfies condition (C); that is, for any sequence $\{u_m\}$ such that $J(u_m)$ is bounded and $\|J'(u_m)\|(1 + \|u_m\|) \to 0$ as $m \to \infty$, there exists a subsequence of $\{u_m\}$ which is convergent in $\mathbf{R}^k$.*

*Proof.* First suppose that ($F_5$)(i) holds. Let $\{u_m\}$ be a sequence in $\mathbf{R}^k$ such that $J(u_m)$ is bounded and $\|J'(u_m)\|(1 + \|u_m\|) \to 0$ as $m \to \infty$. Then there exists a constant $L_1 > 0$ such that

$$|J(u_m)| \leq L_1, \qquad \|J'(u_m)\|(1 + \|u_m\|) \leq L_1 \tag{3.22}$$

for all $m \in \mathbf{Z}[1]$. Hence, we have

$$-3L_1 \leq -\|J'(u_m)\| \cdot (1 + \|u_m\|) - 2J(u_m) \leq (J'(u_m), u_m) - 2J(u_m)$$
$$= \sum_{t=1}^{k} \left[2F(t, u_m(t)) - (f(t, u_m(t)) + \eta(t))u_m(t)\right]. \tag{3.23}$$

Then, $\{u_m\}$ is bounded. In fact, if $\{u_m\}$ is unbounded, there exist a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and $t_0 \in \mathbf{Z}[1,k]$ such that $|u_m(t_0)| \longrightarrow \infty$ as $m \longrightarrow \infty$. By $(\mathrm{F}_5)$(i), we have

$$2F(t_0, u_m(t_0)) - \left(f(t_0, u_m(t_0)) + \eta(t_0)\right)u_m(t_0) \longrightarrow -\infty \quad \text{as } m \longrightarrow \infty. \tag{3.24}$$

The continuity of $2F(t,x) - (f(t,x) + \eta(t))x$ with respect to $x$ and $(\mathrm{F}_5)$(i) implies that there exists a constant $L_2 > 0$ such that for any $(t,x) \in \mathbf{Z}[1,k] \times \mathbf{R}$, $2F(t,x) - (f(t,x) + \eta(t))x \le L_2$. Then, we get

$$\begin{aligned} &\sum_{t=1}^{k} \left[2F(t, u_m(t)) - \left(f(t, u_m(t)) + \eta(t)\right)u_m(t)\right] \\ &\qquad \le 2F(t_0, u_m(t_0)) - \left(f(t_0, u_m(t_0)) + \eta(t_0)\right)u_m(t_0) + (k-1)L_2. \end{aligned} \tag{3.25}$$

Thus,

$$\sum_{t=1}^{k} \left[2F(t, u_m(t)) - \left(f(t, u_m(t)) + \eta(t)\right)u_m(t)\right] \longrightarrow -\infty \quad \text{as } m \longrightarrow \infty, \tag{3.26}$$

which contradicts (3.23). Therefore, $\{u_m\}$ is bounded in $\mathbf{R}^k$ and $J$ satisfies condition $(C)$.

Now, suppose that $(\mathrm{F}_5)$(ii) holds. By a similar argument as above, we know also that $J$ satisfies condition $(C)$. The proof is complete. □

*Proof of Theorem 1.6.* Assume that $(\mathrm{F}_4)$, $(\mathrm{F}_5)$(i), and $(\mathrm{P}_3)$ hold. The proof for the case when $(\mathrm{F}_4)$, $(\mathrm{F}_5)$(ii), and $(\mathrm{P}_3)$ hold is similar and will be omitted here. Due to $(\mathrm{P}_3)$, $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^0 \oplus R^-$, where $R^+, R^-$ and $R^0$ are defined as in (2.4) and (2.5), respectively. We claim that for every $t \in \mathbf{Z}[1,k]$,

$$F(t,x) - \eta(t)x \longrightarrow -\infty \quad \text{as } |x| \longrightarrow \infty. \tag{3.27}$$

Indeed, according to $(\mathrm{F}_5)$(i), we can obtain that for any given $\varepsilon > 0$, there exists a positive constant $G_1$ such that for $t \in \mathbf{Z}[1,k]$, $|x| > G_1$,

$$2F(t,x) - \left(f(t,x) + \eta(t)\right)x < -\frac{1}{\varepsilon}. \tag{3.28}$$

Obviously,

$$\left(f(t,sx) + \eta(t)\right)sx - 2F(t,sx) > \frac{1}{\varepsilon}, \tag{3.29}$$

where $s > 1$, $|x| > G_1$. We have

$$\frac{d}{ds}\left(\frac{F(t,sx) - \eta(t)sx}{s^2}\right) = \frac{\left(f(t,sx) + \eta(t)\right)sx - 2F(t,sx)}{s^3} \ge \frac{d}{ds}\left(-\frac{1}{2\varepsilon s^2}\right). \tag{3.30}$$

By integrating both sides of the above inequality from 1 to $s$, we get

$$\frac{F(t,sx)-\eta(t)sx}{s^2}-F(t,x)+\eta(t)x\geq-\frac{1}{2\varepsilon s^2}+\frac{1}{2\varepsilon}. \tag{3.31}$$

Let $s\to+\infty$ in the above inequality, and it follows from ($F_4$) that

$$F(t,x)-\eta(t)x\leq-\frac{1}{2\varepsilon} \tag{3.32}$$

for $t\in\mathbf{Z}[1,k]$, $|x|>G_1$. From the arbitrariness of $\varepsilon$, we can conclude that (3.27) holds, proving our claim.

Now we prove

$$J(u)\longrightarrow+\infty\quad\text{as } \|u\|\longrightarrow\infty \text{ in } R^+\oplus R^0. \tag{3.33}$$

If (3.33) does not hold, there exist a constant $L_3>0$ and a sequence $\{u_m\}$ in $R^+\oplus R^0$ such that $\|u_m\|\to\infty$ as $m\to\infty$ and

$$J(u_m)\leq L_3 \tag{3.34}$$

for all $m$. Since $\|u_m\|\to\infty$ as $m\to\infty$, there exist a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and $t_0\in\mathbf{Z}[1,k]$ such that $|u_m(t_0)|\longrightarrow\infty$ as $m\longrightarrow\infty$. By (3.27), we have

$$F(t_0,u_m(t_0))-\eta(t_0)u_m(t_0)\longrightarrow-\infty\quad\text{as } m\longrightarrow\infty. \tag{3.35}$$

The continuity of $F(t,x)-\eta(t)x$ with respect to $x$ and (3.27) implies that there exists a constant $L_4>0$ such that for any $(t,x)\in\mathbf{Z}[1,k]\times\mathbf{R}$, $F(t,x)-\eta(t)x\leq L_4$. Then, we get

$$J(u_m)\geq\frac{\lambda_1}{2}\|u_m^+\|^2-\sum_{t=1}^{k}\left[F(t,u_m(t))-\eta(t)u_m(t)\right]\geq-F(t_0,u_m(t_0))+\eta(t_0)u_m(t_0)-(k-1)L_4. \tag{3.36}$$

Thus,

$$J(u_m)\longrightarrow+\infty\quad\text{as } m\longrightarrow\infty, \tag{3.37}$$

which contradicts (3.34). Hence, (3.33) follows.

On the other hand, by ($F_4$), there exists a positive constant $a_2$ such that for any $(t,x)\in\mathbf{Z}[1,k]\times\mathbf{R}$,

$$|F(t,x)|\leq\frac{1}{4}\mu_1|x|^2+a_2. \tag{3.38}$$

Then we have

$$J(u) \leq -\frac{\mu_1}{2}\|u^-\|^2 + \|\eta\|\|u^-\| + \frac{\mu_1}{4}\|u^-\|^2 + a_2 k \tag{3.39}$$

for all $u = u^- \in R^-$. Thus, we can conclude that

$$J(u) \longrightarrow -\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } R^-. \tag{3.40}$$

It follows from (3.33) and (3.40) that $J$ satisfies conditions $(\varphi_1)$ and $(\varphi_2)$. Hence Theorem 1.6 follows from Lemma 3.2, the saddle point theorem, and Remark 2.3. □

The proof of Corollary 1.8 is similar to that of Theorem 1.6 and is omitted.

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