Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2009, Article ID 538431, 15 pages
doi:10.1155/2009/538431

Research Article

Existence of Three Positive Solutions for m-Point Discrete Boundary Value Problems with p-Laplacian

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Received 23 May 2009; Accepted 18 October 2009

Recommended by Leonid Shaikhet

We consider the multi-point discrete boundary value problem with one-dimensional p-Laplacian operator $\Delta(\phi_p(\Delta u(t-1)) + q(t)f(t,u(t),\Delta u(t)) = 0$, $t \in \{1,\ldots,n-1\}$ subject to the boundary conditions: $u(0) = 0$, $u(n) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, where $\phi_p(s) = |s|^{p-2}s, p > 1, \xi_i \in \{2,\ldots,n-2\}$ with $1 < \xi_1 < \cdots < \xi_{m-2} < n-1$ and $a_i \in (0,1), 0 < \sum_{i=1}^{m-2} a_i < 1$. Using a new fixed point theorem due to Avery and Peterson, we study the existence of at least three positive solutions to the above boundary value problem.

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1. Introduction

The second-order differential and difference boundary value problems arise in many branches of both applied and basic mathematics and have been extensively studied in literature. We refer the reader to some recent results for second-order nonlinear two-point [1–6] and multipoint [7–9] boundary value problems. The main tools used in the above works are fixed point theorems.

Recently, Feng and Ge in [9] considered the following multipoint BVPs:

$$\begin{gathered}\left(\phi_p(u(t-1))'\right)' + q(t)f\left(t,u(t),u'(t)\right) = 0, \quad t \in (0,1), \\ u(0) = 0, \qquad u(n) = \sum_{i=1}^{m-2} a_i u(\xi_i).\end{gathered} \tag{1.1}$$

The authors obtained sufficient conditions that guarantee the existence of at least three positive solutions by using fixed point theorems due to Avery-Peterson.

In this work, we study the existence of multiple positive solutions to the discrete boundary value problem for the one-dimensional p-Laplacian:

$$\Delta\left(\phi_p(\Delta u(t-1))\right) + q(t)f(t, u(t), \Delta u(t)) = 0, \quad t \in \{1, \ldots, n-1\}$$
$$u(0) = 0, \qquad u(n) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.2}$$

where $\Delta u(t) = u(t+1) - u(t)$ for $t \in \{0, 1, \ldots, n-1\}$, $\Delta^2 u(t) = u(t+2) - 2u(t+1) + u(t)$, for $t \in \{0, 1, \ldots, n-2\}$, and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in \{2, \ldots, n-2\}$ with $1 < \xi_1 < \cdots < \xi_{m-2} < n-1$.

In order to study the existence of at least three positive solutions to the above boundary value problem, we assume that a_i, f, q satisfy the following.

(H_1) $a_i \in (0,1)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$.

(H_2) $f : \{1, \ldots, n-1\} \times [0, +\infty) \times R \to (0, +\infty)$ is continuous.

(H_3) $q(t) > 0$ for $t \in \{1, 2, \ldots, n-1\}$.

We will depend on an application of a fixed point theorems due to Avery and Peterson, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space to obtain our main results.

2. Preliminaries

For the convenience of readers, we provide some background material from the theory of cones in Banach spaces. In this section, we also state Avery-Peterson's fixed point theorem.

Definition 2.1. Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone of E if it satisfies the following conditions:

(i) $au + bv \in P$ for all $u, v \in p$ and all $a \geq 0$, $b \geq 0$;

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha\left(tx + (1-t)y\right) \geq t\alpha(x) + (1-t)\alpha\left(y\right) \tag{2.1}$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say that the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta\left(tx + (1-t)y\right) \leq t\beta(x) + (1-t)\beta\left(y\right) \tag{2.2}$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on a cone P, let α be a nonnegative continuous concave functional on a cone P, and let ψ be a nonnegative continuous functional on a cone P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$\begin{aligned} P(\gamma, d) &= \{u \in P \mid \gamma(u) < d\}, \\ P(\gamma, \alpha, b, d) &= \{u \in P \mid b \leq \alpha(u), \gamma(u) \leq d\}, \\ P(\gamma, \theta, \alpha, b, c, d) &= \{u \in P \mid b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\} \end{aligned} \tag{2.3}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{u \in P \mid a \leq \psi(u), \gamma(u) \leq d\}. \tag{2.4}$$

To prove our results, we need the following fixed point theorem due to Avery and Peterson in [1].

Theorem 2.4. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\alpha(u) \leq \psi(u), \quad \|u\| \leq M\gamma(u) \tag{2.5}$$

for all $u \in \overline{P(\gamma \quad, d)}$. Suppose that

$$T : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)} \tag{2.6}$$

is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S_1) *$\{u \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for $u \in P(\gamma, \theta, \alpha, b, c, d)$;*

(S_2) *$\alpha(Tu) > b$ for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$;*

(S_3) *$0 \overline{\in} R(\gamma, \psi, a, d)$ and $\psi(Tu) < a$ for $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$.*

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$, such that

$\gamma(u_i) \leq d$ for $i = 1, 2, 3$,

$b < \alpha(u_1)$

$a < \psi(u_2)$, with $\alpha(u_2) < b$,

$\psi(u_3) < a$.

3. Related Lemmas

Let the Banach space $E = \{u : \{0,1,\ldots,n\} \to R\}$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in \{0,1,\ldots,n\}$, and the maximum norm

$$\|u\| = \max\left\{ \max_{t\in\{0,1,\ldots,n\}} |u(t)| \ , \max_{t\in\{0,1,\ldots,n-1\}} |\Delta u(t)| \right\}. \tag{3.1}$$

Then, we define the cone P in E by

$$P = \left\{ u \in E \mid u(t) \geq 0,\ t \in \{0,1,\ldots,n\}; u(0) = 0,\ \Delta^2 u(t) \leq 0,\ t \in \{0,1,\ldots,n-2\} \right\}. \tag{3.2}$$

Let k be a natural number, such that $k < \min\{\xi_1, n - \xi_{m-2}\}$.

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\begin{gathered} \gamma(u) = \max_{t\in\{0,1,\ldots,n-1\}} |\Delta u(t)|, \quad \psi(u) = \theta(u) = \max_{t\in\{0,1,\ldots,n\}} |u(t)|, \\ \alpha(u) = \min_{t\in\{k+1,\ldots,n-k-1\}} |u(t)| \end{gathered} \tag{3.3}$$

for $u \in P$.

In order to prove our main results, we need the following lemma.

Lemma 3.1. *If $u \in P$, then*

$$\max_{t\in\{0,1,\ldots,n\}} |u(t)| \leq n \max_{t\in\{0,1,\ldots,n-1\}} |\Delta u(t)|, \qquad \text{that is}, \theta(u) \leq n\gamma(u). \tag{3.4}$$

Proof. Suppose that the maximum of u occurs at $t_0 \in \{0,1,\ldots,n\}$; by the definition of the cone P, we know $\Delta u(t+1) \leq \Delta u(t)$, and then,

$$\begin{aligned} u(t_0) &= u(t_0) - u(0) = \Delta u(0) + \Delta u(1) + \cdots + \Delta u(t_0 - 1) \\ &\leq t_0 \Delta u(0) \leq n\Delta u(0) \leq n \max_{t\in\{0,1,\ldots,n-1\}} |\Delta u(t)|. \end{aligned} \tag{3.5}$$

So, we have

$$\max_{t\in\{0,1,\ldots,n\}} |u(t)| \leq n \max_{t\in\{0,1,\ldots,n-1\}} |\Delta u(t)|. \tag{3.6}$$

The proof is complete. □

By Lemma 3.1 and the definitions, the functionals defined above satisfy

$$\frac{1}{[n/k]+1}\theta(u) \le \alpha(u) \le \theta(u) = \psi(u), \qquad \|u\| = \max\{\theta(u), \gamma(u)\} \le n\gamma(u) \tag{3.7}$$

for all $u \in \overline{P(\gamma, d)} \subset P$. Therefore, condition (2.5) is satisfied.

Now, we show that $(1/([n/k]+1))\theta(u) \le \alpha(u)$. Here, we also suppose $\theta(u) = u(t_0)$, and by the definitions of α and the cone P, we can distinguish two cases.

(i) $\alpha(u) = u(k+1)$, then we certainly have $t_0 \ge k+1$, and

$$\begin{aligned} u(t_0) &= \Delta u(0) + \cdots + \Delta u(k) + \Delta u(k+1) + \cdots + \Delta u(2k) \\ &\quad + \Delta u(2k+1) + \cdots + \Delta u(3k) + \cdots + \Delta u\left(\left[\frac{t_0}{k}\right]k\right) \\ &\quad + \Delta u\left(\left[\frac{t_0}{k}\right]k+1\right) + \cdots + \Delta u(t_0 - 1) \\ &\le \left(\left[\frac{t_0}{k}\right]+1\right)u(k+1) \le \left(\left[\frac{n}{k}\right]+1\right)u(k+1), \end{aligned} \tag{3.8}$$

that is, $1/([n/k]+1)u(t_0) \le u(k+1)$.

(ii) $\alpha(u) = u(n-k-1)$, then $t_0 \le n-k-1$, $u(n) \le u(n-k-1)$ and

$$\begin{aligned} u(t_0) &= u(n) + \Big(- \Delta u(n-1) - \cdots - \Delta u(n-k-1) \\ &\qquad - \Delta u(n-k-2) - \cdots - \Delta u(n-2k-1) \\ &\qquad - \Delta u(n-2k-2) - \cdots - \Delta u\left(n - \left[\frac{n-t_0}{k}\right]k - 1\right) \\ &\qquad - \Delta u\left(n - \left[\frac{n-t_0}{k}\right]k - 2\right) - \cdots - \Delta u(t_0)\Big) \\ &\le u(n) + \left(\left[\frac{n-t_0}{k}\right]+1\right)[u(n-k-1) - u(n)] \\ &\le \left(\left[\frac{n-t_0}{k}\right]+1\right)u(n-k-1) \le \left(\left[\frac{n}{k}\right]+1\right)u(n-k-1), \end{aligned} \tag{3.9}$$

that is, $(1/([n/k]+1))u(t_0) \le u(n-k-1)$. So, we have $(1/([n/k]+1))\theta(u) \le \alpha(u)$.

Lemma 3.2. *Assume that (H_1)–(H_3) hold. Then, for any $x \in E$, $x(t) \ge 0$,*

$$\Delta(\phi_p(\Delta u(t-1)) + q(t) f(t, x(t), \Delta x(t)) = 0, \quad t \in \{1, \dots, n-1\}, \tag{3.10}$$

$$u(0) = 0, \qquad u(n) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{3.11}$$

has a unique solution $u(t)$ given by

$$u(t) = \sum_{j=0}^{t-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right) \tag{3.12}$$

or

$$u(t) = \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i - 1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right) - \sum_{j=t}^{n-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right), \tag{3.13}$$

where A_u satisfies

$$\sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i - 1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right) = \sum_{j=0}^{n-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right). \tag{3.14}$$

Proof. For any $x \in E$ suppose that u is a solution of the BVPs (3.7) and (3.11). According to the property of the difference operator, it follows that

$$\begin{gathered} \phi_p(\Delta u(1)) - \phi_p(\Delta u(0)) = -q(1) f(1, x(1), \Delta x(1)), \\ \phi_p(\Delta u(2)) - \phi_p(\Delta u(1)) = -q(2) f(2, x(2), \Delta x(2)), \\ \cdots \\ \phi_p(\Delta u(t)) - \phi_p(\Delta u(t-1)) = -q(t) f(t, x(t), \Delta x(t)), \end{gathered} \tag{3.15}$$

then

$$\begin{gathered} \phi_p(\Delta u(t)) - \phi_p(\Delta u(0)) = -\sum_{i=1}^{t} q(i) f(i, x(i), \Delta x(i)), \\ \phi_p(\Delta u(t)) = \phi_p(\Delta u(0)) - \sum_{i=1}^{t} q(i) f(i, x(i), \Delta x(i)). \end{gathered} \tag{3.16}$$

Let $\phi_p(\Delta u(0)) = A_u$,

$$\Delta u(t) = \phi_p^{-1}\left(A_u - \sum_{i=1}^{t} q(i) f(i, x(i), \Delta x(i)) \right), \tag{3.17}$$

and by

$$\begin{gathered}
\Delta u(0) = u(1) - u(0) = \phi_p^{-1}(A_u), \\
\Delta u(1) = u(2) - u(1) = \phi_p^{-1}\left(A_u - \sum_{i=1}^{1} q(i) f(i, x(i), \Delta x(i)) \right), \\
\cdots \\
\Delta u(t-1) = u(t) - u(t-1) = \phi_p^{-1}\left(A_u - \sum_{i=1}^{t-1} q(i) f(i, x(i), \Delta x(i)) \right),
\end{gathered} \tag{3.18}$$

then

$$u(t) - u(0) = \sum_{j=0}^{t-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right). \tag{3.19}$$

By

$$\begin{gathered}
\Delta u(n-1) = u(n) - u(n-1) = \phi_p^{-1}\left(A_u - \sum_{i=1}^{n-1} q(i) f(i, x(i), \Delta x(i)) \right), \\
\Delta u(n-2) = u(n-1) - u(n-2) = \phi_p^{-1}\left(A_u - \sum_{i=1}^{n-2} q(i) f(i, x(i), \Delta x(i)) \right), \\
\cdots \\
\Delta u(t) = u(t+1) - u(t) = \phi_p^{-1}\left(A_u - \sum_{i=1}^{t} q(i) f(i, x(i), \Delta x(i)) \right),
\end{gathered} \tag{3.20}$$

so that

$$\begin{gathered}
u(n) - u(t) = \sum_{j=t}^{n-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right), \\
u(t) = u(n) - \sum_{j=t}^{n-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right),
\end{gathered} \tag{3.21}$$

Using the boundary condition (3.11), we can easily obtain

$$u(t) = \sum_{j=0}^{t-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, x(i), \Delta x(i)) \right) \tag{3.22}$$

or

$$u(t)=\sum_{i=1}^{m-2}a_i\sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(A_u-\sum_{i=1}^{j}q(i)f(i,x(i),\Delta x(i))\right)-\sum_{j=t}^{n-1}\phi_p^{-1}\left(A_u-\sum_{i=1}^{j}q(i)f(i,x(i),\Delta x(i))\right), \tag{3.23}$$

where A_u satisfies (3.14).

On the other hand, it is easy to verify that if u is the solution of (3.12) or (3.13), then u is a solution of (3.10) and (3.11). □

Lemma 3.3. *For any $u\in E$, $u(t)\geq 0$, there exists a unique $A_u\in(-\infty,+\infty)$ satisfying (3.14). Moreover, there is a unique $n_0\in\{1,\ldots,n-1\}$, such that*

$$\sum_{i=1}^{n_0-1}q(i)f(i,u(i),\Delta u(i))<A_u\leq\sum_{i=1}^{n_0}q(i)f(i,u(i),\Delta u(i)). \tag{3.24}$$

Proof. Let

$$\begin{aligned}\varphi(t)=&\sum_{i=1}^{m-2}a_i\sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(t-\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)\\&-\sum_{j=0}^{n-1}\phi_p^{-1}\left(t-\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right),\quad t\in(-\infty,+\infty),\end{aligned} \tag{3.25}$$

so that

$$\begin{aligned}\varphi(0)&=\sum_{i=1}^{m-2}a_i\sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(-\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)-\sum_{j=0}^{n-1}\phi_p^{-1}\left(-\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)\\&=\sum_{j=0}^{n-1}\phi_p^{-1}\left(\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)-\sum_{i=1}^{m-2}a_i\sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)\\&>\sum_{i=1}^{m-2}a_i\left[\sum_{j=0}^{n-1}\phi_p^{-1}\left(\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)-\sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(\sum_{i=1}^{j}q(i)f(i,u(i),\Delta u(i))\right)\right]\\&\geq 0,\end{aligned}$$

$$\begin{aligned}&\varphi\left(\sum_{i=1}^{n-1}q(i)f(i,u(i),\Delta u(i))\right)\\&\quad=\sum_{i=1}^{m-2}a_i\sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(\sum_{i=j+1}^{n-1}q(i)f(i,u(i),\Delta u(i))\right)-\sum_{j=0}^{n-1}\phi_p^{-1}\left(\sum_{i=j+1}^{n-1}q(i)f(i,u(i),\Delta u(i))\right)\end{aligned}$$

$$
\begin{aligned}
&< \sum_{i=1}^{m-2} a_i \left[\sum_{j=0}^{\xi_i - 1} \phi_p^{-1} \left(\sum_{i=j+1}^{n-1} q(i) f(i, u(i), \Delta u(i)) \right) - \sum_{j=0}^{n-1} \phi_p^{-1} \left(\sum_{i=j+1}^{n-1} q(i) f(i, u(i), \Delta u(i)) \right) \right] \\
&\leq 0.
\end{aligned}
\tag{3.26}
$$

By the continuity of $\varphi(t)$, we know that there exists at least one

$$
A_u \in \left(0, \sum_{i=1}^{n-1} q(i) f(i, u(i), \Delta u(i)) \right) \subset (-\infty, +\infty) \tag{3.27}
$$

satisfying (3.14).

On the other hand,

$$
\begin{aligned}
\varphi'(t) = &\sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i - 1} \left(\phi_p^{-1} \right)' \left(t - \sum_{i=1}^{j} q(t) f(i, u(i), \Delta u(i)) \right) \\
&- \sum_{j=0}^{n-1} \left(\phi_p^{-1} \right)' \left(t - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) \\
&< \sum_{i=1}^{m-2} a_i \left[\sum_{j=0}^{\xi_i - 1} \left(\phi_p^{-1} \right)' \left(t - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) \right. \\
&\qquad \left. - \sum_{j=0}^{n-1} \left(\phi_p^{-1} \right)' \left(t - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) \right] \leq 0.
\end{aligned}
\tag{3.28}
$$

Then $\varphi(t)$ is strictly increasing on $t \in (-\infty, +\infty)$.

So, there exists a unique $A_u \in (-\infty, +\infty)$ satisfying (3.14).

Moreover, there exists a $n_0 \in \{1, \ldots, n-1\}$, such that

$$
\sum_{i=1}^{n_0 - 1} q(i) f(i, u(i), \Delta u(i)) < A_u \leq \sum_{i=1}^{n_0} q(i) f(i, u(i), \Delta u(i)). \tag{3.29}
$$

□

Lemma 3.4. *Assume that* (H_1)–(H_3) *hold. If* $x \in E,\ x(t) \geq 0$, *then the unique solution* $u(t)$ *of the BVPs* (3.10) *and* (3.11) *has the following properties:*

(i) $\Delta^2 u(t) \leq 0$, *and* $u(t) \geq 0$;

(ii) *there exists a unique* $n_0 \in \{0, 1, \ldots, n\}$, *such that* $u(n_0) = \max_{t \in \{0,1,\ldots,n\}} u(t)$, *which* n_0 *is given in Lemma 3.3.*

Proof. Suppose that $u(t)$ is the solution of (3.10) and (3.11). Then we have the following.

(i) By Lemma 3.2, it is easy to see that $\Delta^2 u(t) \leq 0$. Without loss of generality, we assume that $u(n) = \min\{u(0), u(n)\}$. By $\Delta^2 u(t) \leq 0$, we know that $u(t) \geq u(n), t \in \{0, 1, \ldots, n\}$. So we get $u(n) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i u(n)$, that is, $(1 - \sum_{i=1}^{m-2} a_i) u(n) \geq 0$. Hence $u(n) \geq 0$. So, from the concavity of $u(t)$, we know that $u(t) \geq 0, t \in \{0, 1, \ldots, n\}$.

(ii) From

$$u(n_0) = \sum_{j=0}^{n_0-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right),$$

$$u(n_0 + 1) = \sum_{j=0}^{n_0-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) + \phi_p^{-1}\left(A_u - \sum_{i=1}^{n_0} q(i) f(i, u(i), \Delta u(i)) \right), \tag{3.30}$$

and by Lemma 3.3, we have $A_u - \sum_{i=1}^{n_0} q(i) f(i, u(i), \Delta u(i)) < 0$, so that $u(n_0) > u(n_0 + 1)$.

Also

$$\begin{aligned} u(n_0 - 1) &= \sum_{j=0}^{n_0-2} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) \\ &= u(n_0) - \phi_p^{-1}\left(A_u - \sum_{i=1}^{n_0-1} q(i) f(i, u(i), \Delta u(i)) \right), \end{aligned} \tag{3.31}$$

and one arrives at $u(n_0) \geq u(n_0 - 1)$. So, $u(n_0) = \max_{t \in \{0,1,\ldots,n\}} u(t)$.

If there exist $n_1, n_2 \in \{0, 1, \ldots, n\}$, $n_1 < n_2$ such that $\Delta u(n_1) = \Delta u(n_2)$, then

$$0 = \phi_p(\Delta u(n_1)) - \phi_p(\Delta u(n_2)) = \sum_{n_1+1}^{n_2} q(i) f(i, x(i), \Delta x(i)) > 0, \tag{3.32}$$

which is a contradiction. □

Lemma 3.5. *For any $u \in P$, define the operator*

$$(Tu)(t) = \begin{cases} \displaystyle\sum_{j=0}^{t-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right), & 0 \leq t \leq n_0, \\ \displaystyle\sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i - 1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) & \\ \displaystyle\quad - \sum_{j=t}^{n-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right), & n_0 + 1 \leq t \leq n. \end{cases} \tag{3.33}$$

Then $T : P \to P$ is completely continuous.

Proof. Using the continuity of f and the definition of T, it is easy to show that $T : P \to P$ is continuous. Next, we prove that T is completely continuous.

Suppose that the sequence $\{u_i\} \subseteq P$ is bounded, then there exists $M > 0$, such that $u_i(j) \leq M$, for any $i = 1, 2, \ldots, j = 0, 1, \ldots, n$. By the continuity of f, ϕ_p^{-1} and A_u are bounded, and we know that there exists $M' > 0$, such that $|Tu_i(t)| \leq M'$, for $t \in \{0, 1, \ldots, n\}$ and $i = 1, 2, \ldots, n, \ldots$. In view of the bounded sequence $\{Tu_i(0)\}$, there exists $\{u_{i0}\} \subseteq \{u_i\}$, such that $\lim_{i\to\infty} Tu_{i0}(0) = a_0$. For the bounded sequence $\{Tu_{i0}(1)\}$, there exists $\{u_{i1}\} \subseteq \{u_{i0}\}$, such that $\lim_{i\to\infty} Tu_{i1}(1) = a_1$. By repetition in this way, we have that there exists $\{u_{ij}\} \subseteq \{u_{ij-1}\}$ for $j = 2, 3, \ldots, n$, such that $\lim_{i\to\infty} Tu_{ij}(j) = a_j$. Let $y = \{a_0, a_1, \ldots, a_n\}$; by the definition of the norm on E, there exists $\{u_{in}\} \subseteq \{u_i\}$, such that $\lim_{i\to\infty} Tu_{in}(j) = y$.

Hence, $T : P \to P$ is completely continuous. □

4. Existence of Triple Positive Solutions to (1.2)

We are now ready to apply Avery-Peterson's fixed point theorem to the operator T to give sufficient conditions for the existence of at least three positive solutions to the BVPs (1.2).

Now for convenience we introduce the following notations. Let

$$L = \phi_p^{-1}\left(\sum_{i=1}^{n-1} q(i)\right),$$

$$M = \min\left\{\sum_{j=k+1}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]} q(i)\right), \sum_{j=[n/2]}^{n-k-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i)\right)\right\},$$

$$N = \max\left\{\sum_{j=0}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]+1} q(i)\right), \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i-1} \phi_p^{-1}\left(\sum_{i=j+1}^{n-1} q(i)\right) + \sum_{j=[n/2]}^{n-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i)\right)\right\}. \tag{4.1}$$

Theorem 4.1. *Assume that conditions (H_1)–(H_3) hold. Let $0 < a < b < ([n/k]+1)b < d$, and suppose that f satisfies the following conditions:*

(A_1) $f(t, u, v) \leq \phi_p(d/L)$, *for* $(t, u, v) \in \{0, 1, \ldots, n\} \times [0, nd] \times [-d, d]$;

(A_2) $f(t, u, v) > \phi_p(([n/k]+1)b/M)$, *for* $(t, u, v) \in \{k+1, \ldots, n-k-1\} \times [b, ([n/k]+1)b] \times [-d, d]$;

(A_3) $f(t, u, v) < \phi_p(a/N)$, *for* $(t, u, v) \in \{0, 1, \ldots, n\} \times [0, a] \times [-d, d]$.

Then BVPs (1.2) *have at least three positive solutions u_1, u_2, and u_3 such that*

$$\begin{gathered}
\max_{t\in\{0,1,\ldots,n-1\}} |\Delta u_i(t)| \leq d, \quad i = 1, 2, 3, \\
b < \min_{t\in\{k+1,\ldots,n-k-1\}} |u_1(t)|, \qquad \max_{t\in\{0,1,\ldots,n\}} |u_1(t)| \leq b, \\
a < \max_{t\in\{0,1,\ldots,n\}} |u_2(t)| < \left(\left[\frac{n}{k}\right]+1\right)b, \qquad \text{with} \min_{t\in\{k+1,\ldots,n-k-1\}} |u_2(t)| < b, \\
\max_{t\in\{0,1,\ldots,n\}} |u_3(t)| < a.
\end{gathered} \tag{4.2}$$

Proof. The BVPs (1.2) have a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies Avery-Peterson's fixed point theorem which will prove the existence of three fixed points of T which satisfy the conclusion of the theorem. Now the proof is divided into some steps.

(1) We will show that (A_1) implies that $T : \overline{P(r,d)} \to \overline{P(r,d)}$.

In fact, for $u \in \overline{P(r,d)}$, there is $\gamma(u) = \max_{t\in\{0,1,\ldots,n-1\}}|\Delta u(t)| \le d$. With Lemma 3.1, there is $\max_{t\in\{0,1,\ldots,n\}}|u(t)| \le nd$, and then condition (A_1) implies $f(t,u,v) \le \phi_p(d/L)$. On the other hand, for $u \in P$, there is $Tu \in P$, then Tu is concave on $t \in \{0,1,\ldots,n\}$, and $\max|\Delta Tu(t)| = \max\{|\Delta Tu(0)|, |\Delta Tu(n-1)|\}$, and so

$$\begin{aligned}\gamma(Tu) &= \max_{t\in\{0,1,\ldots,n-1\}}|\Delta Tu(t)| = \max\{|\Delta Tu(0)|, |\Delta Tu(n-1)|\}\\ &= \max\left\{\phi_p^{-1}(A_u), \left|\phi_p^{-1}\left(A_u - \sum_{i=1}^{n-1} q(i) f(i,u(i),\Delta u(i))\right)\right|\right\}\\ &\le \phi_p^{-1}\left(\sum_{i=1}^{n-1} q(i)\phi_p\left(\frac{d}{L}\right)\right) = \frac{d}{L}L = d.\end{aligned} \tag{4.3}$$

Thus, $T : \overline{P(r,d)} \to \overline{P(r,d)}$ holds.

(2) We show that condition (S_1) in Theorem 2.4 holds.

We take $u(t) = ([n/k]+1)b$, for $t \in \{1,2,\ldots,n\}$, and $u(0) = 0$. It is easy to see that $u(t) \in P(\gamma,\theta,\alpha,b,([n/k]+1)b,d)$ and $\alpha(u) = ([n/k]+1)b > b$. Hence $\{u \in P(\gamma,\theta,\alpha,b,([n/k]+1)b,d) \mid \alpha(u) > b\} \neq \emptyset$. Thus for $u \in P(\gamma,\theta,\alpha,b,([n/k]+1)b,d)$, there is $b \le u(t) \le ([n/k]+1)b$, $|\Delta u(t)| \le d$. Hence by condition (A_2) of this theorem, one has $f(t,u,v) > \phi_p(([n/k]+1)b/M)$ for $t \in [k+1, n-k-1]$. By Lemma 3.4 and combining the conditions on α and P, we have

$$\begin{aligned}&\alpha(Tu)\\ &\quad= \min_{k+1\le t\le n-k-1}|(Tu)(t)|\\ &\quad\ge \frac{1}{([n/k]+1)}\max_{t\in\{0,\ldots,n\}}|Tu(t)| = \frac{1}{([n/k]+1)}(Tu)(n_0)\\ &\quad= \frac{1}{([n/k]+1)}\min\left\{\sum_{j=0}^{n_0-1}\phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i,u(i),\Delta u(i))\right),\right.\\ &\qquad\qquad \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i-1}\phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i,u(i),\Delta u(i))\right)\\ &\qquad\qquad \left. -\sum_{j=n_0}^{n-1}\phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i,u(i),\Delta u(i))\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{([n/k]+1)} \min\left\{ \sum_{j=k+1}^{n_0-1} \phi_p^{-1}\left(\sum_{i=j+1}^{n_0-1} q(i) f(i,u(i),\Delta u(i)) \right), \right. \\
&\qquad\qquad \left. \sum_{j=n_0}^{n-k-1} \phi_p^{-1}\left(\sum_{i=n_0+1}^{j} q(i) f(i,u(i),\Delta u(i)) \right) \right\} \\
&\geq \frac{1}{([n/k]+1)} \min\left\{ \sum_{j=k+1}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]} q(i) f(i,u(i),\Delta u(i)) \right), \right. \\
&\qquad\qquad \left. \sum_{j=[n/2]}^{n-k-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i) f(i,u(i),\Delta u(i)) \right) \right\} \\
&> \frac{1}{([n/k]+1)} \min\left\{ \sum_{j=k+1}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]} q(i) \phi_p\left(\frac{([n/k]+1)b}{M} \right) \right), \right. \\
&\qquad\qquad \left. \sum_{j=[n/2]}^{n-k-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i) \phi_p\left(\frac{([n/k]+1)b}{M} \right) \right) \right\} \\
&= \frac{1}{([n/k]+1)} \frac{([n/k]+1)b}{M} \min\left\{ \sum_{j=k+1}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]} q(i) \right), \sum_{j=[n/2]}^{n-k-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i) \right) \right\} \\
&= \frac{b}{M} M = b.
\end{aligned}
\tag{4.4}
$$

Therefore we have

$$
\alpha(Tu) > b \quad \forall u \in P\left(\gamma, \theta, \alpha, b, \left(\left[\frac{n}{k}\right] + 1 \right) b, d \right). \tag{4.5}
$$

Consequently, condition (S_1) in Theorem 2.4 is satisfied.

(3) We now prove that (S_2) in Theorem 2.4 holds.

With (3.7), we have

$$
\alpha(Tu) \geq \frac{1}{([n/k]+1)} \theta(Tu) > \frac{1}{([n/k]+1)} \left(\left[\frac{n}{k}\right] + 1 \right) b = b \tag{4.6}
$$

for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > ([n/k]+1)b$. Hence, condition (S_2) in Theorem 2.4 is satisfied.

(4) Finally, we prove that (S_3) in Theorem 2.4 is also satisfied.

Since $\psi(0) = 0 < a$, so $0 \overline{\in} R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$. Then, by the condition (S_3) of this theorem,

$$
\begin{aligned}
&\psi(Tu)\\
&= \max_{t \in \{0,1,\dots,n\}} |Tu(t)| = Tu(n_0)\\
&= \max\Bigg\{ \sum_{j=0}^{n_0-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right),\\
&\qquad \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right)\\
&\qquad - \sum_{j=n_0}^{n-1} \phi_p^{-1}\left(A_u - \sum_{i=1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) \Bigg\}\\
&\le \max\Bigg\{ \sum_{j=0}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]+1} q(i) f(i, u(i), \Delta u(i)) \right),\\
&\qquad \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i-1} \phi_p^{-1}\left(\sum_{i=j+1}^{n-1} q(i) f(i, u(i), \Delta u(i)) \right)\\
&\qquad + \sum_{j=[n/2]}^{n-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i) f(i, u(i), \Delta u(i)) \right) \Bigg\}\\
&\le \max\Bigg\{ \sum_{j=0}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]+1} q(i) \phi_p\left(\frac{a}{N}\right) \right),\\
&\qquad \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i-1} \phi_p^{-1}\left(\sum_{i=j+1}^{n-1} q(i) \phi_p\left(\frac{a}{N}\right) \right) + \sum_{j=[n/2]}^{n-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i) \phi_p\left(\frac{a}{N}\right) \right) \Bigg\}\\
&= \frac{a}{N} \max\Bigg\{ \sum_{j=0}^{[n/2]} \phi_p^{-1}\left(\sum_{i=j+1}^{[n/2]+1} q(i) \right), \sum_{i=1}^{m-2} a_i \sum_{j=0}^{\xi_i-1} \phi_p^{-1}\left(\sum_{i=j+1}^{n-1} q(i) \right) + \sum_{j=[n/2]}^{n-1} \phi_p^{-1}\left(\sum_{i=[n/2]+1}^{j} q(i) \right) \Bigg\}\\
&= \frac{a}{N} N = a.
\end{aligned}
\tag{4.7}
$$

Thus condition (S_3) in Theorem 2.4 holds. □

Therefore an application of Theorem 2.4 implies that BVPs (1.2) have at least three positive solutions u_1, u_2, and u_3 such that

$$\max_{t\in\{0,1,\dots,n-1\}}|\Delta u_i(t)| \le d, \quad i = 1,2,3,$$
$$b < \min_{t\in\{k+1,\dots,n-k-1\}}|u_1(t)|, \qquad \max_{t\in\{0,1,\dots,n\}}|u_1(t)| \le b,$$
$$a < \max_{t\in\{0,1,\dots,n\}}|u_2(t)| < \left(\left[\frac{n}{k}\right]+1\right)b, \qquad \text{with } \min_{t\in\{k+1,\dots,n-k-1\}}|\mathrm{u}_2(t)| < b,$$
$$\max_{t\in\{0,1,\dots,n\}}|u_3(t)| < a. \tag{4.8}$$

Acknowledgments

The project is supported by the Natural Science Foundation of China (10971045), and the Natural Science Foundation of Hebei Province (A2009000664).

References

[1] R. I. Avery and A. C. Peterson, "Three positive fixed points of nonlinear operators on ordered Banach spaces," *Computers and Mathematics with Applications*, vol. 42, no. 3–5, pp. 313–322, 2001.
[2] R. I. Avery and J. Henderson, "Three symmetric positive solutions for a second-order boundary value problem," *Applied Mathematics Letters*, vol. 13, no. 3, pp. 1–7, 2000.
[3] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.
[4] Z. Bai, Y. Wang, and W. Ge, "Triple positive solutions for a class of two-point boundary-value problems," *Electronic Journal of Differential Equations*, vol. 2004, no. 6, pp. 1–8, 2004.
[5] Y. Guo, J. Zhang, and Y. Ji, "Existence of triple positive solutions for second-order discrete boundary value problems," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 60534, 10 pages, 2007.
[6] Y. Wang and W. Ge, "Triple positive solutions for two-point boundary-value problems with one-dimensional p-Laplacian," *Applicable Analysis*, vol. 84, no. 8, pp. 821–831, 2005.
[7] Y. Wang and W. Ge, "Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian," *Nonlinear Analysis: Theory, Methods & Application*, vol. 66, no. 6, pp. 1246–1256, 2007.
[8] Y. Wang and C. Hou, "Existence of multiple positive solutions for one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 1, pp. 144–153, 2006.
[9] H. Feng and W. Ge, "Existence of three positive solutions for m-point boundary-value problems with one-dimensional p-Laplacian," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 7, pp. 2017–2026, 2008.