Research Article

Extinction in Nonautonomous Discrete Lotka-Volterra Competitive System with Pure Delays and Feedback Controls

Ling Zhang,[1] Zhidong Teng,[1] Tailei Zhang,[1] and Shujing Gao[2]

[1] *College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China*
[2] *College of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China*

Correspondence should be addressed to Zhidong Teng, zhidong1960@163.com

The paper discusses a nonautonomous discrete time Lotka-Volterra competitive system with pure delays and feedback controls. New sufficient conditions for which a part of the n-species is driven to extinction are established by using the method of multiple discrete Lyapunov functionals.

1. Introduction

The coexistence and global stability of population models are of the interesting subjects in mathematical biology. Many authors have argued that the discrete time models are governed by differential equations which are more appropriate than the continuous ones to describe the dynamics of population when the population has nonoverlapping generations, a lot has been done on discrete Lotka-Volterra systems.

May in [1] firstly considered the following autonomous discrete two-species Lotka-Volterra competitive system:

$$
\begin{aligned}
x(n+1) &= x(n)\exp\{r_1 - a_{11}x(n) - a_{12}y(n)\}, \\
y(n+1) &= y(n)\exp\{r_2 - a_{21}x(n) - a_{22}y(n)\}
\end{aligned}
\tag{1.1}
$$

and studied the stable points, stable cycles, and the chaos behaivor. Further, Lu and Wang [2] studied the permanence and global attractivity of this system.

Chen and Zhou in [3] considered the following periodic discrete two-species Lotka-Volterra competitive system:

$$x(n+1) = x(n) \exp\left\{ r_1\left(1 - \frac{x(n)}{K_1(n)} - \mu_2(n)y(n)\right)\right\},$$
$$y(n+1) = y(n) \exp\left\{ r_2\left(1 - \mu_1 x(n) - \frac{y(n)}{K_2(n)}\right)\right\}$$

(1.2)

and studied the permanence and existence of a periodic solution, and further, sufficient conditions are established on the global stability of the periodic solution.

Zhang and Zhou in [4] investigated the following nonautonomous discrete two-species Lotka-Volterra competitive system:

$$x(n+1) = x(n) \exp\{r_1(n) - a_{11}(n)x(n-k_1) - a_{12}(n)y(n-k_2)\},$$
$$y(n+1) = y(n) \exp\{r_2(n) - a_{21}(n)x(n-l_1) - a_{22}(n)y(n-l_2)\}.$$

(1.3)

Some sufficient conditions were obtained for the permanence of the system.

Wang et al. in [5] studied the following general nonautonomous discrete n-species Lotka-Volterra systems:

$$x_i(k+1) = x_i(k) \exp\left\{ r_i(k) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(k)x_j(k-l) - a_{12}(n)y(n-k_2) \right\}, \quad i = 1, 2, \ldots, n.$$

(1.4)

By applying the linear method and constructing the appropriate Lyapunov functionals, the author established the sufficient conditions which guarantee that any positive solution of this system is stable and attracts others, and obtained some applications of main results.

Muroya in [6, 7] considered the following general nonautonomous discrete n-species Lotka-Volterra systems:

$$N_i(p+1) = N_i(p) \exp\left\{ c_i(p) - a_i(p)N_i(p) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(p)N_j(p-k_l) \right\}, \quad p = 0, 1, 2, \ldots,$$

$$N_i(s) = N_{is} \geq 0 \quad \forall s \leq 0, \ N_{i0} > 0, \quad i = 1, 2, \ldots, n,$$

(1.5)

and related pure delays models, that is, $a_i(p) = 0$ for all $p \geq 0$. The author obtained the permanence and the global asymptotically stable by applying mean-value conditions and the method of constructing discrete Lyapunov functionals.

Liao et al. in [8] discussed the following general discrete nonautonomous n-species competitive system with feedback controls:

$$x_i(k+1) = x_i(k) \exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k) - \sum_{j=1, j\neq i}^{n} c_{ij}(k)x_i(k)x_j(k) - d_i(k)u_i(k)\right\}, \quad k = 0, 1, 2, \ldots,$$

$$\Delta u_i(k) = r_i(k) - e_i(k)u_i(k) + f_i(k)x_i(k), \quad i = 1, 2, \ldots, n.$$

$$(1.6)$$

Some sufficient conditions are established on the permanence and the global stability of the system.

Recently, we see that in [9, 10] the authors studied the following nonautonomous continuous Lotka-Volterra competitive system with pure delays and feedback controls:

$$\dot{x}(t) = x_i(t) \left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t)) - \sum_{j=1}^{n} \int_{-\sigma_{ij}}^{0} b_{ij}(t, s)x_j(t + s)ds \right.$$

$$\left. - c_i(t)u_i(t) - d_i(t)u_i(t - \tau_i(t)) \right],$$

$$(1.7)$$

$$\dot{u}(t) = -e_i(t)u_i(t) + f_i(t)x_i(t) + g_i(t)x_i(t - \delta_i(t)), \quad i = 1, 2, \ldots, n.$$

The sufficient conditions for which a part of the n-species is driven to extinction and the surplus part of the n-species remains permanence are established.

However, we see that for general discrete n-species population systems the results for which a part of the n-species is driven to extinction and the surplus part of the n-species remains the permanence, up to now, are still not obtained. Therefore, motivated by the above works, in this paper we study the following discrete nonautonomous Lotka-Volterra competitive system with pure delays and feedback controls

$$x_i(k + 1) = x_i(k) \exp\left\{ r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k - \tau_{ij}) \right.$$

$$\left. - \sum_{j=1}^{n}\sum_{l=0}^{\sigma_{ij}} b_{ijl}(k)x_j(k - l) - d_i(k)u_i(k - \tau_i) \right\},$$

$$(1.8)$$

$$u_i(k + 1) = (1 - e_i(k))u_i(k) + g_i(k)x_i(k - \delta_i), \quad i = 1, 2, \ldots, n.$$

The main purpose is to establish a criterion for which guarantee the part species $x_{r+1}, x_{r+2}, \ldots, x_n$ in system (1.8) is driven to extinction. The method used in this paper is to constructing the multiple discrete Lyapunov functions. On the permanence of the surplus species $x_1, x_2, \ldots, x_r$, owing to the length of this paper, we will give the discussion in another paper.

This paper is organized as follows. In next section, as preliminaries, some assumptions and useful lemmas are introduced. In Section 3, the main results of this paper on the extinction of a part of the n-species of system (1.8) are established. In Section 4, an example is presented to illustrate the feasibility of our results.

2. Preliminaries

Let Z denote the set of all nonnegative integers. For any bounded sequence $y(k)$, we denote $\overline{y} = \sup_{k \in Z}\{y(k)\}$, $\underline{y} = \inf_{k \in Z}\{y(k)\}$. Throughout this paper, we introduce the following assumptions.

(H_1) $r_i(k)$ is a bounded sequence defined on Z; $a_{ij}(k)$, $d_i(k)$, $g_i(k)$ and $b_{ijl}(k)$ $(i, j = 1, 2, \ldots, n, l = 0, 1, \ldots, \sigma_{ij})$ are nonnegative bounded sequences defined on Z; τ_{ij}, τ_i, δ_i, and σ_{ij} $(i, j = 1, 2, \ldots, n)$ are nonnegative integers.

(H_2) Sequences $e_i(k)$ $(i = 1, 2, \ldots, n)$ satisfy $0 < e_i(k) \leq 1$ for all $k \in Z$.

(H_3) There exist positive integers ω and λ such that for each $i = 1, 2, \ldots, n$

$$\liminf_{n \to \infty} \sum_{s=n}^{n+\omega-1} r_i(s) > 0, \qquad \liminf_{n \to \infty} \sum_{s=n}^{n+\lambda-1} \left[a_{ii}(s) + \sum_{l=0}^{\sigma_{ii}} b_{iil}(s) \right] > 0. \tag{2.1}$$

(H_4) There exists positive integer β such that for each $i = 1, 2, \ldots, n$

$$\limsup_{n \to \infty} \prod_{n}^{n+\beta-1} (1 - e_i(s)) < 1. \tag{2.2}$$

Let $R_+ = [0, \infty)$ and $R_+^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in R_+, i = 1, 2, \ldots, n\}$. We denote by int R_+^n the interior of R_+^n. For any nonnegative constants a and b with $a < b$, we denote by $[a, b]_Z$ the set of all integers in the interval $[a, b]$. For some integer $m \geq 0$, we denote by $D_+[-m, 0]_Z$ the space of all nonnegative discrete time function $\phi : [-m, 0]_Z \to R_+^n$ with norm $\|\phi\| = \sup\{|\phi(s)| : s \in [-m, 0]_Z\}$.

Let $\tau = \max\{\tau_{ij}, \tau_i, \delta_i, \sigma_{ij} : i, j = 1, 2, \ldots, n\}$. Motivated by the biological background of system (1.8), in this paper we only consider all solutions of system (1.8) that satisfy the following initial conditions:

$$x_i(s) = \phi_i(s) \geq 0, \qquad u_i(s) = \psi_i(s) \geq 0, \qquad s \in [-m, 0]_Z, \; i = 1, 2, \ldots, n, \tag{2.3}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in D_+[-m, 0]_Z$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in D_+[-m, 0]_Z$. For any $\phi, \psi \in D_+[-m, 0]_Z$, let $z = (\phi, \psi)$, by the fundamental theory of difference equations, system (1.8) has a unique solution $(x(s, z), (s, z))$ satisfying the initial condition (2.3), where $x(s, z) = (x_1(s, z), x_2(s, z), \ldots, x_n(s, z))$ and $u(s, z) = (u_1(s, z), u_2(s, z), \ldots, u_n(s, z))$. It is obvious that solution $(x(s, z), (s, z))$ is positive, that is, $x_i(s, z) > 0$ and $u_i(s, z) > 0$ $(i = 1, 2, \ldots, n)$ for all $s \in Z$.

We first consider the following nonautonomous difference inequality system:

$$y(n + 1) \leq y(n) \exp\{\alpha(n) - \beta(n)y(n)\}, \quad n \in Z, \tag{2.4}$$

where $\alpha(n)$ and $\beta(n)$ are bounded sequences defined on Z and $\beta(n) \geq 0$ for all $n \in Z$. We have the following result.

Lemma 2.1 (see [11]). *Assume that there exists an integer $\lambda > 0$ such that*

$$\liminf_{n \to \infty} \sum_{k=n}^{n+\lambda} \beta(k) > 0, \tag{2.5}$$

then there exists a constant $M > 0$ such that for any nonnegative solution $y(n)$ of system (2.4) with initial value $y(n_0) = y_0 \geq 0$, where $n_0 \in Z$ is some integer,

$$\limsup_{n \to \infty} y(n) < M. \tag{2.6}$$

Next, we consider the following nonautonomous linear difference equation:

$$v(n + 1) = \gamma(n)v(n) + \omega(n), \tag{2.7}$$

where $\gamma(n)$ and $\omega(n)$ are nonnegative bounded sequences defined on Z. We have the following results.

Lemma 2.2 (see [11]). *Assume that there exists an integer $\lambda > 0$ such that*

$$\limsup_{n \to \infty} \prod_{k=n}^{n+\lambda} \gamma(k) < 1, \tag{2.8}$$

then there exists a constant $M > 0$ such that for any nonnegative solution $v(n)$ of system (2.7) with initial value $v(n_0) = v_0 \geq 0$, where $n_0 \in Z$ is some integer,

$$\limsup_{n \to \infty} v(n) < M. \tag{2.9}$$

Lemma 2.3 (see [11]). *Assume that the conditions of Lemma 2.2 hold, then for any constants $\varepsilon > 0$ and $M_1 > 0$ there exist positive constants $\widehat{\delta} = \widehat{\delta}(\varepsilon)$ and $\widehat{n} = \widehat{n}(\varepsilon, M_1)$ such that for any $\widehat{n}_0 \in Z$ and $0 \leq v_0 \leq M_1$, when $\omega(n) < \widehat{\delta}$ for all $n \geq \widehat{n}_0$, one has*

$$v(n, \widehat{n}_0, v_0) < \varepsilon \quad \forall n \geq \widehat{n}_0 + \widehat{n}, \tag{2.10}$$

where $v(n, \widehat{n}_0, v_0)$ is the solution of (2.7) with initial value $v(\widehat{n}_0) = v_0$.

Lemma 2.4. *Assume that assumptions (H_1)–(H_4) hold, then there exists a constant $M_0 > 0$ such that*

$$\limsup_{k \to \infty} x_i(k) < M_0, \quad \limsup_{k \to \infty} u_i(k) < M_0, \quad i = 1, 2, \ldots, n \tag{2.11}$$

for any positive solution $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ of system (1.8).

Proof. Let $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ be any positive solution of system (1.8). For each $i \in \{1, 2, \ldots, n\}$, we have

$$
\begin{aligned}
x_i(k+1) &= x_i(k) \exp\left\{ r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k - \tau_{ij}) - \sum_{j=1}^{n}\sum_{l=0}^{\sigma_{ij}} b_{ijl}(k)x_j(k - l) - d_i(k)u_i(k - \tau_i) \right\} \\
&\leq x_i(k) \exp\{r_i(k)\},
\end{aligned}
\tag{2.12}
$$

then, for any integer $k \geq 0$ and $\theta \leq 0$ with $k + \theta \geq 0$, summing inequality (2.12) from $k + \theta$ to $k - 1$, we obtain

$$
x_i(k + \theta) \geq x_i(k) \exp\left\{ -\sum_{s=k+\theta}^{k-1} r_i(s) \right\}.
\tag{2.13}
$$

Therefore, for any integer $k \geq \tau$, from (2.13) and the first equation of system (1.8) we obtain

$$
\begin{aligned}
x_i(k+1) &\leq x_i(k) \exp\left\{ r_i(k) - a_{ii}(k)x_i(k - \tau_{ii}) - \sum_{l=0}^{\sigma_{ii}} b_{iil}(k)x_i(k - l) \right\} \\
&\leq x_i(k) \exp\left\{ r_i(k) - a_{ii}(k)x_i(k) \exp\left\{ -\sum_{s=k+\theta}^{k-1} r_i(s) \right\} \right. \\
&\quad \left. - \sum_{l=0}^{\sigma_{ii}} b_{iil}(k)x_i(k) \exp\left\{ -\sum_{s=k-l}^{k-1} r_i(s) \right\} \right\}.
\end{aligned}
\tag{2.14}
$$

Since for any $k \geq \tau$ and $l \in [0, \tau]_Z$

$$
\sum_{s=k+\theta}^{k-1} r_i(s) \leq \tau r_i^u,
\tag{2.15}
$$

where $r_i^u = \sup_{k \in Z} r_i(k)$, we have from (2.14),

$$
x_i(k + i) \leq x_i(k) \exp\left\{ r_i(k) - h_i\left(a_{ii}(k) + \sum_{l=0}^{\sigma_{ii}} b_{iil}(k) \right) x_i(k) \right\}
\tag{2.16}
$$

for any $k \geq \tau$, where $h_i = \exp\{-\tau r_i^u\}$.

We consider the following auxiliary equation:

$$
y_i(k + 1) \leq y_i(k) \exp\{r_i(k) - \beta_i(k)y_i(k)\}, \quad k \in Z,
\tag{2.17}
$$

where $\beta_i(k) = h_i(a_{ii}(k) + \sum_{l=0}^{\sigma_{ii}} b_{iil}(k))$, then by assumption (H_3) and applying Lemma 2.1 there exists a constant $N_i > 0$ such that

$$\limsup_{k \to \infty} x_i(k) < N_i \tag{2.18}$$

for any positive solution $y_i(k)$ of (2.17). Therefore, from the comparison theorem of difference equation, we finally obtain

$$\limsup_{k \to \infty} x_i(k) < N_i. \tag{2.19}$$

Further form inequality (2.19), there exists a positive constant $k_i \geq \tau$ such that

$$x_i(k) < N_i, \quad x_i(k - \delta_i) < N_i \quad \forall k \geq k_i. \tag{2.20}$$

Thus, from the second equation of system (1.8), we obtain

$$u_i(k+1) \leq (1 - e_i(k))u_i(k) + (f_i(k) + g_i(k))N_i \tag{2.21}$$

for all $k \geq k_i$. We consider the following auxiliary equation:

$$v_i(k+1) = (1 - e_i(k))v_i(k) + (f_i(k) + g_i(k))N_i, \tag{2.22}$$

then by assumption (H_4) and applying Lemma 2.2, there exists a constant N_i^* such that

$$\limsup_{k \to \infty} v_i(k) < N_i^* \tag{2.23}$$

for any positive solution $v_i(k)$ of (2.22). Let $v_i^*(k)$ be the solution of (2.22) with initial value $v_i^*(k_i) = u_i(k_i)$, then from the comparison theorem of difference equation, we have $u_i(k) \leq v_i^*(k_i)$ for all $k \geq k_i$. Thus, we finally obtain

$$\limsup_{k \to \infty} u_i(k) < N_i^*. \tag{2.24}$$

Let $N = \max_{1 \leq i \leq n}\{N_i, N_i^*\}$, then from (2.19) and (2.24) we finally see that the conclusions of Lemma 2.4 hold. $\qquad \square$

3. Main Results

In this section, we discuss the extinction of the part of species $x_{r+1}, \ldots, x_n$ of system (1.8). Define functions as follows:

$$A_{ij}(k) = a_{ij}(k + \tau_{ij}) + \sum_{l=0}^{\sigma_{ij}} b_{ijl}(k + l),$$

$$D_i(k) = d_i(k + \tau_i), \qquad R_i(k) = g_i(k + \delta_i), \tag{3.1}$$

where $i, j = 1, 2, \ldots, n$.

Theorem 3.1. *Assume that assumptions (H_1)–(H_4) hold and there exists an integer $1 \leq r < n$ such that for any $h > r$ there exists an integer $i_h < h$ such that*

$$\limsup_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_h(s)}{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)} < \liminf_{k \to \infty} \frac{A_{hj}(k)}{A_{i_h j}(k)} \quad \forall j \leq h,$$

$$\liminf_{k \to \infty} \frac{D_h(k)}{e_h(k)} > \limsup_{k \to \infty} \left(\frac{A_{i_h h}(k)}{R_h(k)} \limsup_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_h(s)}{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)} - \frac{A_{hh}(k)}{R_h(k)} \right), \tag{3.2}$$

$$\limsup_{k \to \infty} \frac{D_{i_h}(k)}{e_{i_h}(k)} < \liminf_{k \to \infty} \left(\frac{A_{h i_h}(k)}{R_{i_h}(k)} \liminf_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)}{\sum_{s=k}^{k+\omega-1} r_h(s)} - \frac{A_{i_h i_h}(k)}{R_{i_h}(k)} \right),$$

then for each $i = r + 1, \ldots, n$ one has

$$\lim_{k \to \infty} x_i(k) = 0, \qquad \lim_{k \to \infty} u_i(k) = 0, \qquad \sum_{k=0}^{\infty} x_i(k) < \infty \tag{3.3}$$

for any positive solution $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ of system (1.8).

Proof. From assumption (H_2), there exist constant $\eta_0 > 0$ and integer $K_0 > 0$ such that

$$\sum_{s=k}^{k+\omega-1} r_i(s) \geq \eta_0 \quad \forall k \geq K_0. \tag{3.4}$$

We first prove that $\lim_{k \to \infty} x_n(k) = 0$. Let $h = n$ and $i_h = p$. From conditions (3.2) we can find positive constants $\alpha_{n1}, \alpha_{n2}, \alpha_{n3}, \alpha_{n4}, \varepsilon_n$ and integer $K_n \geq K_0$ such that

$$
\frac{\sum_{s=k}^{k+\omega-1} r_n(s)}{\sum_{s=k}^{k+\omega-1} r_p(s)} < \frac{\alpha_{n1}}{\alpha_{n2}} - \varepsilon_n < \frac{\alpha_{n1}}{\alpha_{n2}} < \frac{A_{nj}(k)}{A_{pj}(k)},
$$

$$
\frac{D_n(k)}{e_n(k)} > \frac{\alpha_{n4}}{\alpha_{n2}} > \frac{\alpha_{n1} A_{pn}(k) - \alpha_{n2} A_{nn}(k)}{\alpha_{n2} R_n(k)}, \tag{3.5}
$$

$$
\frac{D_p(k)}{e_p(k)} < \frac{\alpha_{n3}}{\alpha_{n1}} < \frac{\alpha_{n2} A_{np}(k) - \alpha_{n1} A_{pp}(k)}{\alpha_{n1} R_p(k)}
$$

for all $k \geq K_n$ and $j = 1, 2, \ldots, n$. Consequently,

$$
\sum_{s=k}^{k+\omega-1} \left(-\alpha_{n1} r_p(s) + \alpha_{n2} r_n(s) \right) < -\varepsilon_n \alpha_{n2} \eta_0, \tag{3.6}
$$

$$
\alpha_{n1} A_{pj}(k) - \alpha_{n2} A_{nj}(k) < 0, \tag{3.7}
$$

$$
\alpha_{n4} e_n(k) - \alpha_{n2} D_n(k) < 0, \tag{3.8}
$$

$$
\alpha_{n1} A_{pp}(k) - \alpha_{n2} A_{np}(k) + \alpha_{n3} R_p(k) < 0, \tag{3.9}
$$

$$
-\alpha_{n3} e_p(k) + \alpha_{n1} D_p(k) < 0 \tag{3.10}
$$

$$
\alpha_{n1} A_{pn}(k) - \alpha_{n2} A_{nn}(k) - \alpha_{n4} R_n(k) < 0 \tag{3.11}
$$

for all $k \geq K_n$ and $j = 1, 2, \ldots, n$.

Let $(x(k), u(k)) = (x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ be any positive solution of system (1.8). Constructing the following discrete Lyapunov functional

$$
\begin{aligned}
V_n(k) = {}& x_p^{-\alpha_{n1}}(k) x_n^{\alpha_{n2}}(k) \\
& \times \exp\Bigg\{ \alpha_{n3} u_p(k) - \alpha_{n4} u_n(k) \\
& + \alpha_{n1} \sum_{j=1}^{n} \sum_{s=k-\tau_{pj}}^{k-1} a_{pj}(s + \tau_{pj}) x_j(s) + \alpha_{n1} \sum_{j=1}^{n} \sum_{l=0}^{\sigma_{pj}} \sum_{s=k-l}^{k-1} b_{pjl}(s + l) x_j(s) \\
& + \alpha_{n1} \sum_{s=k-\tau_p}^{k-1} d_p(s + \tau_p) u_p(s) - \alpha_{n2} \sum_{j=1}^{n} \sum_{s=k-\tau_{nj}}^{k-1} a_{nj}(s + \tau_{nj}) x_j(s) \\
& - \alpha_{n2} \sum_{j=1}^{n} \sum_{l=0}^{\sigma_{nj}} \sum_{s=k-l}^{k-1} b_{njl}(s + l) x_j(s) - \alpha_{n2} \sum_{s=k-\tau_n}^{k-1} d_n(s + \tau_n) u_n(s) \\
& + \alpha_{n3} \sum_{s=k-\delta_p}^{k-1} g_p(s + \delta_p) x_p(s) - \alpha_{n4} \sum_{s=k-\delta_n}^{k-1} g_n(s + \delta_\text{n}) x_n(s) \Bigg\}.
\end{aligned} \tag{3.12}
$$

By calculating, we obtain

$$
\frac{V_n(k+1)}{V_n(k)}
$$

$$
= \exp\left\{ -\alpha_{n1}\left[r_p(k) - \sum_{j=1}^{n} a_{pj}(k)x_j(k-\tau_{pj}) - \sum_{j=1}^{n}\sum_{l=0}^{\sigma_{pj}} b_{pjl}(k)x_j(k+l) - d_p(k)u_p(k-\tau_p) \right] \right.
$$

$$
+ \alpha_{n2}\left[r_n(k) - \sum_{j=1}^{n} a_{nj}(k)x_j(k-\tau_{nj}) - \sum_{j=1}^{n}\sum_{l=0}^{\sigma_{nj}} b_{njl}(k)x_j(k+l) - d_n(k)u_n(k-\tau_n) \right]
$$

$$
+ \alpha_{n3}\left[-e_p(k)u_p(k) + g_p(k)x_p(k-\delta_p) \right] - \alpha_{n4}\left[-e_n(k)u_n(k) + g_n(k)x_n(k-\delta_n) \right]
$$

$$
+ \alpha_{n1}\sum_{j=1}^{n}\left[a_{pj}(k+\tau_{pj})x_j(k) - a_{pj}(k)x_j(k-\tau_{pj}) \right]
$$

$$
+ \alpha_{n1}\sum_{j=1}^{n}\sum_{l=0}^{\sigma_{pj}}\left[b_{pjl}(k+l)x_j(k) - b_{pjl}(k)x_j(k-l) \right]
$$

$$
+ \alpha_{n1}\left[d_p(k+\tau_p)u_p(k) - d_p(k)u_p(k-\tau_p) \right]
$$

$$
- \alpha_{n2}\sum_{j=1}^{n}\left[a_{nj}(k+\tau_{nj})x_j(k) - a_{nj}(k)x_j(k-\tau_{nj}) \right]
$$

$$
- \alpha_{n2}\sum_{j=1}^{n}\sum_{l=0}^{\sigma_{nj}}\left[b_{njl}(k+l)x_j(k) - b_{njl}(k)x_j(k-l) \right]
$$

$$
- \alpha_{n2}\left[d_n(k+\tau_n)u_n(k) - d_n(k)u_n(k-\tau_n) \right]
$$

$$
+ \alpha_{n3}\left[g_p(k+\delta_p)x_p(k) - g_p(k)x_p(k-\delta_p) \right]
$$

$$
\left. - \alpha_{n4}\left[g_n(k+\delta_n)x_n(k) - g_n(k)x_n(k-\delta_n) \right] \right\}
$$

$$
= \exp\left\{ -\alpha_{n1}r_p(k) + \alpha_{n2}r_n(k) - (\alpha_{n3}e_p(k) - \alpha_{n1}D_p(k))u_p(k) \right.
$$

$$
- (\alpha_{n2}D_n(k) - \alpha_{n4}e_n(k))u_n(k) + \sum_{j\neq p}^{n-1}(\alpha_{n1}A_{pj}(k) - \alpha_{n2}A_{nj}(k))x_j(k)
$$

$$
+ (\alpha_{n1}A_{pp}(k) - \alpha_{n2}A_{np}(k) - \alpha_{n3}R_p(k))x_p(k)
$$

$$
\left. + (\alpha_{n1}A_{pn}(k) - \alpha_{n2}A_{nn}(k) - \alpha_{n4}R_n(k))x_n(k) \right\}.
$$

$$(3.13)$$

From inequalities (3.7)–(3.11), we can obtain

$$V_n(k+1) \leq V_n(k) \exp\{-\alpha_{n1} r_p(k) + \alpha_{n2} r_n(k)\} \quad \forall k \geq K_n. \tag{3.14}$$

For any $k > K_n$, we choose an integer $m \geq 0$, such that $k \in [K_n + m\omega, K_n + (m+1)\omega)$, then from (3.6) and (3.14) we further have

$$
\begin{aligned}
V_n(k) &\leq V_n(K_n) \exp\left\{ \sum_{s=K_n}^{k-1} \left(-\alpha_{n1} r_p(s) + \alpha_{n2} r_n(s)\right) \right\} \\
&\leq V_n(K_n) \exp\left\{ \left(\sum_{s=K_n}^{K_n+m\omega-1} + \sum_{s=K_n+m\omega}^{k-1} \right) \left(-\alpha_{n1} r_p(s) + \alpha_{n2} r_n(s)\right) \right\} \\
&\leq V_n(K_n) \exp\{-\varepsilon_n \alpha_{n2} \eta_0 m + M_n\} \\
&\leq V_n(K_n) \exp\{-\lambda_n k + M_n^*\},
\end{aligned}
\tag{3.15}
$$

where

$$
\lambda_n = \frac{\varepsilon_n \alpha_{n2} \eta_0}{\omega}, \qquad M_n^* = M_n + \varepsilon_n \alpha_{n2} \eta_0 \left(1 + \frac{K_n}{\omega}\right),
$$
$$
M_n = \sup_{k \in Z} (-\alpha_{n1} r_p(k) + \alpha_{n2} r_n(k)) \omega.
\tag{3.16}
$$

On the other hand, from assumptions (H_1), we have

$$
\begin{aligned}
V_n(k) &\geq x_p^{-\alpha_{n1}}(k) x_n^{\alpha_{n2}}(k) \\
&\quad \times \exp\left\{ -\alpha_{n4} u_n(k) - \sum_{j=1}^{n} \alpha_{n2} \sum_{s=k-\tau_{nj}}^{k-1} a_{nj}(s + \tau_{nj}) x_j(s) \right. \\
&\qquad\quad - \sum_{j=1}^{n} \alpha_{n2} \sum_{l=0}^{\sigma_{nj}} \sum_{s=k-l}^{k-1} b_{njl}(s+l) x_j(s) - \alpha_{n2} \sum_{s=k-\tau_n}^{k-1} d_n(s+\tau_n) u_n(s) \\
&\qquad\quad \left. - \alpha_{n4} \sum_{s=k-\delta_n}^{k-1} g_n(s+\delta_n) x_n(s) \right\} \\
&\geq x_p^{-\alpha_{n1}}(k) x_n^{\alpha_{n2}}(k) \\
&\quad \times \exp\left\{ -\alpha_{n4} M_0 - \alpha_{n2} \tau \sum_{j=1}^{n} \overline{a_{nj}} M_0 \right. \\
&\qquad\quad \left. - \alpha_{n2} \sum_{j=1}^{n} \sum_{l=0}^{\sigma_{nj}} \overline{b_{njl}} l M_0 - \alpha_{n2} \tau \overline{d_n} M_0 - \alpha_{n4} \tau \overline{g_n} M_0 \right\}
\end{aligned}
\tag{3.17}
$$

for all $k \geq \tau$, where constant M_0 is given in Lemma 2.4. Hence, there exist a positive constant $Q_n > 0$ such that

$$V_n(k) \geq x_p^{-\alpha_{n1}}(k) x_n^{\alpha_{n2}}(k) Q_n \quad \forall k \geq \tau. \tag{3.18}$$

From (3.15) and (3.18), we obtain

$$
\begin{aligned}
x_n(k) &\leq \left[Q_n^{-1} x_p^{\alpha_{n1}}(k) V_n(K_n) \exp\{-\lambda_n k + M_n^*\} \right]^{1/\alpha_{n2}} \\
&\leq Q_n^* \exp\{-\lambda_n^* k\}
\end{aligned}
\tag{3.19}
$$

for all $k \geq K_n + \tau$, where

$$Q_n^* = \sup_{k \in Z} \left[x_p^{\alpha_{n1}}(k) V_n(K_n) Q_n^{-1} \exp M_n^* \right]^{1/\alpha_{n2}}, \qquad \lambda_n^* = \frac{\lambda_n}{\alpha_{n2}}. \tag{3.20}$$

From (3.19), we finally obtain

$$\lim_{k \to \infty} x_n(k) = 0, \qquad \sum_{k=0}^{\infty} x_n(k) < \infty. \tag{3.21}$$

Next, we consider the second equation of system (1.8), applying Lemma 2.3 we can easily obtain $u_n(k) \to 0$ as $k \to \infty$.

Now, we suppose that for any $t > r$, we have obtained

$$\lim_{k \to \infty} x_i(k) = 0, \qquad \lim_{k \to \infty} u_i(k) = 0, \qquad \sum_{k=0}^{\infty} x_n(k) < \infty \tag{3.22}$$

for all $i > t$. We further will prove

$$\lim_{k \to \infty} x_t(k) = 0, \qquad \sum_{k=0}^{\infty} x_t(k) < \infty. \tag{3.23}$$

From conditions (3.2), we can choose positive constants $\alpha_{t1}, \alpha_{t2}, \alpha_{t3}, \alpha_{t4}, \varepsilon_t$ and integer $K_t \geq K_n$ such that

$$
\begin{aligned}
&\frac{\sum_{s=k}^{k+\omega-1} r_t(s)}{\sum_{s=k}^{k+\omega-1} r_q(s)} < \frac{\alpha_{t1}}{\alpha_{t2}} - \varepsilon_t < \frac{\alpha_{t1}}{\alpha_{t2}} < \frac{A_{tj}(k)}{A_{qj}(k)}, \\
&\frac{D_t(k)}{e_t(k)} > \frac{\alpha_{t4}}{\alpha_{t2}} > \frac{\alpha_{t1} A_{qn}(k) - \alpha_{t2} A_{tt}(k)}{\alpha_{t2} R_t(k)}, \\
&\frac{D_q(k)}{e_q(k)} < \frac{\alpha_{t3}}{\alpha_{t1}} > \frac{\alpha_{t2} A_{tq}(k) - \alpha_{t1} A_{qq}(k)}{\alpha_{t1} R_q(k)}
\end{aligned}
\tag{3.24}
$$

for all $k \geq K_t$ and $j = 1, 2, \ldots, t$, where $i_h = q$. Consequently,

$$\sum_{s=k}^{k+\omega-1} \left(-\alpha_{t1} r_q(s) + \alpha_{t2} r_t(s) \right) < -\varepsilon_t \alpha_{t2} \eta_0, \tag{3.25}$$

$$\alpha_{t1} A_{qj}(k) - \alpha_{t2} A_{tj}(k) < 0, \tag{3.26}$$

$$\alpha_{t4} e_t(k) - \alpha_{t2} D_t(k) < 0, \tag{3.27}$$

$$\alpha_{t1} A_{qq}(k) - \alpha_{t2} A_{tq}(k) + \alpha_{t3} R_q(k) < 0, \tag{3.28}$$

$$-\alpha_{t3} e_q(k) + \alpha_{t1} D_q(k) < 0 \tag{3.29}$$

$$\alpha_{t1} A_{qt}(k) - \alpha_{t2} A_{tt}(k) - \alpha_{t4} R_t(k) < 0 \tag{3.30}$$

for all $k \geq K_t$ and $j = 1, 2, \ldots, t$. Constructing the following discrete Lyapunov functional

$$
\begin{aligned}
V_t(k) = {} & x_q^{-\alpha_{t1}}(k) x_t^{\alpha_{t2}}(k) \\
& \times \exp\Bigg\{ \alpha_{t3} u_q(k) - \alpha_{t4} u_t(k) \\
& + \alpha_{t1} \sum_{j=1}^{n} \sum_{s=k-\tau_{qj}}^{k-1} a_{qj}(k+\tau_{qj}) x_j(s) + \alpha_{t1} \sum_{j=1}^{n} \sum_{l=0}^{\sigma_{qj}} \sum_{s=k-l}^{k-1} b_{qjl}(k+l) x_j(s) \\
& + \alpha_{t1} \sum_{s=k-\tau_q}^{k-1} d_q(k+\tau_q) u_q(s) - \alpha_{t2} \sum_{j=1}^{n} \sum_{s=k-\tau_{tj}}^{k-1} a_{tj}(k+\tau_{tj}) x_j(s) \\
& - \alpha_{t2} \sum_{j=1}^{n} \sum_{l=0}^{\sigma_{tj}} \sum_{s=k-l}^{k-1} b_{tjl}(k+l) x_j(s) - \alpha_{t2} \sum_{s=k-\tau_t}^{k-1} d_t(k+\tau_t) u_t(s) \\
& + \alpha_{t3} \sum_{s=k-\delta_q}^{k-1} g_q(k+\delta_q) x_q(s) - \alpha_{t4} \sum_{s=k-\delta_t}^{k-1} g_t(k+\delta_t) x_t(s) \Bigg\}.
\end{aligned}
\tag{3.31}
$$

Calculating $V_t(k+1)/V_t(k)$, similarly to $V_n(k)$, we can obtain

$$
\begin{aligned}
\frac{V_t(k+1)}{V_t(k)} = \exp\bigg\{ & -\alpha_{t1}r_q(k) + \alpha_{t2}r_t(k) \\
& - (\alpha_{t3}e_q(k) - \alpha_{t1}D_q(k))u_q(k) \\
& + (-\alpha_{t2}D_t(k) + \alpha_{t4}e_t(k))u_t(k) \\
& + \sum_{\substack{j=1 \\ j\neq q,t}}^{n-1} (\alpha_{t1}A_{qj}(k) - \alpha_{t2}A_{tj}(k))x_j(k) \\
& + (\alpha_{t1}A_{qq}(k) - \alpha_{t2}A_{tp}(k) - \alpha_{t2}R_q(k))x_q(k) \\
& + (\alpha_{t1}A_{qt}(k) - \alpha_{t2}A_{tt}(k) - \alpha_{t4}R_t(k))x_t(k)\bigg\}
\end{aligned}
\tag{3.32}
$$

for all $k \geq K_t$. From inequalities (3.26)–(3.30), we further obtain

$$
V_t(k+1) \leq V_t(k)\exp\{-\alpha_{t1}r_q(k) + \alpha_{t2}r_t(k) + P(k)\}
\tag{3.33}
$$

for all $k \geq K_t$, where

$$
P(k) = \sum_{j=t+1}^{n} (\alpha_{t1}A_{qj}(k) + \alpha_{t2}A_{tj}(k))x_j(k).
\tag{3.34}
$$

From (3.19) we have $P(k) \to 0$ as $k \to \infty$. Hence,

$$
\lim_{k\to\infty} \sum_{s=k}^{k+\omega-1} P(s) = 0.
\tag{3.35}
$$

Thus, from (3.22) we can obtain that there exists an integer $K_t^* \geq K_t$ such that

$$
\sum_{s=k}^{k+\omega-1} (-\alpha_{t1}r_q(s) + \alpha_{t2}r_t(s) + P(s)) < -\frac{1}{2}\varepsilon_t\alpha_{n2}\eta_0
\tag{3.36}
$$

for all $k > K_t^*$. By calculating, from (3.33) we obtain

$$
V_t(k) \leq V_t(K_t^*)\exp\bigg\{ \sum_{s=K_t^*}^{k-1} (-\alpha_{t1}r_q(s) + \alpha_{t2}r_t(s) + P(s))\bigg\}.
\tag{3.37}
$$

From this, a similar argument as in the proof of (3.15)–(3.19), we further can obtain

$$x_t(k) \leq Q_t^* \exp\{-\lambda_t^* k\} \quad \forall k \geq K_t^*, \tag{3.38}$$

where Q_t^* and λ_t^* are two positive constants. From (3.38), we finally obtain

$$\lim_{k \to \infty} x_t(k) = 0, \qquad \sum_{k=0}^{\infty} x_t(k) < \infty. \tag{3.39}$$

Next, we consider the second equation of system (1.8), applying Lemma 2.3 we can easily obtain $u_t(k) \to 0$ as $k \to \infty$.

Finally, according to the induction principle, we have

$$\lim_{k \to \infty} x_i(k) = 0, \quad \lim_{k \to \infty} u_i(k) = 0, \quad \sum_{k=0}^{\infty} x_i(k) < \infty \quad \forall i > r. \tag{3.40}$$

This completes the proof of Theorem 3.1. $\square$

As consequences of Theorem 3.1, we consider the following two special cases of system (1.8).

Case 1. Nondelayed nonautonomous discrete n-species Lotka-Volterra competitive systems with feedback controls

$$x_i(k+1) = x_i(k) \exp\left\{ r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k) - d_i(k)u_i(k) \right\},$$

$$u_i(k+1) = (1 - e_i(k))u_i(k) + g_i(k)x_i(k), \quad i = 1, 2, \ldots, n. \tag{3.41}$$

For system (3.41), assumptions (H_1) and (H_3) become into the following form

(H_1') $r_i(k)$ is a bounded sequence defined on Z; $a_{ij}(k)$, $d_i(k)$ and $g_i(k)$ $(i, j = 1, 2, \ldots, n)$ are nonnegative bounded sequences defined on Z.

(H_3') There exist positive integers ω and λ such that for each $i = 1, 2, \ldots, n$

$$\liminf_{n \to \infty} \sum_{s=n}^{n+\omega-1} r_i(s) > 0, \qquad \liminf_{n \to \infty} \sum_{s=n}^{n+\lambda-1} a_{ii}(s) > 0. \tag{3.42}$$

Directly from Theorem 3.1, we have the following corollary.

Corollary 3.2. *Assume that assumptions* (H_1'), (H_2), (H_3') *and* (H_4) *hold and there exists a integer* $1 \leq r < n$ *such that for any* $h > r$ *there exists an integer* $i_h < h$ *such that*

$$\limsup_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_h(s)}{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)} < \liminf_{k \to \infty} \frac{a_{hj}(k)}{a_{i_h j}(k)} \quad \forall j \leq h,$$

$$\liminf_{k \to \infty} \frac{d_h(k)}{e_h(k)} > \limsup_{k \to \infty} \left(\frac{a_{i_h h}(k)}{g_h(k)} \limsup_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_h(s)}{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)} - \frac{a_{hh}(k)}{g_h(k)} \right), \tag{3.43}$$

$$\limsup_{k \to \infty} \frac{d_{i_h}(k)}{e_{i_h}(k)} < \liminf_{k \to \infty} \left(\frac{a_{h i_h}(k)}{g_{i_h}(k)} \liminf_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)}{\sum_{s=k}^{k+\omega-1} r_h(s)} - \frac{a_{i_h i_h}(k)}{g_{i_h}(k)} \right),$$

then for each $i = r + 1, \ldots, n$ *one has*

$$\lim_{k \to \infty} x_i(k) = 0, \qquad \lim_{k \to \infty} u_i(k) = 0, \qquad \sum_{k=0}^{\infty} x_i(k) < \infty \tag{3.44}$$

for any positive solution $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ *of system* (3.41).

Case 2. Pure delayed nonautonomous discrete n-species Lotka-Volterra competitive systems without feedback controls

$$x_i(k+1) = x_i(k) \exp \left\{ r_i(k) - \sum_{j=1}^{n} a_{ij}(k) x_j(k - \tau_{ij}) - \sum_{j=1}^{n} \sum_{l=0}^{\sigma_{ij}} b_{ijl}(k) x_j(k-l) \right\}, \quad i = 1, 2, \ldots, n. \tag{3.45}$$

For system (3.41), assumption (H_1) becomes into the following form.

(H_1'') $ri(k)$ is a bounded sequence defined on Z; $a_{ij}(k)$ and $b_{ijl}(k)$ $(i, j = 1, 2, \ldots, n, l = 0, 1, \ldots, \sigma_{ij})$ are nonnegative bounded sequences defined on Z; τ_{ij} and σ_{ij} $(i, j = 1, 2, \ldots, n)$ are nonnegative integers.

Directly from Theorem 3.1, we have the following corollary.

Corollary 3.3. *Assume that assumptions* (H_1'') *and* (H_3) *hold and there exists an integer* $1 \leq r < n$ *such that for any* $h > r$ *there exists an integer* $i_h < h$ *such that*

$$\limsup_{k \to \infty} \frac{\sum_{s=k}^{k+\omega-1} r_h(s)}{\sum_{s=k}^{k+\omega-1} r_{i_h}(s)} < \liminf_{k \to \infty} \frac{A_{hj}(k)}{A_{i_h j}(k)} \quad \forall j \leq h, \tag{3.46}$$

then for each $i = r + 1, \ldots, n$ *one has*

$$\lim_{k \to \infty} x_i(k) = 0, \qquad \sum_{k=0}^{\infty} x_i(k) < \infty \tag{3.47}$$

for any positive solution $(x_1(k), x_2(k), \ldots, x_n(k))$ *of system* (3.45).

Remark 3.4. By comparison, we easily see that the results obtained in this paper are a very good extension of the corresponding results obtained in [9] on the extinction of species for nonautonomous continuous Lotka-Volterra competitive system with pure-delays and feedback controls to discrete ones.

4. An Example

We consider the following periodic discrete three-species Lotka-Volterra competitive system with pure-delays and feedback controls:

$$x_i(k+1) = x_i(k) \exp\left\{ r_i(k) - \sum_{j=1}^{3} a_{ij}(k)x_j(k - \tau_{ij}) - d_i(k)u_i(k - \tau_i) \right\},$$

$$u_i(k+1) = (1 - e_i(k))u_i(k) + g_i(k)x_i(k - \delta_i), \quad i = 1, 2, 3.$$

(4.1)

where

$$r_1(k) = \frac{1}{2} + \sin\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad r_2(k) = \frac{1}{2} + \sin\left(\frac{1}{4} + \frac{\pi}{2}k\right),$$

$$r_3(k) = 3\left(\frac{1}{4} + \sin\left(\frac{1}{5} + \frac{\pi}{2}k\right)\right), \qquad a_{11}(k) = \frac{1}{3}\left(1 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right)\right),$$

$$a_{12}(k) = \frac{1}{6} + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad a_{13}(k) = \frac{1}{4} + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right),$$

$$a_{21}(k) = 1 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad a_{22}(k) = \frac{1}{2} + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right),$$

$$a_{31}(k) = 4 + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad a_{32}(k) = \frac{9}{2} + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right),$$

$$a_{33}(k) = 8 + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad d_1(k) = \frac{1}{10}\left(1 + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right)\right),$$

$$d_2(k) = \frac{1}{2} + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad d_3(k) = 2 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right),$$

$$e_1(k) = \frac{1}{3}\left(1 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right)\right), \qquad e_2(k) = \frac{1}{4}\left(1 + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right)\right),$$

$$e_3(k) = \frac{1}{3}\left(1 + \sin^2\left(\frac{1}{3} + \frac{\pi}{2}k\right)\right), \qquad g_1(k) = \frac{1}{3}\left(1 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right)\right),$$

$$g_2(k) = 1 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right), \qquad g_3(k) = 1 + \cos^2\left(\frac{1}{3} + \frac{\pi}{2}k\right).$$

(4.2)

Clearly, in system (4.2), $r = 2, h = 3$, and $i_h = 1$. By calculating, we obtain

$$\limsup_{k\to\infty}\frac{\sum_{s=k}^{k+3} r_3(s)}{\sum_{s=k}^{k+3} r_1(s)} = \limsup_{k\to\infty}\frac{\sum_{s=k}^{k+3} 3(1/4 + \sin(1/5 + (\pi/2)s))}{\sum_{s=k}^{k+3}[1/2 + \sin(1/3 + (\pi/2)s)]} = \frac{3}{2},$$

$$\liminf_{k\to\infty}\frac{\sum_{s=k}^{k+3} r_1(s)}{\sum_{s=k}^{k+3} r_3(s)} = \liminf_{k\to\infty}\frac{\sum_{s=k}^{k+3}[1/2 + \sin(1/3 + (\pi/2)s)]}{\sum_{s=k}^{k+3} 3(1/4 + \sin(1/5 + (\pi/2)s))} = \frac{2}{3},$$

$$\liminf_{k\to\infty}\frac{A_{33}(k)}{A_{13}(k)} = \liminf_{k\to\infty}\frac{8 + \sin^2(1/3 + (\pi/2)(k + \tau_{33}))}{1/4 + \cos^2(1/3 + (\pi/2)(k + \tau_{13}))} \geq \frac{32}{5},$$

$$\liminf_{k\to\infty}\frac{A_{32}(k)}{A_{12}(k)} = \liminf_{k\to\infty}\frac{9/2 + \sin^2(1/3 + (\pi/2)(k + \tau_{32}))}{1/6 + \sin^2(1/3 + (\pi/2)(k + \tau_{13}))} \geq \frac{54}{14},$$

$$\liminf_{k\to\infty}\frac{A_{31}(k)}{A_{11}(k)} = \liminf_{k\to\infty}\frac{4 + \sin^2(1/3 + (\pi/2)(k + \tau_{31}))}{1/3(1 + \cos^2(1/3 + (\pi/2)(k + \tau_{11})))} \geq 6,$$

$$\limsup_{k\to\infty}\left(\frac{A_{13}(k)}{R_3(k)}\limsup_{k\to\infty}\frac{\sum_{s=k}^{k+3} r_3(s)}{\sum_{s=k}^{k+3} r_1(s)} - \frac{A_{33}(k)}{R_3(k)}\right)$$

$$= \limsup_{k\to\infty}\frac{3/2(1/4 + \cos^2(1/3 + (\pi/2)(k + \tau_{13}))) - \left(8 + \sin^2(1/3 + (\pi/2)(k + \tau_{33}))\right)}{1 + \cos^2(1/3 + (\pi/2)(k + \delta_3))}$$

$$\leq \frac{3/2(1/4 + 1) - 8}{2} < 0,$$

$$\liminf_{k\to\infty}\left(\frac{A_{31}(k)}{R_1(k)}\liminf_{k\to\infty}\frac{\sum_{s=k}^{k+3} r_1(s)}{\sum_{s=k}^{k+3} r_3(s)} - \frac{A_{11}(k)}{R_1(k)}\right)$$

$$= \liminf_{k\to\infty}\frac{2/3\left(4 + \sin^2(1/3 + (\pi/2)(k + \tau_{31}))\right) - (1/3(1 + \cos^2(1/3 + (\pi/2)(k + \tau_{11}))))}{1/3(1 + \cos^2(1/3 + (\pi/2)(k + \delta_1)))}$$

$$\geq \frac{(2/3)4 - (1/3)2}{2/3} = 3,$$

$$\liminf_{k\to\infty}\frac{D_3(k)}{e_3(k)} = \liminf_{k\to\infty}\frac{2 + \cos^2(1/3 + (\pi/2)(k + \tau_3))}{1/3\left(1 + \sin^2(1/3 + (\pi/2)k)\right)} \geq 3,$$

$$\limsup_{k\to\infty}\frac{D_1(k)}{e_1(k)} = \limsup_{k\to\infty}\frac{1/10\left(1 + \sin^2(1/3 + (\pi/2)(k + \tau_1))\right)}{1/3(1 + \cos^2(1/3 + (\pi/2)k))} \leq \frac{3}{5}.$$

$$(4.3)$$

From these inequalities we see that all conditions (3.2) in Theorem 3.1 hold. In addition, we also see that assumptions (H_1)–(H_4) obviously hold. Therefore, by Theorem 3.1 we obtain that species x_3 in system (4.3) is extinct.

Acknowledgment

References

[1] R. May, "Biological population with nonoverlapping generations: stable points, stable cycles and chaos," *Science*, vol. 186, pp. 645–647, 1974.

[2] Z. Lu and W. Wang, "Permanence and global attractivity for Lotka-Volterra difference systems," *Journal of Mathematical Biology*, vol. 39, no. 3, pp. 269–282, 1999.

[3] Y. Chen and Z. Zhou, "Stable periodic solution of a discrete periodic Lotka-Volterra competition system," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 1, pp. 358–366, 2003.

[4] Q. Zhang and Z. Zhou, "Permanence of a nonautonomous Lotka-Volterra difference system with delays," *Mathematical Sciences Research Journal*, vol. 7, no. 3, pp. 99–106, 2003.

[5] W. Wang, G. Mulone, F. Salemi, and V. Salone, "Global stability of discrete population models with time delays and fluctuating environment," *Journal of Mathematical Analysis and Applications*, vol. 264, no. 1, pp. 147–167, 2001.

[6] Y. Muroya, "Persistence and global stability for discrete models of nonautonomous Lotka-Volterra type," *Journal of Mathematical Analysis and Applications*, vol. 273, no. 2, pp. 492–511, 2002.

[7] Y. Muroya, "Persistence and global stability in discrete models of Lotka-Volterra type," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 24–33, 2007.

[8] X. Liao, S. Zhou, and Y. Chen, "Permanence and global stability in a discrete n-species competition system with feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 4, pp. 1661–1671, 2008.

[9] H. Hu, Z. Teng, and S. Gao, "Extinction in nonautonomous Lotka-Volterra competitive system with pure-delays and feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 4, pp. 2508–2520, 2009.

[10] H. Hu, Z. Teng, and H. Jiang, "On the permanence in non-autonomous Lotka-Volterra competitive system with pure-delays and feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1803–1815, 2009.

[11] J. Xu and Z. Teng, "Almost sufficient and necessary conditions for permanece and extinction of nonautonomous discrete logistic systems with time-varying delays and feedback," Submitted to *Applications of Mathematics*.