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Research Article

Periodic Solutions for a System of Difference Equations

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This paper deals with the second-order nonlinear systems of difference equations, we obtain the existence theorems of periodic solutions. The theorems are proved by using critical point theory.

1. Introduction

Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ be the set of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, note that $\mathbb{Z}[a, b] = \{a, a+1, \dots, b\}$, where $a \le b$.

In this paper, we consider the existence of periodic solutions for the system of difference equations of the form

$$\begin{aligned}
\Delta\left(p_{n1}\left(\Delta x_{(n-1)1}\right)^{\delta}\right) + q_{n1}\left(x_{n1}\right)^{\delta} &= f_1\left(n, X_n\right),\\
\Delta\left(p_{n2}\left(\Delta x_{(n-1)2}\right)^{\delta}\right) + q_{n2}\left(x_{n2}\right)^{\delta} &= f_2\left(n, X_n\right),\\
&\vdots\\
\Delta\left(p_{nk}\left(\Delta x_{(n-1)k}\right)^{\delta}\right) + q_{nk}\left(x_{nk}\right)^{\delta} &= f_k\left(n, X_n\right),
\end{aligned} \tag{1.1}$$

which can be recorded as

$$\Delta\left(\overline{P}_n\left(\Delta X_{n-1}^T\right)^{\delta}\right) + \overline{Q}_n\left(X_n^T\right)^{\delta} = f\left(n, X_n\right), \quad n \in \mathbb{Z}, \tag{1.2}$$

where k is a positive integer,

$$\overline{P}_n = \begin{pmatrix} p_{n1} & 0 & \cdots & 0 \\ 0 & p_{n2} & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & p_{nk} \end{pmatrix}, \qquad \overline{Q}_n = \begin{pmatrix} q_{n1} & 0 & \cdots & 0 \\ 0 & q_{n2} & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & q_{nk} \end{pmatrix}, \tag{1.3}$$

and $\overline{P}_{n+\omega} = \overline{P}_n > 0$ (i.e., $p_{n1} > 0, p_{n2} > 0, \ldots, p_{nk} > 0$), $\overline{Q}_{n+\omega} = \overline{Q}_n$, $f = (f_1, f_2, \ldots, f_k)^T$, $f_i = f_i(n, X_n) = f_i(n, x_{n1}, x_{n2}, \ldots, x_{nk})$, $f(n+\omega, U) = f(n, U)$ for any $(n, U) \in \mathbb{Z} \times \mathbb{R}^k$, $\omega > 0$ is a positive integer, $(-1)^\delta = -1$, δ is the ratio of odd positive integers, $\Delta X_n^T = X_{n+1}^T - X_n^T = (x_{(n+1)1} - x_{n1}, x_{(n+1)2} - x_{n2}, \ldots, x_{(n+1)k} - x_{nk})^T$, $\Delta^2 X_{n-1}^T = \Delta(\Delta X_{n-1}^T) = \Delta X_n^T - \Delta X_{n-1}^T$. For $U = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$, define $U^\delta = (u_1^\delta, u_2^\delta, \ldots, u_k^\delta)$. $|U| = (|u_1|, |u_2|, \ldots, |u_k|)$, $|U|^\delta = (|u_1|^\delta, |u_2|^\delta, \ldots, |u_k|^\delta)$. A sequence $X = \{X_n\}_{n\in\mathbb{Z}}$ is a ω-periodic solution of (1.2) if substitution of it into (1.2) yields an identity for all $n \in \mathbb{Z}$.

In [1, 2], the qualitative behavior of linear difference equations

$$\Delta(p_n \Delta x_n) + q_n x_n = 0 \tag{1.4}$$

has been investigated. In [3], the nonlinear difference equation

$$\Delta(p_n \Delta x_{n-1}) + q_n x_n = f(n, x_n) \tag{1.5}$$

has been considered. In [4], by critical point method, the existence of periodic and subharmonic solutions of equation

$$\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z} \tag{1.6}$$

has been studied. Other interesting results can been found in [5–8]. In [9], the authors consider the existence of periodic solutions for second-order nonlinear difference equation

$$\Delta\left(p_n (\Delta x_{n-1})^\delta\right) + q_n x_n^\delta = f(n, x_n), \quad n \in \mathbb{Z}, \tag{1.7}$$

using critical point theory, obtaining some new results. It is a discrete analogues of differential equation

$$\left(p(t)\phi(u')\right)' + f(t, u) = 0. \tag{1.8}$$

They do have physical applications in the study of nuclear physics, gas aerodynamics, and so on (see [10, 11]). In this paper, we obtain some new results of existence of periodic solution for the second-order nonlinear system of difference equations by using critical point theory. We remark, however, the result in [9] is only good for (1.7) which is much less general than our results in what follows.

2. Some Basic Lemmas

Let E be a real Hilbert space, $I \in C^1(E, \mathbb{R})$ mean that I is continuously Fréchet differentiable functional defined on E. I is said to be satisfying Palais-Smale condition (P-S condition) if any bounded sequence $\{I(u_n)\}$ and $I'(u_n) \to 0$ $(n \to \infty)$ possess a convergent subsequence in E. Let B_ρ be the open ball in E with radius ρ and centered at θ, and let ∂B_ρ denote its boundary, θ is null element of E.

Lemma 2.1 (see [12]). *Let E be a real Hilbert space, and assume that $I \in C^1(E, \mathbb{R})$ satisfies the P-S condition and the following conditions:*

(I_1) *there exist constants $\rho > 0$ and $a > 0$ such that $I(x) \geq a$ for all $x \in \partial B_\rho$, where $B_\rho = \{x \in E : \|x\| < \rho\}$;*

(I_2) *$I(0) \leq 0$ and there exists $x_0 \notin B_\rho$ such that $I(x_0) \leq 0$.*

Then $c = \inf_{h\in\Gamma}\sup_{s\in[0,1]} I(h(s))$ is a positive critical value of I, where

$$\Gamma = \{h \in C([0,1], X) : h(0) = \theta, h(1) = x_0\}. \tag{2.1}$$

Let Ω_* be the set of sequences

$$X = \{X_n\}_{n\in\mathbb{Z}} = \{\ldots, X_{-n}, \ldots, X_{-1}, X_0, X_1, \ldots, X_n, \ldots\}, \tag{2.2}$$

where $X_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \in \mathbb{R}^k$, that is,

$$\Omega_* = \left\{X = \{X_n\}_{n\in\mathbb{Z}} : X_n \in \mathbb{R}^k,\ n \in \mathbb{Z}\right\}. \tag{2.3}$$

For any $X, Y \in \Omega_*$, $a, b \in \mathbb{R}$, $aX + bY$ is defined by

$$aX + bY = \{aX_n + bY_n\}_{n=-\infty}^{+\infty}, \tag{2.4}$$

then Ω_* is a vector space. For given positive integer ω, E_ω is defined as a subspace of Ω_* by

$$E_\omega = \left\{X = \{X_n\} \in \Omega_* : X_{n+\omega} = X_n,\ n \in \mathbb{Z}\right\}. \tag{2.5}$$

Obviously, E_ω is isomorphic to $\mathbb{R}^{k\omega}$, for any $X, Y \in E_\omega$, defined inner product

$$\langle X, Y\rangle = \sum_{i=1}^{\omega} \langle X_i, Y_i\rangle, \tag{2.6}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|X\| = \left(\sum_{i=1}^{\omega} \|X_i\|^2\right)^{1/2}, \quad X \in E_\omega. \tag{2.7}$$

where $\|X_i\| = (\sum_{j=1}^{k}|x_{ij}|^2)^{1/2}$. It is obvious that E_ω with the inner product defined by (2.6) is a finite-dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^{k\omega}$. Define the functional J on E_ω as follows:

$$J(X) = \frac{1}{\delta+1}\sum_{n=1}^{\omega}\langle P_n, (\Delta X_{n-1})^{\delta+1}\rangle - \frac{1}{\delta+1}\sum_{n=1}^{\omega}\langle Q_n, X_n^{\delta+1}\rangle + \sum_{n=1}^{\omega}F(n, X_n), \quad X \in E_\omega, \tag{2.8}$$

where $F(n, X_n)$ such that $\nabla_U F(n, U) = f(n, U)$, that is,

$$f_i(n, U) = f_i(n, u_1, u_2, \ldots, u_k) = \frac{\partial}{\partial u_i}F(n, u_1, u_2, \ldots, u_k) \tag{2.9}$$

for any $(n, U) \in \mathbb{Z}[1, \omega] \times \mathbb{R}^k$, $P_n = (p_{n1}, p_{n2}, \ldots, p_{nk})$, $Q_n = (q_{n1}, q_{n2}, \ldots, q_{nk})$. Clearly $J \in C^1(E_\omega, \mathbb{R})$, and for any $X = \{X_n\}_{n\in\mathbb{Z}} \in E_\omega$, by $X_0 = X_\omega$ and $X_1 = X_{\omega+1}$, we have

$$\frac{\partial J(X)}{\partial x_{nl}} = -\Delta\left(p_{nl}\left(\Delta x_{(n-1)l}\right)^{\delta}\right) - q_{nl}\left(x_{nl}\right)^{\delta} + f_l(n, X_n), \quad l \in \mathbb{Z}[1, k], \ n \in \mathbb{Z}[1, \omega]. \tag{2.10}$$

Thus $X = \{X_n\}_{n\in\mathbb{Z}}$ is a critical point of J on E_ω ($J'(X) = 0$) if and only if

$$\Delta\left(p_{nl}\left(\Delta x_{(n-1)l}\right)^{\delta}\right) + q_{nl}\left(x_{nl}\right)^{\delta} = f_l(n, X_n), \quad l \in \mathbb{Z}[1, k], \ n \in \mathbb{Z}[1, \omega]. \tag{2.11}$$

That is,

$$\Delta\left(\overline{P}_n\left(\Delta X_{n-1}^T\right)^{\delta}\right) + \overline{Q}_n\left(X_n^T\right)^{\delta} = f(n, X_n), \quad n \in \mathbb{Z}. \tag{2.12}$$

By the periodicity of X_n and $f(n, X_n)$ in the first variable n, we know that if $X = \{X_n\}_{n\in\mathbb{Z}} \in E_\omega$ is a critical point of the real functional J defined by (2.8), then it is a periodic solution of (1.2).

For $X = \{X_n\}_{n\in\mathbb{Z}} \in E_\omega$, $X_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \in \mathbb{R}^k$, $r > 1$, denote

$$\|X\|_r = \left(\sum_{i=1}^{\omega}\|X_i\|^r\right)^{1/r}, \qquad \|X_n\|_r = \left(\sum_{i=1}^{k}\|x_{ni}\|^r\right)^{1/r}. \tag{2.13}$$

Clearly, $\|X\|_2 = \|X\|$, $\|X_n\|_2 = \|X_n\|$. Because of $\|\cdot\|_{r_1}$ and $\|\cdot\|_{r_2}$ being equivalent when r_1, $r_2 > 1$, so there exist constants $c_1, c_2, c_3, c_4, \hbar_1, \hbar_2, \hbar_3$, and $\hbar_4$ such that $c_2 \geq c_1 > 0$, $c_4 \geq c_3 > 0$, $\hbar_2 \geq \hbar_1 > 0$, and $\hbar_4 \geq \hbar_3 > 0$,

$$\begin{gathered} c_1\|X\| \leq \|X\|_{\delta+1} \leq c_2\|X\|, \\ c_3\|X\| \leq \|X\|_{\beta} \leq c_4\|X\|, \\ \hbar_1\|X_n\| \leq \|X_n\|_{\delta+1} \leq \hbar_2\|X_n\|, \\ \hbar_3\|X_n\| \leq \|X_n\|_{\beta} \leq \hbar_4\|X_n\|, \end{gathered} \tag{2.14}$$

for all $X \in E_\omega$, $\delta > 0$, and $\beta > 1$.

Lemma 2.2. *Suppose that*

(F_1) *there exist constants* $a_1 > 0$, $a_2 > 0$, $\beta > \delta + 1$ *such that*

$$F(n, U) \leq -a_1 \|U\|^{\beta} + a_2 \tag{2.15}$$

for any $(n, U) \in \mathbb{Z}[1, \omega] \times R^k$;

(F_2)

$$q_{ni} \leq 0, \quad n \in \mathbb{Z},\ i \in \mathbb{Z}[1, k]. \tag{2.16}$$

Then

$$J(X) = \frac{1}{\delta+1} \sum_{n=1}^{\omega} \langle P_n, (\Delta X_{n-1})^{\delta+1} \rangle - \frac{1}{\delta+1} \sum_{n=1}^{\omega} \langle Q_n, X_n^{\delta+1} \rangle + \sum_{n=1}^{\omega} F(n, X_n) \tag{2.17}$$

satisfies P-S condition.

Proof. For any sequence $\{X^{(l)}\} = \{\dots, X_{-n}^{(l)}, \dots, X_{-1}^{(l)}, X_0^{(l)}, X_1^{(l)}, \dots, X_n^{(l)}, \dots\} \in E_{\omega}$, $J(X^{(l)})$ is bounded and $J'(X^{(l)}) \to 0$ $(l \to \infty)$. Then there exists a positive constant $M > 0$, such that $|J(X^{(l)})| \leq M$. From (F_1), we have

$$\begin{aligned}
-M &\leq J(X^{(l)}) \\
&= \frac{1}{\delta+1} \sum_{n=1}^{\omega} \left[\left\langle P_n, (X_n^{(l)} - X_{n-1}^{(l)})^{\delta+1} \right\rangle - \left\langle Q_n, (X_n^{(l)})^{\delta+1} \right\rangle \right] \\
&\quad + \sum_{n=1}^{\omega} F(n, X_n^{(l)}) \\
&\leq \frac{1}{\delta+1} \sum_{n=1}^{\omega} 2^{\delta+1} \left\langle P_n, \left(|X_n^{(l)}|^{\delta+1} + |X_{n-1}^{(l)}|^{\delta+1} \right) \right\rangle \\
&\quad - \frac{1}{\delta+1} \sum_{n=1}^{\omega} \left\langle Q_n, |X_n^{(l)}|^{\delta+1} \right\rangle + \sum_{n=1}^{\omega} F(n, X_n^{(l)}) \\
&\leq \frac{2^{\delta+1}}{\delta+1} \sum_{n=1}^{\omega} \left\langle P_n + P_{n+1}, |X_n^{(l)}|^{\delta+1} \right\rangle - \frac{1}{\delta+1} \sum_{n=1}^{\omega} \left\langle Q_n, |X_n^{(l)}|^{\delta+1} \right\rangle \\
&\quad + \sum_{n=1}^{\omega} F(n, X_n^{(l)}) \\
&\leq \frac{2^{\delta+1}}{\delta+1} \sum_{n=1}^{\omega} \left\langle P_n + P_{n+1}, |X_n^{(l)}|^{\delta+1} \right\rangle - \frac{1}{\delta+1} \sum_{n=1}^{\omega} \left\langle Q_n, |X_n^{(l)}|^{\delta+1} \right\rangle \\
&\quad - a_1 \sum_{n=1}^{\omega} \|X_n^{(l)}\|^{\beta} + a_2 \omega \\
&= \frac{1}{\delta+1} \sum_{n=1}^{\omega} \left\langle 2^{\delta+1}(P_n + P_{n+1}) - Q_n, |X_n^{(l)}|^{\delta+1} \right\rangle - a_1 \|X^{(l)}\|_{\beta}^{\beta} + a_2 \omega.
\end{aligned} \tag{2.18}$$

Set

$$A_0 = \max_{n\in\mathbb{Z}[1,\omega],i\in\mathbb{Z}[1,k]}\left\{2^{\delta+1}\left(p_{ni}+p_{(n+1)i}\right)-q_{ni}\right\}. \tag{2.19}$$

Then $A_0 > 0$, and

$$\begin{aligned}
-M &\le J\left(X^{(l)}\right) \\
&\le \frac{A_0}{\delta+1}\sum_{n=1}^{\omega}\left\|X_n^{(l)}\right\|_{\delta+1}^{\delta+1} - a_1\left\|X^{(l)}\right\|_{\beta}^{\beta} + a_2\omega \\
&\le \frac{A_0\hbar_2^{\delta+1}}{\delta+1}\sum_{n=1}^{\omega}\left\|X_n^{(l)}\right\|^{\delta+1} - a_1\left\|X^{(l)}\right\|_{\beta}^{\beta} + a_2\omega \\
&= \frac{A_0\hbar_2^{\delta+1}}{\delta+1}\left\|X^{(l)}\right\|_{\delta+1}^{\delta+1} - a_1\left\|X^{(l)}\right\|_{\beta}^{\beta} + a_2\omega.
\end{aligned} \tag{2.20}$$

Because of $\beta > \delta+1$, and $(\beta-\delta-1)/\beta + (\delta+1)/\beta = 1$, in view of Hölder inequality, we have

$$\sum_{n=1}^{\omega}\left\|X_n^{(l)}\right\|^{\delta+1} \le \omega^{(\beta-\delta-1)/\beta}\left(\sum_{n=1}^{\omega}\left\|X_n^{(l)}\right\|^{\beta}\right)^{(\delta+1)/\beta}. \tag{2.21}$$

Thus

$$\left\|X^{(l)}\right\|_{\beta}^{\beta} \ge \omega^{(\delta+1-\beta)/(\delta+1)}\left\|X^{(l)}\right\|_{\delta+1}^{\beta}. \tag{2.22}$$

Then we have

$$\begin{aligned}
-M &\le J(X^{(l)}) \\
&\le \frac{A_0\hbar_2^{\delta+1}}{\delta+1}\left\|X^{(l)}\right\|_{\delta+1}^{\delta+1} - a_1\left\|X^{(l)}\right\|_{\beta}^{\beta} + a_2\omega \\
&\le \frac{A_0\hbar_2^{\delta+1}}{\delta+1}\left\|X^{(l)}\right\|_{\delta+1}^{\delta+1} - a_1\omega^{(\delta+1-\beta)/(\delta+1)}\left\|X^{(l)}\right\|_{\delta+1}^{\beta} + a_2\omega.
\end{aligned} \tag{2.23}$$

Thus, for any $l \in \mathbb{N}$,

$$a_1\omega^{(\delta+1-\beta)/(\delta+1)}\left\|X^{(l)}\right\|_{\delta+1}^{\beta} - \frac{A_0\hbar_2^{\delta+1}}{\delta+1}\left\|X^{(l)}\right\|_{\delta+1}^{\delta+1} \le M + a_2\omega. \tag{2.24}$$

Because of $\beta > \delta+1$, it is easily seen that the inequality (2.24) implies that $\{X^{(l)}\}$ is a bounded sequence in E_ω. Thus $\{X^{(l)}\}$ possesses convergent subsequences. The proof is complete. □

3. Main Result

Theorem 3.1. *Suppose that condition* (F_1) *holds, and*

(F_3) *for each* $n \in \mathbb{Z}$,

$$\lim_{\|U\| \to 0} \frac{F(n,U)}{\|U\|^{\delta+1}} = 0; \tag{3.1}$$

(F_4) *for any* $i \in \mathbb{Z}[1,k], n \in \mathbb{Z}[1,\omega]$,

$$q_{ni} < 0; \tag{3.2}$$

(F_5) $F(n,\theta) = 0$.

Then (1.2) *has at least two nontrivial* ω*-periodic solutions.*

Proof. By Lemma 2.2, J satisfies P-S condition. Next, we will verify the conditions (I_1) and (I_2) of Lemma 2.1. By (F_3), there exists $\rho > 0$, such that

$$|F(n,U)| \leq -\frac{q_{\max}\hbar_1^{\delta+1}}{2(\delta+1)}\|U\|^{\delta+1} \tag{3.3}$$

for any $\|U\| < \rho$ and $n \in \mathbb{Z}[1,\omega]$, where $q_{\max} = \max_{n\in\mathbb{Z}[1,\omega],\ i\in\mathbb{Z}[1,k]} q_{ni} < 0$. Thus

$$\begin{aligned}
J(X) &\geq -\frac{1}{\delta+1}\sum_{n=1}^{\omega}\langle Q_n, X_n^{\delta+1}\rangle + \sum_{n=1}^{\omega}F(n,X_n) \\
&\geq -\frac{q_{\max}}{\delta+1}\sum_{n=1}^{\omega}\|X_n\|_{\delta+1}^{\delta+1} + \frac{q_{\max}\hbar_1^{\delta+1}}{2(\delta+1)}\sum_{n=1}^{\omega}\|X_n\|^{\delta+1} \\
&\geq -\frac{q_{\max}\hbar_1^{\delta+1}}{\delta+1}\sum_{n=1}^{\omega}\|X_n\|^{\delta+1} + \frac{q_{\max}\hbar_1^{\delta+1}}{2(\delta+1)}\sum_{n=1}^{\omega}\|X_n\|^{\delta+1} \\
&= -\frac{q_{\max}\hbar_1^{\delta+1}}{2(\delta+1)}\|X\|_{\delta+1}^{\delta+1} \\
&\geq -\frac{q_{\max}\hbar_1^{\delta+1}c_1^{\delta+1}}{2(\delta+1)}\|X\|^{\delta+1}
\end{aligned} \tag{3.4}$$

for any $X \in E_\omega$ with $\|X\| \leq \rho$. We choose $a = -\hbar_1^{\delta+1}c_1^{\delta+1}(q_{\max}/2(\delta+1))\rho^{\delta+1}$, then we have

$$J(X)|_{\partial B_\rho} \geq a > 0, \tag{3.5}$$

that is, the condition (I_1) of Lemma 2.1 holds.

Obviously, $J(0)=0$. For any given $V\in E_\omega$ with $\|V\|=1$ and constant $\alpha>0$,

$$
\begin{aligned}
J(\alpha V) &= \frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\langle P_n,\left(\alpha V_n-\alpha V_{n-1}\right)^{\delta+1}\right\rangle-\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\langle Q_n,\left(\alpha V_n\right)^{\delta+1}\right\rangle+\sum_{n=1}^{\omega}F(n,\alpha V_n)\\
&=\frac{1}{\delta+1}\sum_{n=1}^{\omega}\Big\{p_{n1}\left(\alpha v_{n1}-\alpha v_{(n-1)1}\right)^{\delta+1}+p_{n2}\left(\alpha v_{n2}-\alpha v_{(n-1)2}\right)^{\delta+1}\\
&\qquad\qquad+\cdots+p_{nk}\left(\alpha v_{nk}-\alpha v_{(n-1)k}\right)^{\delta+1}\Big\}\\
&\quad-\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\{q_{n1}\left(\alpha v_{n1}\right)^{\delta+1}+q_{n2}\left(\alpha v_{n2}\right)^{\delta+1}+\cdots+q_{nk}\left(\alpha v_{nk}\right)^{\delta+1}\right\}\\
&\quad+\sum_{n=1}^{\omega}F(n,\alpha V_n)\\
&\le\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\{p_{n1}(2\alpha)^{\delta+1}+p_{n2}(2\alpha)^{\delta+1}+\cdots+p_{nk}(2\alpha)^{\delta+1}\right\}\\
&\quad-\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\{q_{n1}\alpha^{\delta+1}+q_{n2}\alpha^{\delta+1}+\cdots+q_{nk}\alpha^{\delta+1}\right\}\\
&\quad-a_1\sum_{n=1}^{\omega}\|\alpha V_n\|^{\beta}+a_2\omega\\
&\le\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\{2^{\delta+1}\|P_n\|_1+\|Q_n\|_1\right\}\alpha^{\delta+1}-a_1\alpha^{\beta}\sum_{n=1}^{\omega}\|V_n\|^{\beta}+a_2\omega\\
&\le\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\{2^{\delta+1}\|P_n\|_1+\|Q_n\|_1\right\}\alpha^{\delta+1}-a_1\alpha^{\beta}\|V\|_{\beta}^{\beta}+a_2\omega\\
&\le\frac{1}{\delta+1}\sum_{n=1}^{\omega}\left\{2^{\delta+1}\|P_n\|_1+\|Q_n\|_1\right\}\alpha^{\delta+1}-a_1c_3^{\beta}\alpha^{\beta}+a_2\omega\\
&\longrightarrow-\infty,\quad(\alpha\longrightarrow+\infty).
\end{aligned}
\tag{3.6}
$$

Thus we can choose a sufficiently large α such that $\alpha>\rho$, and $\overline{X}=\alpha V\in E_\omega$, $J(\overline{X})<0$. According to Lemma 2.1, there exists at least one critical value $c\ge a>0$. We suppose that X^* is a critical point corresponding to c, then $J(X^*)=c$ and $J'(X^*)=0$.

By similar argument of Lemma 2.2, we know that $J(X)$ is bounded from above, so there exists $X^{**}\in E_\omega$ such that $J(X)\le J(X^{**})=c_{\max}$ for any $X\in E_\omega$. Obviously, $X^{**}\ne 0$. If $X^{**}\ne X^*$, then the proof is complete. Otherwise, $X^{**}=X^*$, $c=c_{\max}$. In view of Lemma 2.1,

$$c=\inf_{h\in\Gamma}\sup_{s\in[0,1]}J(h(s)),\tag{3.7}$$

where $\Gamma=\{h\in C([0,1],E_\omega):h(0)=\theta,h(1)=\overline{X}\}$. Then $c_{\max}=\max_{s\in[0,1]}J(h(s))$ for any $h\in\Gamma$ holds. In view of the continuity of $J(h(s))$ in s, $J(\theta)\le 0$, and $J(\overline{X})<0$, we know that there exists some $s_0 \in (0,1)$ such that $J(h(s_0)) = c_{\max}$. If we choose $h_1, h_2 \in \Gamma$ such that

$$\{h_1(s) : s \in (0,1)\} \cap \{h_2(s) : s \in (0,1)\} = \phi, \tag{3.8}$$

then there exist $s_1, s_2 \in (0,1)$ such that $J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}$. Then J possesses two different critical points $\widehat{Y} = h_1(s_1)$ and $\check{Z}= h_2(s_2)$ in E_ω, hence, we obtain at least two nontrivial critical points which correspond to the critical value $c_{\max}$. Thus (1.2) possesses at least two nontrivial ω-periodic solutions. The proof is complete. □

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