Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2009, Article ID 812674, 5 pages
doi:10.1155/2009/812674

Research Article

On the Global Attractivity of a Max-Type Difference Equation

Ali Gelişken and Cengiz Çinar

Mathematics Department, Faculty of Education, Selcuk University, 42090 Konya, Turkey

Correspondence should be addressed to Ali Gelişken, aligelisken@yahoo.com.tr

Received 27 December 2008; Accepted 5 February 2009

Recommended by Yong Zhou

We investigate asymptotic behavior and periodic nature of positive solutions of the difference equation $x_n = \max\{A/x_{n-1}, 1/x_{n-3}^{\alpha}\}, n = 0, 1, \ldots$, where $A > 0$ and $0 < \alpha < 1$. We prove that every positive solution of this difference equation approaches $\overline{x} = 1$ or is eventually periodic with period 2.

1. Introduction

In the recent years, there has been a lot of interest in studying the global attractivity and the periodic nature of, so-called, max-type difference equations (see, e.g., [1–17] and references therein).

In [10], the following difference equation was proposed by Ladas:

$$x_n = \max\left\{\frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \ldots, \frac{A_p}{x_{n-p}}\right\}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $A_1, A_2, \ldots, A_p$ are real numbers and initial conditions are nonzero real numbers.

In [17], asymptotic behavior of positive solutions of the difference equation was investigated

$$x_n = \max\left\{\frac{1}{x_{n-1}^{\alpha}}, \frac{A}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $0 < \alpha < 1$ and $0 < A$. It was showed that every positive solution of this difference equation approaches $\overline{x} = 1$ or is eventually periodic with period 4.

In [14], it was proved that every positive solution of the difference equation,

$$x_n = \max\left\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\right\}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where $0 < \alpha, \beta < 1, 0 < A$, and $0 < B$, converges to $\overline{x} = \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

In this paper, we investigate the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^{\alpha}}\right\}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where $0 < A$, $0 < \alpha < 1$ and initial conditions are positive real numbers. We prove that every positive solution of this difference equation approaches $\overline{x} = 1$ or is eventually periodic with period 2.

2. The Case $A = 1$

In this section, we consider the difference equation

$$x_n = \max\left\{\frac{1}{x_{n-1}}, \frac{1}{x_{n-3}^{\alpha}}\right\}, \quad n = 0, 1, \ldots, \tag{2.1}$$

where $0 < \alpha < 1$.

Theorem 2.1. *Let x_n be a solution of* (2.1). *Then x_n approaches $\overline{x} = 1$.*

Proof. Choose a number B such that $0 < B < 1$, let $x_n = B^{y_n}$ for $n \geq -3$. Then (2.1) implies the difference equation

$$y_n = \min\{-y_{n-1}, -\alpha y_{n-3}\}, \quad n = 0, 1, \ldots, \tag{2.2}$$

where $0 < \alpha < 1$ and initial conditions are real numbers.

Let y_n be a solution of (2.2). Then it suffices to prove $y_n \to 0$. Observe that there exists a positive integer N such that

$$y_N = -y_{N-1}, \quad y_{N+1} = -\alpha y_{N-2} \quad \text{for } N \geq 0. \tag{2.3}$$

By computation, we get that

$$y_{N+2} = -y_{N+1}, \qquad y_{N+3} = -\alpha y_N, \qquad y_{N+4} = -y_{N+3} \tag{2.4}$$

and then

$$y_{4n+N} = \alpha^n y_N, \quad y_{4n+N+1} = \alpha^n y_{N+1}, \quad y_{4n+N+2} = -y_{4n+N+1}, \quad y_{4n+N+3} = -\alpha y_{4n+N} \quad \forall n \geq 0. \tag{2.5}$$

So, $y_{4n+N} \to 0$, $y_{4n+N+1} \to 0$, $y_{4n+N+2} \to 0$, $y_{4n+N+3} \to 0$. This implies $y_n \to 0$. □

3. The Case $0 < A < 1$

In this section, we consider (1.4), where $0 < \alpha < 1$.

Let $x_n = A^{y_n}$, $n \geq -3$. Equation (1.4) implies the difference equation

$$y_n = \min\{1 - y_{n-1}, -\alpha y_{n-3}\}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where initial conditions are real numbers.

Lemma 3.1. *Let y_n be a solution of* (3.1). *Then for all $n \geq 0$,*

$$|y_n| \leq \max\{|y_{n-1}| - 1, \alpha|y_{n-3}|\}. \tag{3.2}$$

Proof. From (3.1), we have the following statements:

if $y_{n-1} \geq 0$ and $y_{n-3} \geq 0$, then $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha|y_{n-3}|\}$;
if $y_{n-1} \leq 0$ and $y_{n-3} \leq 0$, then $|y_n| \leq \alpha|y_{n-3}|$;
if $y_{n-1} \geq 0$ and $y_{n-3} \leq 0$, then $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha|y_{n-3}|\}$;
if $y_{n-1} \leq 0$ and $y_{n-3} \geq 0$, then $|y_n| = \alpha|y_{n-3}|$.

In general, we have $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha|y_{n-3}|\}$ for all $n \geq 0$. □

Theorem 3.2. *if x_n is a solution of* (1.4), *x_n approaches $\overline{x} = 1$.*

Proof. Let y_n be a solution of (3.1). To prove $x_n \to 1$, it suffices to prove $y_n \to 0$.

Choose a number β such that $0 < |y_{n-1}| - 1 \leq \beta|y_n|$. Then from inequality (3.2), we get that

$$|y_n| \leq \max\{\beta|y_{n-1}|, \alpha|y_{n-3}|\}, \quad n \geq 0. \tag{3.3}$$

Let $\gamma = \max\{\beta, \alpha\}$, then $0 < \gamma < 1$ and

$$|y_n| \leq \gamma \max\{|y_{n-1}|, |y_{n-3}|\}, \quad n \geq 0. \tag{3.4}$$

From (3.4) and by induction, we get that

$$|y_n| \leq \gamma^{[n/3]+1} \max\{|y_{-1}|, |y_{-2}|, |y_{-3}|\}, \quad n \geq 0. \tag{3.5}$$

This implies $y_n \to 0$. □

4. The Case $A > 1$

In this section, we consider (1.4). Let $x_n = A^{y_n}$ for $n \geq -3$. Equation (1.4) implies the difference equation

$$y_n = \max\{1 - y_{n-1}, -\alpha y_{n-3}\}, \quad n = 0, 1, \ldots, \tag{4.1}$$

where $0 < \alpha < 1$ and initial conditions are real numbers.

Theorem 4.1. *If x_n is a solution of* (1.4), *then the following statements are true:*

(a) *x_n approaches $\overline{x} = 1$, if there is an integer N such that*

$$x_N = \frac{A}{x_{N-1}}, \quad x_{N+1} = \frac{1}{x_{N-2}^{\alpha}} \quad \textit{for } 0 \leq n \leq N. \tag{4.2}$$

(b) *x_n is eventually periodic with period 2, if there is an integer N such that*

$$x_N = \frac{A}{x_{N-1}}, \quad x_{N+1} = x_{N-1}, \quad A^{-\alpha/(1-\alpha)} \leq x_{N-1} \leq A^{1/(1-\alpha)} \quad \textit{for } 0 \leq n \leq N. \tag{4.3}$$

Proof. (a) Change of variables $x_n = A^{y_n}$, $n \geq -3$. If $x_N = A/x_{N-1}$ and $x_{N+1} = 1/x_{N-2}^{\alpha}$ for $0 \leq n \leq N$, then $y_N = 1 - y_{N-1}$ and $y_{N+1} = -\alpha y_{N-2}$. Let y_n be a solution of (4.1). So, to prove $x_n \to 1$, it suffices to prove $y_n \to 0$. From (4.1), there is at least an integer N such that $y_N = 1 - y_{N-1} > 0$ and $y_{N+1} = -\alpha y_{N-2}$ for $0 \leq n \leq N$. By computation from (4.1), we get that $y_{N+1} < 0$, $y_{N-2} \geq y_N > 0$ and then

$$\begin{gathered} y_{N+2} = 1 + \alpha y_{N-2}, \quad y_N > y_{N+2} > 0, \\ y_{N+3} = -\alpha y_N, \quad y_{N+1} < y_{N+3} < 0, \\ y_{N+4} = 1 + \alpha y_N, \quad y_N > y_{N+2} \geq y_{N+4} > 0, \\ y_{N+5} = -\alpha y_{N+2}, \quad y_{N+1} \leq y_{N+3} < y_{N+5} < 0. \end{gathered} \tag{4.4}$$

So, we have

$$y_{4n+N} > y_{4n+N+2} \geq y_{4n+N+4} > 0, \qquad y_{4n+N+1} \leq y_{4n+N+3} < y_{4n+N+5} < 0 \tag{4.5}$$

for all $n \geq 0$. This implies $y_n \to 0$.

(b) Change of variables $x_n = A^{y_n}$, $n \geq -3$. Let y_n be a solution of (4.1).

If $x_N = A/x_{N-1}$, $x_{N+1} = x_{N-1}$, and $A^{-\alpha/(1-\alpha)} \leq x_{N-1} \leq A^{1/(1-\alpha)}$ for $0 \leq n \leq N$, then

$$y_N = 1 - y_{N-1}, \quad y_{N+1} = y_{N-1}, \quad \frac{-\alpha}{1-\alpha} \leq y_{N-1} \leq \frac{1}{1-\alpha}. \tag{4.6}$$

Clearly, there is at least an integer N such that $y_N = 1 - y_{N-1}$ for $0 \leq n \leq N$.

Suppose that $y_{N+1} = y_{N-1}$ and $y_{N-1} \notin [-\alpha/(1-\alpha), 1/(1-\alpha)]$.

If $y_{N-1} > 1/(1-\alpha)$ then from (4.1), we have $y_{N+2} = -\alpha y_{N-1}$. So, from (a) we get immediately that $y_n \to 0$.

If $y_{N-1} < -\alpha/(1-\alpha)$, then we have $y_{N+2} = 1 - y_{N+1}$ and $y_{N+3} = -\alpha y_N$. Then we get that $y_n \to 0$, from (a).

We assume that $y_{N+1} = y_{N-1}$ and $-\alpha/(1-\alpha) \leq y_{N-1} \leq 1/(1-\alpha)$. To prove the desired result, it suffices to show that y_n is eventually periodic with period 2. By computation from (4.1), we get immediately $y_N = 1 - y_{N-1}$ for all $0 \leq n \leq N$. This is the desired result. □

References

[1] R. M. Abu-Saris and F. M. Allan, "Periodic and nonperiodic solutions of the difference equation $x_{n+1} = \max\{x_n^2, A\}/x_n x_{n-1}$," in *Advances in Difference Equations (Veszprém, 1995)*, pp. 9–17, Gordon and Breach, Amsterdam, The Netherlands, 1997.

[2] A. M. Amleh, J. Hoag, and G. Ladas, "A difference equation with eventually periodic solutions," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 401–404, 1998.

[3] K. S. Berenhaut, J. D. Foley, and S. Stević, "Boundedness character of positive solutions of a max difference equation," *Journal of Difference Equations and Applications*, vol. 12, no. 12, pp. 1193–1199, 2006.

[4] W. J. Briden, E. A. Grove, C. M. Kent, and G. Ladas, "Eventually periodic solutions of $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$," *Communications on Applied Nonlinear Analysis*, vol. 6, no. 4, pp. 31–43, 1999.

[5] W. J. Briden, E. A. Grove, G. Ladas, and L. C. McGrath, "On the nonautonomous equation $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$," in *New Developments in Difference Equations and Applications (Taipei, 1997)*, pp. 49–73, Gordon and Breach, Amsterdam, The Netherlands, 1999.

[6] C. Çinar, S. Stević, and I. Yalçinkaya, "On positive solutions of a reciprocal difference equation with minimum," *Journal of Applied Mathematics & Computing*, vol. 17, no. 1-2, pp. 307–314, 2005.

[7] A. Gelişken, C. Çinar, and R. Karatas, "A note on the periodicity of the Lyness max equation," *Advances in Difference Equations*, vol. 2008, Article ID 651747, 5 pages, 2008.

[8] A. Gelişken, C. Çinar, and I. Yalçinkaya, "On the periodicity of a difference equation with maximum," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 820629, 11 pages, 2008.

[9] E. A. Grove, C. M. Kent, G. Ladas, and M. A. Radin, "On $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ with a period 3 parameter," in *Fields Institute Communications, 29*, pp. 161–180, American Mathematical Society, Providence, RI, USA, 2001.

[10] G. Ladas, "On the recursive sequence $x_n = \max\{A_1/x_{n-1}, A_2/x_{n-2}, \ldots, A_p/x_{n-p}\}$," *Journal of Difference Equations and Applications*, vol. 2, no. 3, pp. 339–341, 1996.

[11] D. P. Mishev, W. T. Patula, and H. D. Voulov, "A reciprocal difference equation with maximum," *Computers & Mathematics with Applications*, vol. 43, no. 8-9, pp. 1021–1026, 2002.

[12] I. Szalkai, "On the periodicity of the sequence $x_{n+1} = \max\{A_0/x_n, A_1/x_{n-1}, \ldots, A_k/x_{n-k}\}$," *Journal of Difference Equations and Applications*, vol. 5, no. 1, pp. 25–29, 1999.

[13] S. Stević, "On the recursive sequence $x_{n+1} = \max\{c, x_n^p/x_{n-1}^p\}$," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 791–796, 2008.

[14] F. Sun, "On the asymptotic behavior of a difference equation with maximum," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 243291, 6 pages, 2008.

[15] H. D. Voulov, "On the periodic nature of the solutions of the reciprocal difference equation with maximum," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 1, pp. 32–43, 2004.

[16] I. Yalçinkaya, B. D. Iričanin, and C. Çinar, "On a max-type difference equation," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 47264, 10 pages, 2007.

[17] X. Yang, X. Liao, and C. Li, "On a difference equation with maximum," *Applied Mathematics and Computation*, vol. 181, no. 1, pp. 1–5, 2006.