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Research Article

Boundedness, Attractivity, and Stability of a Rational Difference Equation with Two Periodic Coefficients

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We study the boundedness, the attractivity, and the stability of the positive solutions of the rational difference equation $x_{n+1} = (p_n x_{n-2} + x_{n-3})/(q_n + x_{n-3})$, $n = 0, 1, \ldots$, where p_n, q_n, $n = 0, 1, \ldots$ are positive sequences of period 2.

1. Introduction

In [1], Camouzis et al. studied the global character of the positive solutions of the difference equation:

$$x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $\delta,\ A$ are positive parameters and the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are positive real numbers.

The mathematical modeling of a physical, physiological, or economical problem very often leads to difference equations (for partial review of the theory of difference equations and their applications see [2–12]). Moreover, a lot of difference equations with periodic coefficients have been applied in mathematical models in biology (see [13–15]). In addition, between others in [16–19], we can see some more difference equations with periodic coefficients that have been studied.

In this paper, we investigate the difference equation

$$x_{n+1} = \frac{p_n x_{n-2} + x_{n-3}}{q_n + x_{n-3}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where p_n, q_n, $n = 0, 1, \ldots$ are positive sequences of period 2 and the initial values x_i, $i = -3, -2, -1, 0$ are positive numbers.

Our goal in this paper is to extend some results obtained in [1]. More precisely, we study the existence of a unique positive periodic solution of (1.2) of prime period 2. In the sequel, we investigate the boundedness, the persistence, and the convergence of the positive solutions to the unique periodic solution of (1.2). Finally, we study the stability of the positive periodic solution and the zero solution of (1.2).

If we set $y_n = x_{2n-1}$, $z_n = x_{2n}$, it is easy to prove that (1.2) is equivalent to the following system of difference equations:

$$y_{n+1} = \frac{p_0 z_{n-1} + y_{n-1}}{q_0 + y_{n-1}}, \quad z_{n+1} = \frac{p_1 y_n + z_{n-1}}{q_1 + z_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where p_i, q_i, $i = 0, 1$ are positive constants and the initial values y_i, z_i, $i = -1, 0$ are positive numbers. So in order to study (1.2) we investigate system (1.3).

2. Existence of the Unique Positive Equilibrium of System (1.3)

In the following proposition, we study the existence of the unique positive equilibrium of system (1.3).

Proposition 2.1. *Consider system* (1.3) *where* p_i, q_i, $i = 0, 1$ *are positive constants and the initial values* y_i, z_i, $i = -1, 0$ *are positive numbers. Suppose that*

$$q_0 - 1 < p_0, \qquad q_1 - 1 < p_1 \tag{2.1}$$

are satisfied. Then system (1.3) *possesses a unique positive equilibrium.*

Proof. Let (y, z) be a positive equilibrium of system (1.3) then

$$y = \frac{p_0 z + y}{q_0 + y}, \qquad z = \frac{p_1 y + z}{q_1 + z}. \tag{2.2}$$

Equations (2.2) imply that z is a solution of the equation

$$f(x) = x^3 + 2(q_1 - 1)x^2 + \left[(q_1 - 1)^2 + p_1(q_0 - 1)\right]x + (q_1 - 1)(q_0 - 1)p_1 - p_0 p_1^2 = 0. \tag{2.3}$$

Suppose that

$$q_1 \geq 1. \tag{2.4}$$

Let $\lambda_1,\ \lambda_2$, and λ_3 be the solutions of (2.3). Then from (2.1), (2.3), and (2.4) we take

$$\lambda_1 + \lambda_2 + \lambda_3 = 2(1 - q_1) \leq 0,$$
$$\lambda_1\lambda_2\lambda_3 = -(q_1 - 1)(q_0 - 1)p_1 + p_0 p_1^2 > 0, \tag{2.5}$$

and so (2.3) has unique positive solution z. Then from (2.2) and (2.4) we have

$$z > 1 - q_1, \qquad y = \frac{z^2 + (q_1 - 1)z}{p_1} > 0, \tag{2.6}$$

and so system (1.3) has a unique positive equilibrium.

Now suppose that

$$q_1 < 1, \quad (q_1 - 1)(q_0 - 1) > p_1 p_0. \tag{2.7}$$

If $\lambda_1,\ \lambda_2$, and λ_3 are the solutions of (2.3), then from (2.3) and (2.7) we take

$$\lambda_1 + \lambda_2 + \lambda_3 = 2(1 - q_1) > 0,$$
$$\lambda_1\lambda_2\lambda_3 = -(q_1 - 1)(q_0 - 1)p_1 + p_0 p_1^2 < 0, \tag{2.8}$$

and so (2.3) has a negative solution, but also (2.3) has a solution in the interval $(0, 1 - q_1)$, since

$$f(0) = (q_1 - 1)(q_0 - 1)p_1 - p_0 p_1^2 > 0,$$
$$f(1 - q_1) = -p_0 p_1^2 < 0. \tag{2.9}$$

Moreover, (2.3) has a solution z in the interval $(1 - q_1, \infty)$, since

$$\lim_{x \to \infty} f(x) = \infty, \tag{2.10}$$

therefore, we get (2.6) and so system (1.3) has a unique positive equilibrium.

Finally, suppose that

$$q_1 < 1, \quad (q_1 - 1)(q_0 - 1) < p_1 p_0. \tag{2.11}$$

If $\lambda_1,\ \lambda_2$, and λ_3 are the solutions of (2.3), then from (2.3) and (2.11), we take

$$\lambda_1 + \lambda_2 + \lambda_3 = 2(1 - q_1) > 0,$$
$$\lambda_1\lambda_2\lambda_3 = -(q_1 - 1)(q_0 - 1)p_1 + p_0 p_1^2 > 0. \tag{2.12}$$

We have $\lim_{x\to\infty} f(x) = \infty$, and since $f(1-q_1) < 0$, it is obvious that (2.3) has a solution z in the interval $(1-q_1, \infty)$. From (2.3), we get

$$f'(x) = 3x^2 + 4x(q_1 - 1) + (q_1 - 1)^2 + p_1(q_0 - 1). \tag{2.13}$$

If equation $f'(x) = 0$ has complex roots, then it is obvious that z is the unique solution of (2.3). Therefore, we get (2.6), and so system (1.3) has a unique positive equilibrium.

Now, suppose that the roots of $f'(x) = 0$

$$\mu_1 = \frac{2(1-q_1) - \sqrt{D}}{3}, \quad \mu_2 = \frac{2(1-q_1) + \sqrt{D}}{3}, \quad D = (1-q_1)^2 + 3p_1(1-q_0), \tag{2.14}$$

are real numbers.

Suppose that $q_0 < 1$, then it is obvious that

$$\mu_1 < 1 - q_1 < \mu_2, \tag{2.15}$$

and so we have that (2.3) has a unique solution $z \in (1-q_1, \infty)$.

If $q_0 \geq 1$, then it holds that

$$0 < \mu_1 \leq \mu_2 \leq 1 - q_1, \tag{2.16}$$

which implies that (2.3) has a unique solution $z \in (1-q_1, \infty)$.

Therefore, we can take (2.6) and so system (1.3) has a unique positive equilibrium. This completes the proof of the proposition. □

3. Boundedness and Persistence of the Solutions of System (1.3)

In the following propositions we study the boundedness and the persistence of the positive solutions of system (1.3). In the sequel we will use the following result which has proved in [20].

Theorem 3.1. *Assume that all roots of the polynomial*

$$P(t) = t^N - s_1 t^{N-1} - \cdots - s_N, \tag{3.1}$$

where $s_1, s_2, \ldots, s_N \geq 0$ *have absolute value less than 1, and let* y_n *be a nonnegative solution of the inequality*

$$y_{n+N} \leq s_1 y_{n+N-1} + \cdots + s_N y_n + z_n. \tag{3.2}$$

Then, the following statements are true.

(i) *If* z_n *is a nonnegative bounded sequence, then* y_n *is also bounded.*

(ii) *If* $\lim_{n\to\infty} z_n = 0$*, then* $\lim_{n\to\infty} y_n = 0$.

Proposition 3.2. *One considers the system of difference equations* (1.3) *where* p_i, q_i, $i = 0, 1$ *are positive constants and the initial values* y_i, z_i, $i = -1, 0$ *are positive numbers. Then the following statements are true.*

(i) *If*

$$\frac{q_0 q_1}{p_0 p_1} \geq 1, \tag{3.3}$$

then every solution of (1.3) *is bounded.*

(ii) *If*

$$q_0 - 1 < p_0 \leq q_0, \qquad q_1 - 1 < p_1 \leq q_1, \tag{3.4}$$

then every solution of (1.3) *is bounded and persists.*

Proof. Let (y_n, z_n) be an arbitrary solution of (1.3).

(i) From (3.3), we get that one of the three following conditions holds:

$$\frac{q_0}{p_0} > 1, \tag{3.5}$$

$$\frac{q_1}{p_1} > 1, \tag{3.6}$$

$$p_0 = q_0 = p, \qquad p_1 = q_1 = q. \tag{3.7}$$

We assume that (3.5) is satisfied. We prove that there exists a positive integer N such that

$$y_n < 1, \quad z_n < \frac{q_0}{p_0}, \quad n \geq N. \tag{3.8}$$

First, we show that if there exists a positive integer n_0 such that

$$z_{n_0} < \frac{q_0}{p_0}, \tag{3.9}$$

then

$$z_{n_0+3p} < \frac{q_0}{p_0}, \quad p = 0, 1, \ldots. \tag{3.10}$$

In contradiction, we assume that

$$z_{n_0+3} = \frac{p_1 y_{n_0+2} + z_{n_0+1}}{q_1 + z_{n_0+1}} \geq \frac{q_0}{p_0}. \tag{3.11}$$

Using relations (1.3), (3.5), and (3.11), we get that

$$y_{n_0+2} = \frac{p_0 z_{n_0} + y_{n_0}}{q_0 + y_{n_0}} > \frac{q_0 q_1}{p_0 p_1}, \tag{3.12}$$

and so relations (1.3) and (3.3) imply that

$$z_{n_0} > \frac{q_0{}^2 q_1}{p_0{}^2 p_1} > \frac{q_0}{p_0}, \tag{3.13}$$

which contradicts (3.9). So $z_{n_0+3} < q_0/p_0$ and working inductively, we get (3.10).

If $z_{-1} < q_0/p_0$, then from the analogous relations (3.9) and (3.10), we get

$$z_{-1+3p} < \frac{q_0}{p_0}, \quad p = 0, 1, \dots. \tag{3.14}$$

Now, suppose that

$$z_{-1} \geq \frac{q_0}{p_0}, \tag{3.15}$$

we prove that there exists a positive integer q such that

$$z_{-1+3q} < \frac{q_0}{p_0}. \tag{3.16}$$

From (3.3), there exists a positive integer h such that

$$z_{-1} < \left(\frac{q_0 q_1}{p_0 p_1}\right)^h. \tag{3.17}$$

If $z_2 < q_0/p_0$, then (3.16) is true for $q = 1$.

Now, suppose that

$$z_2 \geq \frac{q_0}{p_0}. \tag{3.18}$$

Then from (1.3), (3.5), and (3.18), we get $y_1 > q_0 q_1 / p_0 p_1$ and so from (1.3), (3.3), and (3.5), we have that

$$z_{-1} > \frac{q_1 q_0{}^2}{p_1 p_0{}^2} > \frac{q_1 q_0}{p_1 p_0}. \tag{3.19}$$

If $z_5 < q_0/p_0$, then (3.16) is true for $q = 2$.

Now, suppose that

$$z_5 \geq \frac{q_0}{p_0}. \tag{3.20}$$

Using (1.3), (3.3), (3.5), (3.20) and arguing as to prove (3.19) we get

$$z_{-1} > \left(\frac{q_1 q_0}{p_1 p_0}\right)^2. \tag{3.21}$$

Working inductively, we get that

$$\text{if } z_{-1+3w} \geq \frac{q_0}{p_0}, \quad w = 1, 2, \ldots, \quad \text{then } z_{-1} > \left(\frac{q_1 q_0}{p_1 p_0}\right)^w. \tag{3.22}$$

From (3.22) for $w = h$, we get $z_{-1} > (q_1 q_0 / p_1 p_0)^h$ which contradicts (3.17). So $z_{-1+3h} < q_0/p_0$ which means that (3.16) holds for $q = h$.

Arguing as for z_{-1}, we can prove that there exist positive integers $k,\ l$ such that

$$z_{0+3k} < \frac{q_0}{p_0}, \qquad z_{1+3l} < \frac{q_0}{p_0}. \tag{3.23}$$

From (3.16) and (3.23), we get that there exists a positive integer r such that

$$z_r < \frac{q_0}{p_0}, \qquad n \geq r. \tag{3.24}$$

Finally, from (1.3) and (3.24), we get $y_{r+2} < 1$ and so (3.8) is true for $N = r + 2$.

Similarly, we can prove that if (3.6) holds, then there exists a positive integer N such that

$$z_n < 1, \quad y_n < \frac{q_1}{p_1}, \quad n \geq N. \tag{3.25}$$

Finally, suppose that (3.7) hold. From (1.3) and (3.7), we have

$$y_{n+1} - 1 = \frac{p(z_{n-1} - 1)}{p + y_{n-1}}, \qquad z_{n+1} - 1 = \frac{q(y_n - 1)}{q + z_{n-1}}, \tag{3.26}$$

and so,

$$y_{n+1} - 1 = \frac{p}{p + y_{n-1}} \frac{q}{q + z_{n-3}} (y_{n-2} - 1). \tag{3.27}$$

From (3.27), we get

$$0 \le y_{n+1} - 1 \le y_{n-2} - 1, \quad \text{or} \quad 0 \ge y_{n+1} - 1 \ge y_{n-2} - 1, \tag{3.28}$$

and so the subsequences y_{3n}, y_{3n+1}, y_{3n+2} either are bounded from below by 1 and decreasing or bounded from above by 1 and increasing. Hence, y_n is bounded and persists. Similarly, we can prove that z_n is bounded and persists. This completes the proof of part (i) of the proposition.

(ii) In statement (i), we have already proved that if (3.7) hold, then every solution of (1.3) is bounded and persists. So, from (3.4), it remains to show that if either

$$q_0 - 1 < p_0 < q_0, \qquad q_1 - 1 < p_1 \le q_1, \tag{3.29}$$

or

$$q_0 - 1 < p_0 \le q_0, \qquad q_1 - 1 < p_1 < q_1, \tag{3.30}$$

holds, then the solution (y_n, z_n) persists. From (3.3), (3.8), (3.25), (3.29), and (3.30), we get that

$$y_n < \frac{q_1}{p_1}, \quad z_n < \frac{q_0}{p_0}, \quad n \ge N. \tag{3.31}$$

We consider the positive number m such that

$$m < \min\{y_N, z_N, y_{N+1}, z_{N+1}, p_0 + 1 - q_0, p_1 + 1 - q_1\}. \tag{3.32}$$

Moreover, if

$$f(y,z) = \frac{p_0 z + y}{q_0 + y}, \qquad g(y,z) = \frac{p_1 y + z}{q_1 + z}, \tag{3.33}$$

then it is easy to see that for the functions (3.33), f is increasing with respect to y for any z, $z < q_0/p_0$ and g is increasing with respect to z for any y, $y < q_1/p_1$.

Therefore, from (1.3), (3.31), and (3.32) we have

$$y_{N+2} > \frac{(p_0+1)m}{q_0+m} > m, \qquad z_{N+2} > \frac{(p_1+1)m}{q_1+m} > m, \tag{3.34}$$

and working inductively, we take

$$y_{N+s} \ge m, \quad z_{N+s} \ge m, \quad s = 0, 1, \ldots. \tag{3.35}$$

Therefore, (y_n, z_n) persists and using statement (i), then (y_n, z_n) is bounded and persists. This completes the proof of the proposition. □

Proposition 3.3. *One considers the system of difference equations* (1.3) *where* p_i, q_i, $i = 0, 1$ *are positive constants, and the initial values* y_i, z_i, $i = -1, 0$ *are positive numbers. Then, the following statements are true.*

(i) *If*

$$\frac{q_0 q_1}{p_0 p_1} < 1, \tag{3.36}$$

then every solution of (1.3) *persists.*

(ii) *If*

$$q_0 \leq p_0 \leq q_0 + 1, \qquad q_1 \leq p_1 \leq q_1 + 1, \tag{3.37}$$

then every solution of (1.3) *is bounded and persists.*

Proof. Let (y_n, z_n) be an arbitrary solution of (1.3).

(i) From (3.36), we have

$$\frac{q_0}{p_0} < 1, \tag{3.38}$$

or

$$\frac{q_1}{p_1} < 1. \tag{3.39}$$

Arguing as in the proof of statement (i) of Proposition 3.2, we can easily prove that if (3.38) holds, then there exists a positive integer M such that

$$y_n > 1, \quad z_n > \frac{q_0}{p_0}, \quad n \geq M, \tag{3.40}$$

and if (3.39) holds, then there exists a positive integer M such that

$$z_n > 1, \quad y_n > \frac{q_1}{p_1}, \quad n \geq M. \tag{3.41}$$

(ii) From Proposition 3.2, we have that if (3.7) holds, then every solution of (1.3) is bounded and persists. So, from (3.37), it remains to show that if either

$$q_0 < p_0 \leq q_0 + 1, \qquad q_1 \leq p_1 \leq q_1 + 1, \tag{3.42}$$

or

$$q_0 \le p_0 \le q_0 + 1, \qquad q_1 < p_1 \le q_1 + 1, \tag{3.43}$$

holds, then the solution (y_n, z_n) is bounded and persists.

From (3.36), (3.40), (3.41), (3.42), and (3.43), we get that

$$y_n > \frac{q_1}{p_1}, \quad z_n > \frac{q_0}{p_0}, \quad n \ge M. \tag{3.44}$$

Suppose that

$$p_0 \ne q_0 + 1 \quad \text{or} \quad p_1 \ne q_1 + 1. \tag{3.45}$$

From (1.3) and (3.44), we have

$$z_{M+1} > 1, \qquad y_{M+3} > 1. \tag{3.46}$$

We have for the functions (3.33) that f is decreasing with respect to y for any z, $z > q_0/p_0$ and g is decreasing with respect to z for any y, $y > q_1/p_1$. Therefore, relations (1.3), (3.44), and (3.46) imply that

$$z_{M+3} \le \frac{p_1 y_{M+2} + 1}{q_1 + 1}, \tag{3.47}$$

and so from (1.3) and (3.46),

$$y_{M+5} \le \frac{p_0 p_1}{(q_0+1)(q_1+1)} y_{M+2} + \frac{p_0}{(q_0+1)(q_1+1)} + 1. \tag{3.48}$$

Working inductively, we can prove that

$$y_{n+5} \le \frac{p_0 p_1}{(q_0+1)(q_1+1)} y_{n+2} + \frac{p_0}{(q_0+1)(q_1+1)} + 1, \quad n \ge M. \tag{3.49}$$

Then from (3.42), (3.43), (3.45), and Theorem 3.1, y_n is bounded. Similarly, we take that z_n is bounded. Therefore, from (3.44), the solution (y_n, z_n) is bounded and persists.

Now, suppose that

$$p_0 = q_0 + 1, \qquad p_1 = q_1 + 1. \tag{3.50}$$

We claim that y_n is bounded. For the sake of contradiction, we assume that y_n is not bounded. Then, there exists a subsequence n_i such that

$$\lim_{i\to\infty} y_{n_i+1} = \infty, \tag{3.51}$$

$$y_{n_i+1} > \max\{y_j,\ j < n_i\}. \tag{3.52}$$

Moreover, from (1.3) and (3.50), we get

$$y_{n_i+1} < \frac{q_0+1}{q_0} z_{n_i-1} + 1, \tag{3.53}$$

and so from (3.51),

$$\lim_{i\to\infty} z_{n_i-1} = \infty. \tag{3.54}$$

Moreover, from (1.3) and (3.50),

$$z_{n_i-1} < \frac{q_1+1}{q_1} y_{n_i-2} + 1, \tag{3.55}$$

and so from (3.54),

$$\lim_{i\to\infty} y_{n_i-2} = \infty. \tag{3.56}$$

Working inductively, we can prove that

$$\lim_{i\to\infty} y_{n_i+1-3s} = \infty, \quad \lim_{i\to\infty} z_{n_i-1-3s} = \infty, \quad s = 0,1,\dots. \tag{3.57}$$

We claim that y_{n_i-6} is a bounded sequence. Suppose on the contrary that there exists an unbounded subsequence of y_{n_i-6} and without loss of generality, we may suppose that

$$\lim_{i\to\infty} y_{n_i-6} = \infty. \tag{3.58}$$

Arguing as above, we can easily prove that

$$\lim_{i\to\infty} y_{n_i-9} = \lim_{i\to\infty} y_{n_i-12} = \infty. \tag{3.59}$$

Also, since from (1.3),

$$y_{n_i-6} = \frac{(q_0+1)\left((z_{n_i-8})/(y_{n_i-8})\right)+1}{q_0/y_{n_i-8}+1} < \frac{(q_0+1)z_{n_i-8}}{y_{n_i-8}} + 1, \tag{3.60}$$

from (3.58), we have that $\lim_{i\to\infty}(z_{n_i-8}/y_{n_i-8}) = \infty$ and so eventually,

$$z_{n_i-8} > y_{n_i-8}. \tag{3.61}$$

From (1.3), (3.50), and (3.61), we have

$$\begin{aligned}
y_{n_i+1} &= \frac{(q_0+1)z_{n_i-1} + y_{n_i-1}}{q_0 + y_{n_i-1}} \\
&< \frac{q_0+1}{q_0} z_{n_i-1} + 1 \\
&= \frac{q_0+1}{q_0}\left(\frac{(q_1+1)y_{n_i-2} + z_{n_i-3}}{q_1 + z_{n_i-3}}\right) + 1 \\
&< 1 + \frac{q_0+1}{q_0} + \frac{q_0+1}{q_0}\frac{q_1+1}{q_1} y_{n_i-2} \\
&< \cdots < A + B y_{n_i-8} \\
&< A + B z_{n_i-8},
\end{aligned} \tag{3.62}$$

where

$$\begin{aligned}
A &= 1 + \frac{q_0+1}{q_0} + \frac{q_0+1}{q_0}\frac{q_1+1}{q_1} + \left(\frac{q_0+1}{q_0}\right)^2 \frac{q_1+1}{q_1} \\
&\quad + \left(\frac{q_0+1}{q_0}\right)^2\left(\frac{q_1+1}{q_1}\right)^2 + \left(\frac{q_0+1}{q_0}\right)^3\left(\frac{q_1+1}{q_1}\right)^2, \\
B &= \left(\frac{q_0+1}{q_0}\right)^3\left(\frac{q_1+1}{q_1}\right)^3.
\end{aligned} \tag{3.63}$$

Therefore, using (1.3) and (3.50), we get

$$y_{n_i+1} < A + B\left(\frac{(q_1+1)y_{n_i-9} + z_{n_i-10}}{q_1 + z_{n_i-10}}\right), \tag{3.64}$$

and since from (3.57) and (3.59), we have that $y_{n_i-9} \to \infty$, $z_{n_i-10} \to \infty$ as $i \to \infty$, we can easily prove that eventually,

$$y_{n_i+1} < y_{n_i-9}, \tag{3.65}$$

which contradicts to (3.52).

Therefore, y_{n_i-6} is a bounded sequence. From (1.3), (3.50), and (3.57), we get

$$z_{n_i-5} = \frac{(q_1+1)y_{n_i-6}+z_{n_i-7}}{q_1+z_{n_i-7}} = \frac{(q_1+1)(y_{n_i-6}/z_{n_i-7})+1}{q_1/z_{n_i-7}+1} \longrightarrow 1, \quad i \longrightarrow \infty. \tag{3.66}$$

Similarly, from (1.3), (3.50) and (3.57) and (3.66) follows,

$$y_{n_i-3} = \frac{(q_0+1)z_{n_i-5}+y_{n_i-5}}{q_0+y_{n_i-5}} = \frac{(q_0+1)(z_{n_i-5}/y_{n_i-5})+1}{q_0/y_{n_i-5}+1} \longrightarrow 1, \quad i \longrightarrow \infty. \tag{3.67}$$

Now, we prove that

$$\liminf_{i\to\infty} y_{n_i-1} > 1. \tag{3.68}$$

Otherwise, and without loss of generality, we may suppose that $\lim_{i\to\infty} y_{n_i-1} \leq 1$. So, relations (1.3), (3.50), and (3.67) imply that

$$\lim_{i\to\infty} y_{n_i-1} = \frac{(q_0+1)\lim_{i\to\infty} z_{n_i-3}+\lim_{i\to\infty} y_{n_i-3}}{q_0+\lim_{i\to\infty} y_{n_i-3}} \leq 1, \tag{3.69}$$

and so

$$\lim_{i\to\infty} z_{n_i-3} \leq \frac{q_0}{q_0+1}. \tag{3.70}$$

Moreover, from (1.3), (3.44), and (3.50), we get eventually

$$z_{n_i-3} = \frac{(q_1+1)y_{n_i-4}+z_{n_i-5}}{q_1+z_{n_i-5}} > \frac{(q_1+1)(q_1/(q_1+1))+z_{n_i-5}}{q_1+z_{n_i-5}} = 1, \tag{3.71}$$

and so from (3.66), $\lim_{i\to\infty} z_{n_i-3} \geq 1$ which contradicts to (3.70).

Hence, (3.68) is true.

Similarly, we can prove that

$$\liminf_{i\to\infty} z_{n_i-3} > 1. \tag{3.72}$$

Therefore, from (3.68) and (3.72), we have eventually

$$y_{n_i-1} > 1+k, \qquad z_{n_i-3} > 1+m, \tag{3.73}$$

where $k,\ m$ are positive real numbers.

Hence, from (1.3), (3.50), and (3.73) we have

$$\begin{aligned} y_{n_i+1} &= \frac{(q_0+1)\left[\left((q_1+1)y_{n_i-2}+z_{n_i-3}\right)/\left(q_1+z_{n_i-3}\right)\right]+y_{n_i-1}}{q_0+y_{n_i-1}} \\ &< \frac{(q_0+1)(q_1+1)}{(q_1+1+m)(q_0+1+k)}y_{n_i-2}+\frac{q_0+1}{q_0}+1. \end{aligned} \tag{3.74}$$

Then from (3.57), we can prove that eventually

$$y_{n_i+1} < y_{n_i-2}, \tag{3.75}$$

which contradicts to (3.52).

Therefore, y_n is a bounded sequence. Moreover, from (1.3), (3.50), we take that z_n is bounded. Therefore, the solution (y_n, z_n) is bounded and persists. This completes the proof of the proposition. □

4. Attractivity of the Positive Equilibrium of System (1.3)

In the following propositions, we study the convergency of the solutions of system (1.3) to its positive equilibrium.

Proposition 4.1. *One considers the system of difference equations* (1.3) *where* p_i, q_i, $i = 0, 1$ *are positive constants, and the initial values* y_i, z_i, $i = -1, 0$ *are positive numbers. If either* (3.29) *or* (3.30) *hold, then every solution of* (1.3) *tents to the positive equilibrium of* (1.3).

Proof. Let (y_n, z_n) be an arbitrary solution of (1.3). From Proposition 3.2, there exist

$$\begin{gathered} L_1 = \limsup_{n\to\infty} y_n, \qquad L_2 = \limsup_{n\to\infty} z_n, \qquad l_1 = \liminf_{n\to\infty} y_n, \qquad l_2 = \liminf_{n\to\infty} z_n, \\ 0 < L_1, L_2, l_1, l_2 < \infty. \end{gathered} \tag{4.1}$$

From (1.3), (3.31), and the monotony of functions (3.33), we have

$$L_1 \le \frac{p_0 L_2 + L_1}{q_0 + L_1}, \qquad L_2 \le \frac{p_1 L_1 + L_2}{q_1 + L_2}, \qquad l_1 \ge \frac{p_0 l_2 + l_1}{q_0 + l_1}, \qquad l_2 \ge \frac{p_1 l_1 + l_2}{q_1 + l_2}, \tag{4.2}$$

and hence

$$\begin{aligned} L_1^2 + L_1(q_0-1) - p_0 L_2 \le 0, &\qquad L_2^2 + L_2(q_1-1) - p_1 L_1 \le 0, \\ l_1^2 + l_1(q_0-1) - p_0 l_2 \ge 0, &\qquad l_2^2 + l_2(q_1-1) - p_1 l_1 \ge 0. \end{aligned} \tag{4.3}$$

The third inequality of (4.3), implies that

$$l_1 \geq \frac{1 - q_0 + \sqrt{(1 - q_0)^2 + 4p_0 l_2}}{2}, \tag{4.4}$$

and so from the last inequality of (4.3), we have

$$2l_2^2 + 2l_2(q_1 - 1) + (q_0 - 1)p_1 \geq p_1\sqrt{(1 - q_0)^2 + 4p_0 l_2}. \tag{4.5}$$

Hence, we get

$$\left(2l_2^2 + 2l_2(q_1 - 1) + (q_0 - 1)p_1\right)^2 \geq \left(p_1\sqrt{(1 - q_0)^2 + 4p_0 l_2}\right)^2, \tag{4.6}$$

or

$$l_2^3 + 2l_2^2(q_1 - 1) + l_2\left[(q_1 - 1)^2 + p_1(q_0 - 1)\right] + p_1(q_1 - 1)(q_0 - 1) - p_0 p_1^2 \geq 0. \tag{4.7}$$

The first inequality of (4.3), implies that

$$0 < L_1 \leq \frac{1 - q_0 + \sqrt{(1 - q_0)^2 + 4p_0 L_2}}{2}, \tag{4.8}$$

and so from second inequality of (4.3), we get

$$2L_2^2 + 2L_2(q_1 - 1) + (q_0 - 1)p_1 \leq p_1\sqrt{(1 - q_0)^2 + 4p_0 L_2}. \tag{4.9}$$

Using (4.3), we have

$$L_1 \geq l_1 > 1 - q_0, \qquad L_2 \geq l_2 > 1 - q_1. \tag{4.10}$$

Therefore, from (4.5) and (4.10), we get

$$\begin{aligned} 2L_2^2 + 2L_2(q_1 - 1) + (q_0 - 1)p_1 &= 2L_2(L_2 + q_1 - 1) + (q_0 - 1)p_1 \\ &\geq 2l_2^2 + 2l_2(q_1 - 1) + (q_0 - 1)p_1 \\ &> 0. \end{aligned} \tag{4.11}$$

Using (4.9) and (4.11), we have

$$L_2^3 + 2L_2^2(q_1 - 1) + L_2\left[(q_1 - 1)^2 + p_1(q_0 - 1)\right] + p_1(q_1 - 1)(q_0 - 1) - p_0 p_1^2 \leq 0. \tag{4.12}$$

In Proposition 2.1, we proved that (2.3) has a unique positive solution z, $z \in (1 - q_1, \infty)$. We can write

$$f(x) = (x - z)\left(x^2 + ax + b\right), \quad a, b \in \mathbb{R}, \tag{4.13}$$

where $f(x)$ is defined in (2.3) and $x^2 + ax + b > 0$ for any $x > 1 - q_1$. Then from (4.7), (4.12), and (4.13), we have

$$\left(L_2 - z\right)\left(L_2^2 + aL_2 + b\right) \leq 0, \qquad \left(l_2 - z\right)\left(l_2^2 + al_2 + b\right) \geq 0. \tag{4.14}$$

Therefore, from (4.10) and (4.14),

$$L_2 \leq z \leq l_2,$$

which implies that

$$L_2 = l_2 = z. \tag{4.15}$$

In addition, using (4.15), the first and the third inequalities of (4.3), we have

$$L_1^2 + (q_0 - 1)L_1 \leq l_1^2 + (q_0 - 1)l_1, \tag{4.16}$$

and so (4.10) implies that

$$L_1 = l_1. \tag{4.17}$$

This completes the proof of the proposition. □

Proposition 4.2. *One considers the system of difference equations* (1.3) *where p_i, q_i, $i = 0, 1$ are positive constants, and the initial values y_i, z_i, $i = -1, 0$ are positive numbers. If either* (3.42) *or* (3.43) *hold, then every solution of* (1.3) *tents to the positive equilibrium of* (1.3).

Proof. Let (y_n, z_n) be an arbitrary solution of (1.3). From Proposition 3.3, there exist L_i, l_i, $i = 1, 2$ such that (4.1) are satisfied.

From (1.3), the monotony of functions (3.33) and (3.44), we have

$$L_1 \leq \frac{p_0 L_2 + l_1}{q_0 + l_1}, \qquad L_2 \leq \frac{p_1 L_1 + l_2}{q_1 + l_2}, \qquad l_1 \geq \frac{p_0 l_2 + L_1}{q_0 + L_1}, \qquad l_2 \geq \frac{p_1 l_1 + L_2}{q_1 + L_2}, \tag{4.18}$$

and hence

$$\begin{aligned} L_1 l_1 + L_1 q_0 &\leq p_0 L_2 + l_1, \qquad L_1 l_1 + l_1 q_0 \geq p_0 l_2 + L_1, \\ L_2 l_2 + L_2 q_1 &\leq p_1 L_1 + l_2, \qquad L_2 l_2 + l_2 q_1 \geq p_1 l_1 + L_2, \end{aligned} \tag{4.19}$$

which implies that

$$(1+q_0)(L_1-l_1) \le p_0(L_2-l_2), \qquad (1+q_1)(L_2-l_2) \le p_1(L_1-l_1). \tag{4.20}$$

Therefore,

$$[(1+q_0)(1+q_1)-p_0p_1](L_1-l_1) \le 0. \tag{4.21}$$

First suppose that (3.45) holds. Then from (3.42) or (3.43), and (3.45), we get $L_1-l_1 \le 0$, which means that

$$L_1 = l_1. \tag{4.22}$$

Using (4.20), it is obvious that

$$L_2 = l_2. \tag{4.23}$$

So if (3.45) holds, the proof is completed.

Now, suppose that (3.50) hold. Then from (4.20), we have

$$L_2 - l_2 = L_1 - l_1. \tag{4.24}$$

Moreover, from (4.24), it follows that

$$(q_0+1)l_2 + L_1 - l_1q_0 = (q_0+1)L_2 + l_1 - L_1q_0. \tag{4.25}$$

In addition, from (3.50), the first and the second inequalities of (4.19), we get

$$(q_0+1)l_2 + L_1 - l_1q_0 \le L_1l_1 \le (q_0+1)L_2 + l_1 - L_1q_0. \tag{4.26}$$

Therefore, from (4.25) and (4.26), we have

$$L_1 = \frac{(q_0+1)L_2 + l_1}{q_0 + l_1}. \tag{4.27}$$

We may assume that there exists a positive integer n_i such that

$$\lim_{i\to\infty} y_{n_i-j} = A_j, \qquad \lim_{i\to\infty} z_{n_i-j} = B_j, \qquad \lim_{i\to\infty} y_{n_i+1} = L_1. \tag{4.28}$$

Moreover, from (1.3), (3.50), and (4.28), we get

$$L_1 = \frac{(q_0+1)B_1 + A_1}{q_0 + A_1}. \tag{4.29}$$

Since $f(x,y) = ((q_0+1)x+y)/(q_0+y)$ is decreasing with respect to y, for any $x > (q_0/(q_0+1))$, if $B_1 < L_2$ or $l_1 < A_1$, then from (3.44), and (3.50), we get

$$L_1 < \frac{(q_0+1)L_2 + l_1}{q_0 + l_1}, \tag{4.30}$$

which contradicts to (4.27). So,

$$B_1 = L_2, \qquad l_1 = A_1. \tag{4.31}$$

Using the same argument, we can prove that

$$\begin{aligned} A_2 &= L_1, & B_3 &= l_2, \\ B_3 &= l_2, & A_3 &= L_1, \\ B_4 &= L_2, & A_4 &= l_1, \\ A_4 &= l_1, & B_5 &= L_2, \\ B_5 &= L_2, & A_5 &= l_1, \\ A_5 &= L_1, & B_6 &= l_2, \end{aligned} \tag{4.32}$$

and so $L_1 = l_1 = A$. Also, from (4.24), we have $L_2 = l_2 = B$. Therefore,

$$\lim_{n\to\infty} y_n = A, \qquad \lim_{n\to\infty} z_n = B, \tag{4.33}$$

where obviously $A = B = 2$. This completes the proof of the proposition. □

Proposition 4.3. *One considers the system of difference equations* (1.3) *where* p_i, q_i, $i = 0, 1$ *are positive constants, and the initial values* y_i, z_i, $i = -1, 0$ *are positive numbers. If relations* (3.7) *hold, then every solution of* (1.3) *tents to the positive equilibrium* $(1,1)$ *of* (1.3).

Proof. Let (y_n, z_n) be an arbitrary solution of (1.3). From the proof of Proposition 3.2, the subsequences y_{3n}, y_{3n+1}, y_{3n+2}, z_{3n}, z_{3n+1}, and z_{3n+2} are monotone and y_n, z_n are bounded and persist. So, there exist positive numbers L_1, L_2, L_3, M_1, M_2, and M_3 such that

$$\begin{aligned} L_1 &= \lim_{n\to\infty} y_{3n}, & L_2 &= \lim_{n\to\infty} y_{3n+1}, & L_3 &= \lim_{n\to\infty} y_{3n+2}, \\ M_1 &= \lim_{n\to\infty} z_{3n}, & M_2 &= \lim_{n\to\infty} z_{3n+1}, & M_3 &= \lim_{n\to\infty} z_{3n+2}, \end{aligned} \tag{4.34}$$

and from (1.3) and (3.7), we get

$$L_1 = \frac{pM_2 + L_2}{p + L_2}, \qquad M_1 = \frac{qL_3 + M_2}{q + M_2},$$
$$L_2 = \frac{pM_3 + L_3}{p + L_3}, \qquad M_2 = \frac{qL_1 + M_3}{q + M_3}, \tag{4.35}$$
$$L_3 = \frac{pM_1 + L_1}{p + L_1}, \qquad M_3 = \frac{qL_2 + M_1}{q + M_1}.$$

Then, we have

$$L_1p + L_1L_2 = pM_2 + L_2, \qquad M_1q + M_1M_2 = qL_3 + M_2,$$
$$L_2p + L_2L_3 = pM_3 + L_3, \qquad M_2q + M_2M_3 = qL_1 + M_3, \tag{4.36}$$
$$L_3p + L_1L_3 = pM_1 + L_1, \qquad M_3q + M_3M_1 = qL_2 + M_1,$$

and hence,

$$(L_1 - M_2)p = L_2(1 - L_1), \qquad (M_1 - L_3)q = M_2(1 - M_1),$$
$$(L_2 - M_3)p = L_3(1 - L_2), \qquad (M_2 - L_1)q = M_3(1 - M_2), \tag{4.37}$$
$$(L_3 - M_1)p = L_1(1 - L_3), \qquad (M_3 - L_2)q = M_1(1 - M_3).$$

Therefore, we take

$$\frac{1}{p}L_2(1 - L_1) = \frac{1}{q}M_3(M_2 - 1),$$
$$\frac{1}{p}L_3(1 - L_2) = \frac{1}{q}M_1(M_3 - 1),$$
$$\frac{1}{p}L_1(1 - L_3) = \frac{1}{q}M_2(M_1 - 1).$$

So,

$$\text{if } L_1 \geq 1 \ (\text{resp., } L_1 \leq 1), \quad \text{then } M_2 \leq 1 \ (\text{resp., } M_2 \geq 1),$$
$$\text{if } L_2 \geq 1 \ (\text{resp., } L_2 \leq 1), \quad \text{then } M_3 \leq 1 \ (\text{resp., } M_3 \geq 1), \tag{4.38}$$
$$\text{if } L_3 \geq 1 \ (\text{resp., } L_3 \leq 1), \quad \text{then } M_1 \leq 1 \ (\text{resp., } M_1 \geq 1).$$

Therefore, if $L_1 \geq 1,\ M_2 \leq 1$ (resp., $L_1 \leq 1,\ M_2 \geq 1$), we have $L_1 - M_2 \geq 0$ (resp., $L_1 - M_2 \leq 0$) and so from (4.37), $L_1 \leq 1$ (resp., $L_1 \geq 1$). Hence, $L_1 = 1$ and from (4.37), $M_2 = 1$. Similarly, we can prove that $L_2 = 1,\ L_3 = 1,\ M_1 = 1,\ M_3 = 1$. This completes the proof of the proposition. □

5. Stability of System (1.3)

In this section we find conditions so that the positive equilibrium (y, z) and the zero equilibrium of (1.3) are stable.

Proposition 5.1. *Consider system* (1.3) *where* p_i, q_i, $i = 0, 1$ *are positive constants and the initial values* y_i, z_i, $i = -1, 0$ *are positive numbers. Then, the following statements are true.*

(i) *If*

$$q_0 - 1 < p_0 \leq q_0, \qquad q_1 - 1 < p_1 \leq q_1, \qquad q_0 + q_1 + p_0 p_1 + q_0 q_1 < 1, \tag{5.1}$$

then the unique positive equilibrium (y, z) *of* (1.3) *is globally asymptotically stable.*

(ii) *If*

$$q_0 + q_1 + p_0 p_1 + 1 < q_0 q_1, \tag{5.2}$$

then the zero equilibrium of (1.3) *is locally asymptotically stable.*

Proof. (i) Since (y, z) is the unique positive positive equilibrium of (1.3), we have

$$y = \frac{p_0 z + y}{q_0 + y}, \qquad z = \frac{p_1 y + z}{q_1 + z}. \tag{5.3}$$

Then from (5.1) and (5.3), we get

$$y \leq \frac{q_0 z + y}{q_0 + y}, \qquad z \leq \frac{q_1 y + z}{q_1 + z}. \tag{5.4}$$

Without loss of generality we assume that $z \leq y$. Then from (5.4), it results that

$$y \leq \frac{q_0 y + y}{q_0 + y}, \tag{5.5}$$

which means that

$$y \leq 1. \tag{5.6}$$

Moreover, from (5.4) and (5.6), we get

$$z \leq \frac{q_1 + z}{q_1 + z} = 1. \tag{5.7}$$

In addition, from (5.3), we have

$$y > \frac{y}{q_0 + y}, \qquad z > \frac{z}{q_1 + z}, \tag{5.8}$$

and so

$$y > 1 - q_0, \qquad z > 1 - q_1. \tag{5.9}$$

Then the linearized system of (1.3) about the positive equilibrium (y, z) is

$$Z_{n+1} = AZ_n, \tag{5.10}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{q_0 - p_0 z}{(q_0 + y)^2} & \dfrac{p_0}{q_0 + y} & 0 & 0 \\ 0 & \dfrac{q_1 - p_1 y}{(q_1 + z)^2} & \dfrac{p_1}{q_1 + z} & 0 \end{pmatrix}, \qquad Z_n = \begin{pmatrix} w_{n-1} \\ v_{n-1} \\ w_n \\ v_n \end{pmatrix}. \tag{5.11}$$

The characteristic equation of A is

$$\lambda^4 - \left(\frac{q_0 - p_0 z}{(q_0 + y)^2} + \frac{q_1 - p_1 y}{(q_1 + z)^2} \right) \lambda^2 - \frac{p_1 p_0}{(q_0 + y)(q_1 + z)} \lambda + \frac{(q_0 - p_0 z)(q_1 - p_1 y)}{(q_0 + y)^2 (q_1 + z)^2} = 0. \tag{5.12}$$

According to Remark 1.3.1 of [7], all the roots of (5.12) are of modulus less than 1 if and only if

$$\left| \frac{q_0 - p_0 z}{(q_0 + y)^2} + \frac{q_1 - p_1 y}{(q_1 + z)^2} \right| + \frac{p_1 p_0}{(q_0 + y)(q_1 + z)} + \left| \frac{(q_0 - p_0 z)(q_1 - p_1 y)}{(q_0 + y)^2 (q_1 + z)^2} \right| < 1. \tag{5.13}$$

From (5.3), we get

$$q_0 - p_0 z = (1 - y)(y + q_0), \qquad q_1 - p_1 y = (1 - z)(z + q_1). \tag{5.14}$$

Then from (5.6), (5.7), and (5.14), inequality (5.13) is equivalent to

$$\frac{1 - y}{q_0 + y} + \frac{1 - z}{q_1 + z} + \frac{p_1 p_0}{(q_0 + y)(q_1 + z)} + \frac{(1 - y)(1 - z)}{(q_0 + y)(q_1 + z)} < 1. \tag{5.15}$$

Using (5.9), inequality (5.15) holds if (5.1) are satisfied. Using Propositions 4.1 and 4.3, we have that the unique positive equilibrium (y, z) of (1.3) is globally asymptotically stable.

(ii) Arguing as above, we can prove that the linearized system of (1.3) about the zero equilibrium is

$$Z_{n+1} = AZ_n, \tag{5.16}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{1}{q_0} & \dfrac{p_0}{q_0} & 0 & 0 \\ 0 & \dfrac{1}{q_1} & \dfrac{p_1}{q_1} & 0 \end{pmatrix}, \qquad Z_n = \begin{pmatrix} w_{n-1} \\ v_{n-1} \\ w_n \\ v_n \end{pmatrix}. \tag{5.17}$$

The characteristic equation of A is

$$\lambda^4 - \left(\frac{1}{q_0} + \frac{1}{q_1}\right)\lambda^2 - \frac{p_1 p_0}{q_0 q_1}\lambda + \frac{1}{q_0 q_1} = 0. \tag{5.18}$$

Using [7, Remark 1.3.1], all the roots of (5.18) are of modulus less than 1 if and only if relation (5.2) holds. This completes the proof of the proposition. □

6. Conclusion

In this paper, in order to investigate (1.2), we study the equivalent system (1.3). Summarizing the results of Sections 2, 3, 4, we get the following statements, concerning (1.2).

(i) If (2.1) hold, then (1.2) has a unique positive periodic solution of period 2.

(ii) If either (3.4) or (3.37) holds, then every positive solution of (1.2) is bounded and persists and tends to the unique positive periodic solution.

(iii) If (5.1) hold, then the unique periodic solution of (1.2) is globally asymptotically stable and if (5.2) holds, then the zero solution of (1.2) is locally asymptotically stable.

Open Problem

Consider the difference equation (1.2) where p_n, q_n, $n = 0, 1, \ldots$ are positive sequences of period 2, and the initial values x_i, $i = -3, -2, -1, 0$ are positive numbers. Prove that

(i) if

$$q_0 - 1 < p_0 \leq q_0 + 1, \qquad q_1 - 1 < p_1 \leq q_1 + 1 \tag{6.1}$$

are satisfied, then every positive solution of (1.2) is bounded and persists;

(ii) if relations (6.1) are satisfied, then every positive solution of (1.2) tends to the unique positive equilibrium (y, z) of (1.2) as $n \to \infty$.

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References

[1] E. Camouzis, E. Chatterjee, and G. Ladas, "On the dynamics of $x_{n+1} = (\delta x_{n-2} + x_{n-3})/(A + x_{n-3})$," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 230–239, 2007.

[2] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Application*, vol. 155 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1992.

[3] D. Benest and C. Froeschlé, Eds., *Analysis and Modelling of Discrete Dynamical Systems*, vol. 1 of *Advances in Discrete Mathematics and Applications*, Gordon and Breach Science, Amsterdam, The Netherlands, 1998.

[4] L. Edelstein-Keshet, *Mathematical Models in Biology*, The Random House/Birkhäuser Mathematics Series, Random House, New York, NY, USA, 1988.

[5] S. N. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 1996.

[6] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, vol. 4 of *Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2005.

[7] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[8] S. Stević, "On a discrete epidemic model," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 87519, 10 pages, 2007.

[9] S. Stević, "On global periodicity of a class of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 23503, 10 pages, 2007.

[10] S. Stević, "On the recursive sequence $x_{n+1} = A + x_n^p / x_{n-1}^p$," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 34517, 9 pages, 2007.

[11] S. Stević, "Asymptotics of some classes of higher-order difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.

[12] S. Stević, "Asymptotic periodicity of a higher-order difference equation," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 13737, 9 pages, 2007.

[13] J. M. Cushing, "Periodically forced nonlinear systems of difference equations," *Journal of Difference Equations and Applications*, vol. 3, no. 5-6, pp. 547–561, 1998.

[14] J. M. Cushing and S. M. Henson, "Global dynamics of some periodically forced, monotone difference equations," *Journal of Difference Equations and Applications*, vol. 7, no. 6, pp. 859–872, 2001.

[15] J. M. Cushing and S. M. Henson, "A periodically forced Beverton-Holt equation," *Journal of Difference Equations and Applications*, vol. 8, no. 12, pp. 1119–1120, 2002.

[16] M. R. S. Kulenović, G. Ladas, and C. B. Overdeep, "On the dynamics of $x_{n+1} = p_n + x_{n-1}/x_n$ with a period-two coefficient," *Journal of Difference Equations and Applications*, vol. 10, no. 10, pp. 905–914, 2004.

[17] G. Papaschinopoulos and C. J. Schinas, "On a $(k+1)$-th order difference equation with a coefficient of period $k+1$," *Journal of Difference Equations and Applications*, vol. 11, no. 3, pp. 215–225, 2005.

[18] G. Papaschinopoulos and C. J. Schinas, "Periodic and asymptotic character of a difference equation with periodic coefficient," *Mathematical Sciences Research Journal*, vol. 9, no. 2, pp. 32–40, 2005.

[19] G. Papaschinopoulos, C. J. Schinas, and G. Stefanidou, "On a difference equation with 3-periodic coefficient," *Journal of Difference Equations and Applications*, vol. 11, no. 15, pp. 1281–1287, 2005.

[20] M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations*, vol. 32 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1990.