Research Article

Stability Analysis of a Delayed SIR Epidemic Model with Stage Structure and Nonlinear Incidence

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We investigate the stability of an SIR epidemic model with stage structure and time delay. By analyzing the eigenvalues of the corresponding characteristic equation, the local stability of each feasible equilibrium of the model is established. By using comparison arguments, it is proved when the basic reproduction number is less than unity, the disease free equilibrium is globally asymptotically stable. When the basic reproduction number is greater than unity, sufficient conditions are derived for the global stability of an endemic equilibrium of the model. Numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

Let $S(t)$ denote the number of members of a population susceptible to a disease, $I(t)$ the number of infective members, and $R(t)$ the number of members who have been removed from the possibility of infection through full immunity, a standard SIR compartmental model is of the form [1]

$$
\begin{aligned}
\dot{S}(t) &= A - \mu S(t) - \beta S(t) I(t), \\
\dot{I}(t) &= \beta S(t) I(t) - (\mu + \gamma + \varepsilon) I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t),
\end{aligned}
\tag{1.1}
$$

where the parameters A, β, γ, μ, ε are positive constants in which A is the recruitment rate of susceptible population, μ represents the natural death rate of the population, ε is the disease-induced death rate of the infectives, and γ is the recovery rate from the infected compartment. It is assumed further that susceptibles become infectious by contact with infectious individuals. Later they may recover and join the group of immune (or dead)

individuals. Based on the previous idea, different types of SIR epidemic models have been investigated (see, e.g., [2–6]). We note that most of the previous works assume that each species has the same contact and recovery rates ignoring the effect of stage structure. In the real world, any species has a process of growth and development, such as from immature to mature, and growth at various stages of life history showed differences in physiology. In the recent years, there have been a fair amount of work on epidemiological models with stage structure (see, e.g., [7–10]). In fact, the spread of disease is related to the species stage structure. Some diseases, such as measles, mumps, chickenpox and scarlet fever, only spread or have more opportunities to spread in children, and some other diseases, such as diphtheria, leptospirosis, a variety of sexually transmitted diseases, may spread in adult. By assuming that the mature population does not contract the disease and the immature population is susceptible to the infection in [9], Xiao et al. proposed an SIR disease transmission model with stage structure and bilinear incidence rate as follows:

$$
\begin{aligned}
\dot{x}_1(t) &= a(1-\gamma)y(t) - r_1 x_1(t) - \beta x_1(t)x_2(t), \\
\dot{x}_2(t) &= \beta x_1(t)x_2(t) - bx_2(t) - r_2 x_2(t), \\
\dot{x}_3(t) &= bx_2(t) + a\gamma y(t) - a\gamma e^{-r_3\tau} y(t-\tau) - r_3 x_3(t), \\
\dot{y}(t) &= a\gamma e^{-r_3\tau} y(t-\tau) - ry^2(t),
\end{aligned}
\tag{1.2}
$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the densities of the immature population that are susceptible, infectious population and recovered population with immunity, respectively; $y(t)$ denotes the density of the mature population which does not contract the disease. The parameters a, b, β, γ, r_1, r_2, r_3, r are positive constants. a is the birth rate of the immature population. It is assumed that newborn individuals are the recovered population with immunity with probability γ ($0 < \gamma < 1$) and are susceptible population with probability $1-\gamma$. β is the rate that the susceptible population become infective, and b is the rate that the infective population becomes recovered with immunity. r_1, r_2, r_3 are the death rates of the susceptible, infective, recovered population, respectively, and $r_1 \leq r_2$ is reasonable for biological meaning. r is the death rate of the mature population. Finally, it is assumed that those immatures born at time $t-\tau$ that survive to time t exit from the immature population and enter the mature population. Xiao et al. [9] proved that if the basic reproduction number is less than unity, the disease-free equilibrium of system (1.2) is globally asymptotically stable; if the basic reproduction number is greater than unity, sufficient conditions were derived for the global stability of an endemic equilibrium.

Incidence rate plays a very important role in the research of epidemiological models; it should generally be written as $\beta U(N)S/N$, where N is the total population size (see [1]). In classical epidemic models, bilinear incidence rate βSI and standard incidence rate $\beta SI/N$ are frequently used. The bilinear incidence rate is based on the law of mass action. This contact law is more appropriate for communicable diseases such as influenza., but not for sexually transmitted diseases. For standard incidence rate, it may be a good approximation if the number of available partners is large enough and everybody could not make more contacts than is practically feasible [11]. After a study of the cholera epidemic spread in Bari in 1973, Capasso and Serio [12] introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I)$ tends to a saturation level when I gets large, that is,

$$
g(I) = \frac{\beta I}{1 + \alpha I},
\tag{1.3}
$$

where βI measures the force of infection of the disease, and $1/(1+\alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the susceptible individuals. This incidence rate seems more reasonable than the bilinear incidence rate $g(I)S = \beta IS$, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters [13].

Motivated by the work of Capasso and Serio [12] and Xiao et al. [9], in this paper, we are concerned with the effect of stage structure and saturation incidence on the dynamic of an SIR epidemic model. To this end, we study the following delayed differential system

$$\dot{x}_1(t) = a(1-\gamma)y(t) - r_1 x_1(t) - \frac{\beta x_1(t) x_2(t)}{1 + \alpha x_2(t)},$$

$$\dot{x}_2(t) = \frac{\beta x_1(t) x_2(t)}{1 + \alpha x_2(t)} - b x_2(t) - r_2 x_2(t),$$

$$\dot{x}_3(t) = b x_2(t) + a\gamma y(t) - a\gamma e^{-r_3\tau} y(t-\tau) - r_3 x_3(t),$$

$$\dot{y}(t) = a\gamma e^{-r_3\tau} y(t-\tau) - r y^2(t). \tag{1.4}$$

The initial conditions for system (1.4) take the form

$$x_i(\theta) = \phi_1(\theta), \qquad x_2(\theta) = \phi_2(\theta), \qquad x_3(\theta) = \phi_3(\theta), \qquad y(\theta) = \psi(\theta),$$

$$\phi_1(\theta) \geq 0, \qquad \phi_2(\theta) \geq 0, \qquad \phi_3(\theta) \geq 0, \qquad \psi(\theta) \geq 0, \quad \theta \in [-\tau, 0], \tag{1.5}$$

$$\phi_1(0) > 0, \qquad \phi_2(0) > 0, \qquad \phi_3(0) > 0, \qquad \psi(0) > 0,$$

where

$$\phi_i, \psi \in C\left([-\tau, 0], \mathbb{R}_{+0}^4\right), \qquad \phi_i(0) > 0 \ (i = 1, 2, 3), \ \psi(0) > 0, \tag{1.6}$$

here $\mathbb{R}_{+0}^4 = \{(x_1, x_2, x_3, x_4) : x_i \geq 0, \ i = 1, 2, 3, 4\}$.

For continuity of initial conditions, we require

$$\phi_3(0) = \int_{-\tau}^{0} a\gamma\psi(s) e^{r_3 s} ds. \tag{1.7}$$

It is easy to show that all solutions of system (1.4) with initial conditions (1.5) and (1.7) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, by analyzing the corresponding characteristic equations, the local stability of each of nonnegative equilibria of system (1.4) is discussed. In Section 3, we study the global stability of the disease-free equilibrium and the endemic equilibrium of system (1.4), respectively. Numerical simulations are carried out in Section 4 to illustrate the main theoretical results. A brief discussion is given in Section 5 to conclude this work.

2. Local Stability

In this section, we discuss the local stability of each of nonnegative equilibria of system (1.4) by analyzing the eigenvalues of the corresponding characteristic equations, respectively.

System (1.4) always has a trivial equilibrium $E_0(0, 0, 0, 0)$, and a disease free equilibrium $E_1(x_1^0, 0, x_3^0, y^0)$, where

$$x_1^0 = \frac{a^2\gamma(1-\gamma)e^{-r_3\tau}}{rr_1}, \qquad x_3^0 = \frac{a^2\gamma^2 e^{-r_3\tau}(1-e^{-r_3\tau})}{rr_3}, \qquad y^0 = \frac{a\gamma e^{-r_3\tau}}{r}. \tag{2.1}$$

The basic reproduction number is given as

$$\mathcal{R}_0 = \frac{a^2\beta\gamma(1-\gamma)e^{-r_3\tau}}{rr_1(b+r_2)}. \tag{2.2}$$

It is easy to prove that if $\mathcal{R}_0 > 1$, system (1.4) has an endemic equilibrium $E^*(x_1^*, x_2^*, x_3^*, y^*)$, where

$$x_1^* = \frac{a\alpha(1-\gamma)y^* + b + r_2}{\alpha r_1 + \beta}, \qquad x_2^* = \frac{a\beta(1-\gamma)y^* - r_1(b+r_2)}{(b+r_2)(\alpha r_1 + \beta)},$$

$$x_3^* = \frac{bx_2^* + a\gamma(1-e^{-r_3\tau})y^*}{r_3}, \qquad y^* = \frac{a\gamma e^{-r_3\tau}}{r}. \tag{2.3}$$

The characteristic equation of system (1.4) at the equilibrium $E_0(0, 0, 0, 0)$ is of the form

$$(\lambda + r_1)(\lambda + b + r_2)(\lambda + r_3)\left(\lambda - a\gamma e^{-(\lambda + r_3)\tau}\right) = 0. \tag{2.4}$$

Obviously, (2.4) always has three negative real roots $\lambda = -r_1$, $\lambda = -b_2 - r_2$, and $\lambda = -r_3$. Noting that $y = \lambda$ and $y = a\gamma e^{-\tau(\lambda + r_3)}$ must intersect at a positive value of λ, hence, the equation $\lambda - a\gamma e^{-(\lambda + r_3)\tau} = 0$ has a positive real root. Accordingly, E_0 is unstable.

The characteristic equation of system (1.4) at the equilibrium $E_1(x_1^0, 0, x_3^0, y^0)$ takes the form

$$(\lambda + r_1)(\lambda + r_3)\left[\lambda - \frac{a^2\beta\gamma(1-\gamma)e^{-r_3\tau} - rr_1(b+r_2)}{rr_1}\right]\left[\lambda + 2a\gamma e^{-r_3\tau} - a\gamma e^{-(\lambda + r_3)\tau}\right] = 0. \tag{2.5}$$

Obviously, (2.5) always has three real roots $\lambda_1 = -r_1 < 0$, $\lambda_2 = -r_3 < 0$, and $\lambda_3 = [a^2\beta\gamma(1-\gamma)e^{-r_3\tau} - rr_1(b+r_2)]/(rr_1)$. Clearly, if $\mathcal{R}_0 < 1$, $\lambda_3 < 0$. Other roots are given by the roots of equation

$$\lambda + 2a\gamma e^{-r_3\tau} - a\gamma e^{-(\lambda + r_3)\tau} = 0. \tag{2.6}$$

Let $f(\lambda) = \lambda + 2a\gamma e^{-r_3\tau} - a\gamma e^{-(\lambda+r_3)\tau}$. Now, we claim that the roots of $f(\lambda) = 0$ have only negative real parts. Suppose that $\mathrm{Re}\,\lambda \geq 0$, then it follows from (2.6) that

$$\mathrm{Re}\,\lambda = a\gamma e^{-r_3\tau}\left[e^{-\tau\,\mathrm{Re}\,\lambda}\cos(\tau\mathrm{Im}\,\lambda) - 2\right]$$

$$\leq -a\gamma e^{-r_3\tau} < 0,$$

(2.7)

which leads to a contradiction. Hence, we have $\mathrm{Re}\,\lambda < 0$. Therefore, if $\mathcal{R}_0 < 1$, the disease-free equilibrium $E_1(x_1^0, 0, x_3^0, y^0)$ is locally asymptotically stable. If $\mathcal{R}_0 > 1$, (2.5) has a positive root, then the disease-free equilibrium E_1 is unstable.

The characteristic equation of system (1.4) at the endemic equilibrium $E^*(x_1^*, x_2^*, x_3^*, y^*)$ takes the form

$$(\lambda + r_3)\left(\lambda^2 + p\lambda + q\right)\left(\lambda + 2a\gamma e^{-r_3\tau} - a\gamma e^{-(\lambda+r_3)\tau}\right) = 0,$$

(2.8)

where

$$p = \frac{r_1 + [\alpha(r_1 + b + r_2) + \beta]x_2^*}{1 + \alpha x_2^*} > 0,$$

$$q = \frac{(b + r_2)(\alpha r_1 + \beta)x_2^*}{1 + \alpha x_2^*} > 0.$$

(2.9)

Clearly, (2.8) always has a negative real root $\lambda = -r_3$. Noting that $p > 0$, $q > 0$, roots of equation $\lambda^2 + p\lambda + q = 0$ have only negative real parts. In addition, from the discussion above, we see that roots of the (2.6) have only negative real parts. By the general theory on characteristic equations of delay differential equations from [14], we see that if $\mathcal{R}_0 > 1$, the endemic equilibrium E^* is locally asymptotically stable.

Based on the discussions above, we have the following result.

Theorem 2.1. *For system* (1.4), *one has the following:*

(i) *if $\mathcal{R}_0 > 1$, the endemic equilibrium $E^*(x_1^*, x_2^*, x_3^*, y^*)$ is locally asymptotically stable,*

(ii) *if $\mathcal{R}_0 < 1$, the disease-free equilibrium $E_1(x_1^0, 0, x_3^0, y^0)$ is locally asymptotically stable.*

3. Global Stability

In this section, we discuss the global stability of the disease-free equilibrium and the endemic equilibrium of system (1.4), respectively. The technique of proofs is to use a comparison argument and an iteration scheme.

We first consider the subsystem of (1.4)

$$\dot{x}_1(t) = a(1 - \gamma)y(t) - r_1 x_1(t) - \frac{\beta x_1(t)x_2(t)}{1 + \alpha x_2(t)},$$

$$\dot{x}_2(t) = \frac{\beta x_1(t)x_2(t)}{1 + \alpha x_2(t)} - bx_2(t) - r_2 x_2(t).$$

(3.1)

Letting $z(t) = x_1(t) + x_2(t)$, $x(t) = x_2(t)$, system (3.1) becomes

$$\dot{z}(t) = a(1 - \gamma)y(t) - r_1 z(t) + (r_1 - r_2 - b)x(t),$$

$$\dot{x}(t) = x(t)\left[\frac{\beta z(t)}{1 + \alpha x(t)} - \frac{\beta x(t)}{1 + \alpha x(t)} - (b + r_2)x(t)\right]. \tag{3.2}$$

The initial conditions for system (3.2) take the form

$$z(\theta) = \varphi_1(\theta), \qquad x(\theta) = \varphi_2(\theta), \qquad \varphi_i(\theta) \geq 0, \qquad \varphi_i(0) > 0, \quad i = 1, 2. \tag{3.3}$$

Clearly, system (3.2) has a nonnegative equilibrium $A_1(z^0, 0)$, where $z^0 = a^2\gamma(1-\gamma)e^{-r_3\tau}/(rr_1)$; when $\mathcal{R}_0 > 1$, system (3.2) has a positive equilibrium $A^*(z^*, x^*)$, where

$$z^* = \frac{a^2\gamma(1 - \gamma)e^{-r_3\tau}}{rr_1} + \frac{(r_1 - r_2 - b)x^*}{r_1},$$

$$x^* = \frac{a^2\beta\gamma(1 - \gamma)e^{-r_3\tau} - rr_1(b + r_2)}{r(b + r_2)(\alpha r_1 + \beta)}. \tag{3.4}$$

Moreover, from Theorem 2.1, we see that A_1 is locally asymptotically stable if $\mathcal{R}_0 < 1$, and A^* is locally asymptotically stable if $\mathcal{R}_0 > 1$.

To study the global dynamics of system (1.4), we need only to discuss the global behavior of solutions of system (3.2). In the following, we investigate the global asymptotic stability of the equilibria A_1 and A^* by using the comparison arguments and the iteration scheme [15], respectively. To this end, we need the following result developed by Song and Chen in [16].

Lemma 3.1. *Consider the following equation:*

$$\dot{x}(t) = ax(t - \tau) - bx(t) - cx^2(t), \tag{3.5}$$

where $a, b, c, \tau > 0$, $x(t) > 0$ for $t \in [-\tau, 0]$. One has the following:

　　(i) *if $a > b$, then $\lim_{t \to +\infty} x(t) = (a - b)/c$;*
　　(ii) *if $a < b$, then $\lim_{t \to +\infty} x(t) = 0$.*

Lemma 3.2. *Let $\mathcal{R}_0 > 1$. If $\alpha r_1 > \beta$, then $A^*(z^*, x^*)$ is globally asymptotically stable.*

Proof. Let $(z(t), x(t))$ be any positive solution of system (3.2) with initial condition (3.3). Let

$$U_1 = \limsup_{t \to +\infty} z(t), \qquad V_1 = \liminf_{t \to +\infty} z(t),$$
$$U_2 = \limsup_{t \to +\infty} x(t), \qquad V_2 = \liminf_{t \to +\infty} x(t). \tag{3.6}$$

Now we claim that $U_1 = V_1 = z^*$, $U_2 = V_2 = x^*$.

From Lemma 3.1, it is easy to show that

$$\lim_{t \to \infty} y(t) = y^0 = y^* = \frac{\alpha \beta e^{-r_3 \tau}}{r}. \tag{3.7}$$

Hence, we know that for $\varepsilon > 0$, there exists a $T_0 > 0$ such that, if $t > T_0$,

$$y^* - \varepsilon < y(t) < y^* + \varepsilon. \tag{3.8}$$

We derive from the first equation of system (3.2) that

$$\dot{z}(t) \le a(1 - \gamma)(y^* + \varepsilon) - r_1 z. \tag{3.9}$$

By comparison, we have

$$\limsup_{t \to +\infty} z(t) \le \frac{a(1 - \gamma)(y^* + \varepsilon)}{r_1}. \tag{3.10}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $U_1 \le M_1^z$, where

$$M_1^z = \frac{a(1 - \gamma)y^*}{r_1}. \tag{3.11}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > T_0$ such that, if $t > T_1$, $z(t) \le M_1^z + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second equation of system (3.2) that, for $t > T_1$,

$$\dot{x}(t) \le \frac{x(t)}{1 + \alpha x(t)} \left[(\beta(M_1^z + \varepsilon) - (b + r_2)) - (\beta + \alpha(b + r_2))x(t) \right]. \tag{3.12}$$

Consider the following auxiliary system:

$$\dot{u}(t) = u(t) \left[(\beta(M_1^z + \varepsilon) - (b + r_2)) - (\beta + \alpha(b + r_2))u(t) \right]. \tag{3.13}$$

By Lemma 3.1 it follows from (3.13) that

$$\lim_{t \to +\infty} u(t) = \frac{\beta(M_1^z + \varepsilon) - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.14}$$

By comparison, we obtain that

$$\limsup_{t \to +\infty} x(t) \le \frac{\beta(M_1^z + \varepsilon) - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.15}$$

Since the inequality is true for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $U_2 \leq M_1^x$, where

$$M_1^x = \frac{\beta M_1^z - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.16}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1$ such that, if $t > T_2$, $x(t) \leq M_1^x + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (3.2) that, for $t > T_2$,

$$\dot{z}(t) \geq a(1 - \gamma)(y^* - \varepsilon) - r_1 z(t) + (r_1 - r_2 - b)(M_1^x + \varepsilon). \tag{3.17}$$

By comparison and by Lemma 3.1, we have

$$\liminf_{t \to +\infty} z(t) \geq \frac{a(1 - \gamma)(y^* - \varepsilon) + (r_1 - r_2 - b)(M_1^x + \varepsilon)}{r_1}. \tag{3.18}$$

Since the inequality is true for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $V_1 \geq N_1^z$, where

$$N_1^z = \frac{a(1 - \gamma)y^* + (r_1 - r_2 - b)M_1^x}{r_1}. \tag{3.19}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_3 > T_2$ such that, if $t > T_3$, $z(t) \geq N_1^z - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second equation of system (3.2) that, for $t > T_3$,

$$\dot{x}(t) \geq \frac{x(t)}{1 + \alpha x(t)} \left[(\beta(N_1^z - \varepsilon) - (b + r_2)) - (\beta + \alpha(b + r_2))x(t) \right]. \tag{3.20}$$

By comparison and by Lemma 3.1, we have

$$\liminf_{t \to +\infty} x(t) \geq \frac{\beta(N_1^z - \varepsilon) - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.21}$$

Since the inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $V_2 \geq N_1^x$, where

$$N_1^x = \frac{\beta N_1^z - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.22}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_4 > T_3$ such that if $t > T_4$, $x(t) \geq N_1^x - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (3.2) that, for $t > T_4$,

$$\dot{z}(t) \leq a(1 - \gamma)(y^* + \varepsilon) - r_1 z(t) + (r_1 - r_2 - b)(N_1^x - \varepsilon). \tag{3.23}$$

By comparison and by Lemma 3.1, we have

$$\limsup_{t \to +\infty} z(t) \le \frac{a(1-\gamma)(y^* + \varepsilon) + (r_1 - r_2 - b)(N_1^x - \varepsilon)}{r_1}. \tag{3.24}$$

Since the inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $U_1 \le M_2^z$, where

$$M_2^z = \frac{a(1-\gamma)y^* + (r_1 - r_2 - b)N_1^x}{r_1}. \tag{3.25}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_5 > T_4$ such that, if $t > T_5$, $z(t) \le M_2^z + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second equation of system (3.2) that, for $t > T_5$,

$$\dot{x}(t) \le \frac{x(t)}{1 + \alpha x(t)} \left[(\beta(M_2^z + \varepsilon) - (b + r_2)) - (\beta + \alpha(b + r_2))x(t) \right]. \tag{3.26}$$

By comparison and by Lemma 3.1, we have

$$\limsup_{t \to +\infty} x(t) \le \frac{\beta(M_2^z + \varepsilon) - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.27}$$

Since the inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_2 \le M_2^x$, where

$$M_2^x = \frac{\beta M_2^z - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.28}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_6 > T_5$ such that, if $t > T_6$, $x(t) \le M_2^x + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (3.2) that, for $t > T_6$,

$$\dot{z}(t) \ge a(1-\gamma)(y^* - \varepsilon) - r_1 z(t) + (r_1 - r_2 - b)(M_2^x + \varepsilon). \tag{3.29}$$

By comparison and by Lemma 3.1 it follows that

$$\liminf_{t \to +\infty} z(t) \ge \frac{a(1-\gamma)(y^* - \varepsilon) + (r_1 - r_2 - b)(M_2^x + \varepsilon)}{r_1}. \tag{3.30}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \ge N_2^z$, where

$$N_2^z = \frac{a(1-\gamma)y^* + (r_1 - r_2 - b)M_2^x}{r_1}. \tag{3.31}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_7 > T_6$ such that, if $t > T_7$, $z(t) \ge N_2^z - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second equation of system (3.2) that, for $t > T_7$,

$$\dot{x}(t) \geq \frac{x(t)}{1 + \alpha x(t)} \left[\left(\beta(N_2^z - \varepsilon) - (b + r_2) \right) - \left(\beta + \alpha(b + r_2) \right) x(t) \right]. \tag{3.32}$$

By comparison and by Lemma 3.1 it follows that

$$\liminf_{t \to +\infty} x(t) \geq \frac{\beta(N_2^z - \varepsilon) - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.33}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_2^x$, where

$$N_2^x = \frac{\beta N_2^z - (b + r_2)}{\beta + \alpha(b + r_2)}. \tag{3.34}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_8 > T_7$ such that, if $t > T_8$, $x(t) \geq N_2^x - \varepsilon$.

Continuing this process, we derive four sequences $M_n^z, M_n^x, N_n^z, N_n^x$ $(n = 1, 2, \ldots)$ such that for $n \geq 2$,

$$\begin{aligned}
M_n^z &= \frac{a(1 - \gamma)y^* + (r_1 - r_2 - b)N_{n-1}^x}{r_1}, \\[2mm]
N_n^z &= \frac{a(1 - \gamma)y^* + (r_1 - r_2 - b)M_n^x}{r_1}, \\[2mm]
M_n^x &= \frac{\beta M_n^z - (b + r_2)}{\beta + \alpha(b + r_2)}, \\[2mm]
N_n^x &= \frac{\beta N_n^z - (b + r_2)}{\beta + \alpha(b + r_2)}.
\end{aligned} \tag{3.35}$$

Clearly, we have

$$N_n^x \leq V_2 \leq U_2 \leq M_n^x, \qquad N_n^z \leq V_1 \leq U_1 \leq M_n^z. \tag{3.36}$$

It follows from (3.36) that

$$\begin{aligned}
M_{n+1}^x &= \frac{a\beta(1 - \gamma)y^* - r_1(b + r_2)}{r_1[\beta + \alpha(b + r_2)]} \left[1 + \frac{\beta(r_1 - r_2 - b)}{r_1(\beta + \alpha(b + r_2))} \right] \\[2mm]
&\quad + \frac{\beta^2(r_1 - r_2 - b)^2}{r_1^2[\beta + \alpha(b + r_2)]^2} M_n^x.
\end{aligned} \tag{3.37}$$

Noting that $M_n^x \geq x^*$ and $\alpha r_1 > \beta$, we derive from (3.37) that

$$
\begin{aligned}
M_{n+1}^x - M_n^x &= \frac{a\beta(1-\gamma)y^* - r_1(b+r_2)}{r_1[\beta + \alpha(b+r_2)]}\left[1 + \frac{\beta(r_1 - r_2 - b)}{r_1(\beta + \alpha(b+r_2))}\right] \\
&\quad + \left[\frac{\beta^2(r_1 - r_2 - b)^2}{r_1^2[\beta + \alpha(b+r_2)]^2} - 1\right]M_n^x \\
&\leq \frac{a\beta(1-\gamma)y^* - r_1(b+r_2)}{r_1[\beta + \alpha(b+r_2)]}\left[1 + \frac{\beta(r_1 - r_2 - b)}{r_1(\beta + \alpha(b+r_2))}\right] \\
&\quad + \left[\frac{\beta^2(r_1 - r_2 - b)^2}{r_1^2[\beta + \alpha(b+r_2)]^2} - 1\right]x^* \\
&= 0.
\end{aligned}
\tag{3.38}
$$

Hence, the sequence M_n^x is monotonically nonincreasing. Therefore, $\lim_{n\to+\infty} M_n^x$ exists. Taking $n \to +\infty$, it follows from (3.37) that

$$
\lim_{n\to+\infty} M_n^x = \frac{a^2\beta\gamma(1-\gamma)e^{-r_3\tau} - rr_1(b+r_2)}{r(b+r_2)(r_1\alpha + \beta)} = x^*.
\tag{3.39}
$$

We therefore obtain from (3.35) and (3.39) that

$$
\lim_{n\to+\infty} N_n^x = x^*, \qquad \lim_{n\to+\infty} M_n^z = z^*, \qquad \lim_{n\to+\infty} N_n^z = z^*.
\tag{3.40}
$$

It follows from (3.36), (3.39), and (3.40) that

$$
U_1 = V_1 = z^*, \qquad U_2 = V_2 = x^*.
\tag{3.41}
$$

We therefore have

$$
\lim_{t\to+\infty} z(t) = z^*, \qquad \lim_{t\to+\infty} x(t) = x^*.
\tag{3.42}
$$

Noting that if $\mathcal{R}_0 > 1$ and $\alpha r_1 > \beta$ hold, the positive equilibrium A^* is locally asymptotically stable, we conclude that A^* is globally asymptotically stable. The proof is complete. □

Theorem 3.3. *If $\mathcal{R}_0 > 1$ and $\alpha r_1 > \beta$ hold, then the endemic equilibrium $E^*(x_1^*, x_2^*, x_3^*, y^*)$ of system (1.4) is globally asymptotically stable; that is, the disease remains endemic.*

Proof. From (3.7), we know that $\lim_{t\to+\infty} y(t) = y^* = y^0 = a\gamma e^{-r_3\tau}/r$. According to the results of Lemma 3.2, we prove that

$$
\lim_{t\to+\infty} x_1(t) = x_1^*, \qquad \lim_{t\to+\infty} x_2(t) = x_2^*.
\tag{3.43}
$$

In the following, we show the existence of $\lim_{t\to+\infty} x_3(t)$.

By Lemma 3.2, it follows from (3.43) that for $\varepsilon > 0$ sufficiently small, there exists a $T > 0$, such that, if $t > T$,

$$y^* - \varepsilon < y(t) < y^* + \varepsilon, \qquad x_1^* - \varepsilon < x_1(t) < x_1^* + \varepsilon, \qquad x_2^* - \varepsilon < x_2(t) < x_2^* + \varepsilon. \tag{3.44}$$

Therefore, we derive from the third equation of system (1.4) that, for $t > T + \tau$,

$$\dot{x}_3(t) \leq b(x_2^* + \varepsilon) + a\gamma(y^* + \varepsilon) - a\gamma e^{-r_3\tau}(y^* - \varepsilon) - r_3 x_3(t). \tag{3.45}$$

By comparison, we have

$$\limsup_{t\to+\infty} x_3(t) \leq \frac{b(x_2^* + \varepsilon) + a\gamma(y^* + \varepsilon) - a\gamma e^{-r_3\tau}(y^* - \varepsilon)}{r_3}. \tag{3.46}$$

Since the inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, we have $\limsup_{t\to+\infty} x_3(t) \leq M^{x_3}$, where

$$M^{x_3} = \frac{bx_2^* + a\gamma(1 - e^{-r_3\tau})y^*}{r_3}. \tag{3.47}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > T$ such that, if $t > T_1$,

$$x_3(t) \leq M^{x_3} + \varepsilon. \tag{3.48}$$

Again, for $\varepsilon > 0$ sufficiently small, it follows from the third equation of system (1.4) that, for $t > T_1 + \tau$,

$$\dot{x}_3(t) \geq b(x_2^* - \varepsilon) + a\gamma(y^* - \varepsilon) - a\gamma e^{-r_3\tau}(y^* + \varepsilon) - r_3 x_3(t). \tag{3.49}$$

By comparison, we have

$$\liminf_{t\to+\infty} x_3(t) \geq \frac{b(x_2^* - \varepsilon) + a\gamma(y^* - \varepsilon) - a\gamma e^{-r_3\tau}(y^* + \varepsilon)}{r_3}. \tag{3.50}$$

Since the inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\liminf_{t\to+\infty} x_3(t) \geq N^{x_3}$, where

$$N^{x_3} = \frac{bx_2^* + a\gamma(1 - e^{-r_3\tau})y^*}{r_3}. \tag{3.51}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1$ such that, if $t > T_2$,

$$x_3(t) \geq N^{x_3} - \varepsilon. \tag{3.52}$$

It follows from (3.48) and (3.52) that

$$\lim_{t \to +\infty} x_3(t) = \frac{bx_2^* + a\gamma(1 - e^{-r_3\tau})y^*}{r_3} = x_3^*. \tag{3.53}$$

Noting that if $\mathcal{R}_0 > 1$ and $ar_1 > \beta$ hold, the endemic equilibrium E^* is locally asymptotically stable, we see that E^* is globally asymptotically stable. This completes the proof. $\square$

Theorem 3.4. *If $\mathcal{R}_0 < 1$ holds, the disease-free equilibrium $E_1(x_1^0, 0, x_3^0, y^0)$ of system (1.4) is globally asymptotically stable; that is, the disease fades out.*

Proof. Choose $\varepsilon > 0$ sufficiently small satisfying

$$\beta\left(\frac{a(1 - \gamma)y^0}{r_1} + \varepsilon\right) < b + r_2. \tag{3.54}$$

From (3.7), we know that for $\varepsilon > 0$ sufficiently small, there exists a $t_0 > 0$ such that if $t > t_0$,

$$y^0 - \varepsilon < y(t) < y^0 + \varepsilon. \tag{3.55}$$

We derive from the first equation of system (3.2) that

$$\dot{z}(t) \le a(1 - \gamma)\left(y^0 + \varepsilon\right) - r_1 z. \tag{3.56}$$

By Lemma 3.1 and by a comparison argument, we get

$$\limsup_{t \to +\infty} z(t) \le \frac{a(1 - \gamma)(y^0 + \varepsilon)}{r_1}. \tag{3.57}$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that

$$\limsup_{t \to +\infty} z(t) \le \frac{a(1 - \gamma)y^0}{r_1}. \tag{3.58}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $t_1 > t_0$ such that, for $t > t_1$,

$$z(t) \le \frac{a(1 - \gamma)y^0}{r_1} + \varepsilon. \tag{3.59}$$

For $\varepsilon > 0$ sufficiently small satisfying (3.54), it follows from (3.59) and the second equation of system (3.2) that

$$\dot{x}(t) \le \frac{x(t)}{1 + ax(t)}\left[\beta\left(\frac{a(1 - \gamma)y^0}{r_1} + \varepsilon\right) - (b + r_2) - (\beta + a(b + r_2))x(t)\right]. \tag{3.60}$$

Noting that (3.54) holds, we conclude that

$$\lim_{t \to +\infty} x(t) = 0. \tag{3.61}$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying (3.54), there is a $t_2 > t_1$ such that, if $t > t_2$, $x(t) < \varepsilon$.

On the other hand, we derive from the first equation of system (3.2) that, for $t > t_2$,

$$\dot{z}(t) \geq a(1 - \gamma)(y^0 - \varepsilon) - r_1 z(t) + (r_1 - r_2 - b)\varepsilon. \tag{3.62}$$

By Lemma 3.1 and by a comparison argument, we have

$$\liminf_{t \to +\infty} z(t) \geq \frac{a(1 - \gamma)(y^0 - \varepsilon) + (r_1 - r_2 - b)\varepsilon}{r_1}. \tag{3.63}$$

Since this inequality is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that

$$\liminf_{t \to +\infty} z(t) \geq \frac{a(1 - \gamma)y^0}{r_1}, \tag{3.64}$$

which, together with (3.58), yields

$$\lim_{t \to +\infty} z(t) = \frac{a(1 - \gamma)y^0}{r_1} = z^0. \tag{3.65}$$

According to (3.61) and (3.65), we can easily prove that

$$\lim_{t \to +\infty} x_1(t) = x_1^0, \qquad \lim_{t \to +\infty} x_2(t) = 0. \tag{3.66}$$

Using a similar argument as in the proof of Theorem 3.3, we can show that if $\mathcal{R}_0 < 1$, then

$$\lim_{t \to +\infty} x_3(t) = \frac{a\gamma(1 - e^{-r_3\tau})y^0}{r_3} = x_3^0. \tag{3.67}$$

Noting that if $\mathcal{R}_0 < 1$, the disease-free equilibrium E_1 is locally stable, we conclude that E_1 is globally asymptotically stable. This completes the proof. $\qquad \square$

4. Numerical Examples

In this section, we give two examples to illustrate the main theoretical results above.

Example 4.1. In system (1.4), let $\alpha = 1$, $\beta = 2$, $\gamma = 0.5$, $r_1 = 3$, $r_2 = 1$, $r_3 = 0.5$, $r = 1$, $a = 6$, $b = 2$, $\tau = 1$. Computation gives the following value for the basic reproduction
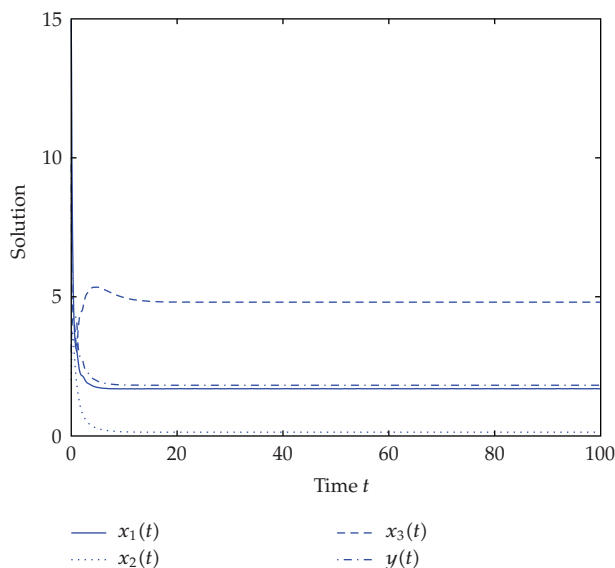
Figure 1: The numerical solution of system (1.4) with $\alpha = 1$, $\beta = 2$, $\gamma = 0.5$, $r_1 = 3$, $r_2 = 1$, $r_3 = 0.5$, $r = 1$, $a = 6$, $b = 2$, $\tau = 1$; $(\phi_1, \phi_2, \phi_3, \psi) = (15, 3, 4, 10)$.

number $\mathcal{R}_0 = 2e^{-1/2} > 1$, and system (1.4) has a unique endemic equilibrium $E^*((9e^{-1/2} + 3)/5$, $(4e^{-1/2}-3)/5$, $-3e^{-1}+31e^{-1/2}/5-12/5$, $3e^{-1/2})$. Clearly, $\alpha r_1 - \beta = 1 > 0$. By Theorem 3.3, we see that the endemic equilibrium E^* of system (1.4) is globally asymptotically stable. Numerical simulation illustrates the previous result (see Figure 1).

Example 4.2. In system (1.4), let $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$, $r_1 = 0.5$, $r_2 = 1$, $r_3 = 1$, $r = 0.5$, $a = 2$, $b = 2$, $\tau = 1$. Computation gives the following value for the basic reproduction number $\mathcal{R}_0 = 4e^{-1}/3 < 1$, system (1.4) has only a disease-free equilibrium $E_1(4e^{-1}$, 0, $2e^{-1}(1-e^{-1})$, $2e^{-1})$. By Theorem 3.4, we see that the disease-free equilibrium E_1 of system (1.4) is globally asymptotically stable. Numerical simulation illustrates this fact (see Figure 2).

5. Discussion

In this paper, we have discussed the effect of stage structure and saturation incidence rate on an SIR epidemic model with time delay. The basic reproduction number $\mathcal{R}_0$ was found. The local stability of each of feasible equilibria of system (1.4) was investigated. When the basic reproduction number is greater than unity, by using the iteration scheme, we have established sufficient conditions for the global stability of the endemic equilibrium of system (1.4). By Theorem 3.3, we see that when $\mathcal{R}_0 > 1$ and $\alpha r_1 > \beta$, the endemic equilibrium is globally stable. Biologically, these indicate that when the proportionality (infection) constant and /or the birth rate of the immature population is sufficiently large and the death rates of susceptible population and the mature population are sufficiently small such that $\mathcal{R}_0 > 1$, then the disease remains endemic. On the other hand, by Theorem 3.4, we see that, if the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable. Biologically, if the proportionality (infection) constant and /or the birth rate of the immature population is small enough and the death rates of susceptible population and the mature
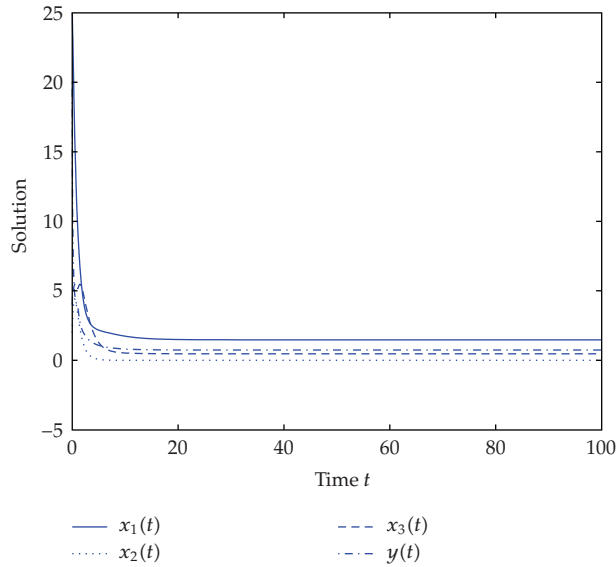
Figure 2: The numerical solution of system (1.4) with $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$, $r_1 = 0.5$, $r_2 = 1$, $r_3 = 1$, $r = 0.5$, $a = 2$, $b = 2$, $\tau = 1$; $(\phi_1, \phi_2, \phi_3, \psi) = (20, 2, 6, 15)$.

population are large enough such that $\mathcal{R}_0 < 1$, then the disease fades out. We would like to point out here that Theorem 3.3 has room for improvement, we leave this for future work.

Acknowledgments

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