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Research Article

Oscillation Criteria of Solution for a Second Order Difference Equation with Forced Term

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We will consider oscillation criteria for the second order difference equation with forced term $\Delta(a_n\Delta(x_n + \lambda x_{n-\tau})) + q_n x_{n-\sigma} = r_n$ $(n \geq 0)$. We establish sufficient conditions which guarantee that every solution is oscillatory or eventually positive solutions converge to zero.

In the last thirty years, there has been an increasing interest in the study of oscillation and asymptotic behavior of solutions of second order difference equations (see [1–11]). In [1], Arul and Thandapani considered the equation

$$\Delta\left(p_n\phi(\Delta x_n)\right) + f(n, x_{n+1}) = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

and gave some sufficient conditions for the existence of positive solutions. In [3], Saker considered the equation

$$\Delta\left(p_n\Delta x_n\right) + q_n f(x_{n-\sigma}) = 0, \quad n = 0, 1, 2, \ldots, \tag{2}$$

and gave some sufficient conditions which guarantee that every solution is oscillatory. Following this trend, we are concerned with oscillation criteria of solutions for a second order difference equation with forced term

$$\Delta(a_n\Delta(x_n + \lambda x_{n-\tau})) + q_n x_{n-\sigma} = r_n, \quad n = 1, 2, \ldots, \tag{3}$$

where $\{a_n\}$ is a positive sequence, $\{q_n\}$ is a nonnegative sequence and not identically zero for all large n, $\{r_n\}$ is a real sequence, λ is a real number, and σ, τ are nonnegative integers, $\mu = \max\{\sigma, \tau\}$.

A solution $\{x_n\}$ of (3) is said to be eventually positive if $x_n > 0$ for all large n and eventually negative if $x_n < 0$ for all large n. Equation (3) is said to be oscillatory if it is neither eventually positive nor eventually negative.

In order to obtain our conclusions, we first give two lemmas.

Lemma 0.1. *If difference inequality*

$$\Delta(a_n \Delta z_n) + q_n z_{n-\sigma} \leq r_n, \quad n > 0, \tag{4}$$

is oscillation, then difference equation

$$\Delta(a_n \Delta z_n) + q_n z_{n-\sigma} = r_n, \quad n > 0, \tag{5}$$

is oscillation.

Otherwise, if (5) has eventually positive solution, then (4) has eventually positive solution; this is contradictory.

Lemma 0.2. *Suppose that $\{x_n\}$ is an eventually positive solution of* (3), *$\lambda \geq 0$, and*

(i) $\sum_{n=1}^{\infty}(1/a_n) = +\infty$,

(ii) $\sum_{n=1}^{\infty} q_n = +\infty$,

(iii) $\sum_{n=1}^{\infty} r_n < \infty$.

Set $z_n = x_n + \lambda x_{n-\tau}$. Then $z_n > 0$ and $\lim_{n\to\infty} a_n \Delta z_n = 0$

Proof. Suppose that $\{x_n\}$ is an eventually positive solution of (3), then there exists $n_1 > \mu$, such that $x_n > 0$, $x_{n-\tau} > 0$, and $x_{n-\sigma} > 0$ for $n > n_1$, then $z_n > 0$ for $n > n_1$. By summing up (3) from n_1 to n, we obtain

$$a_{n+1}\Delta z_{n+1} - a_{n_1}\Delta z_{n_1} + \sum_{s=n_1}^{n} q_s x_{s-\sigma} = \sum_{s=n_1}^{n} r_s. \tag{6}$$

From (6), we know that $\lim_{n\to\infty}\sum_{n=n_1}^{n} q_s x_{s-\sigma} = \alpha$, where α is a positive limited number or $\alpha = +\infty$. Thus $\lim_{n\to\infty} a_n \Delta z_n = \beta$, β is a limited number or $\beta = -\infty$.

If $\beta < c < 0$ (c is a constant), then there exist $n_2 \geq n_1$, $a_n \Delta z_n \leq c$ for $n \geq n_2$, so that

$$z_{n+1} \leq z_{n_2} + c\sum_{s=n_2}^{n} \frac{1}{a_s}, \tag{7}$$

which is contrary to $z_n > 0$.

If $\beta > 0$, then there exist $n_3 \geq n_1,\ a_n \Delta z_n > \beta/2$ for $n \geq n_3$; hence,

$$z_{n+1} \geq z_{n_3} + \frac{\beta}{2} \sum_{s=n_3}^{n} \frac{1}{a_s} \longrightarrow +\infty \quad (n \longrightarrow \infty), \tag{8}$$

therefore, $\lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} x_{n-\sigma} = \infty$; thus, there exist $n_4 \geq n_3$, $x_n \geq M$, and $x_{n-\sigma} \geq M$ $(M > 0)$ for $n \geq n_4$. By summing up (3) from n_4 to, we obtain

$$a_{n+1} \Delta z_{n+1} - a_{n_4} \Delta z_{n_4} + M \sum_{s=n_4}^{n} q_s \leq \sum_{s=n_4}^{n} r_s. \tag{9}$$

As $n \to \infty$, the right-hand side of (9) is bounded, but the left-hand side of (9) tends to ∞; this is contradictory.

Then $\beta = 0$; thus $\lim_{n\to\infty} a_n \Delta z_n = 0$. This completes the proof. □

By means of Lemma 0.2, we obtain the following.

Theorem 0.3. *If conditions (i), (ii), and (iii) hold and* $\{x_n\}$ *is an eventually positive solution of* (3), *then* $\lim_{n\to\infty} x_n = 0$.

Proof. Making use of (6) and the conclusion of Lemma 0.2, we know

$$\lim_{n\to\infty} \sum_{s=n_1}^{n} q_s x_{s-\sigma} = \alpha \quad (0 < \alpha < +\infty), \tag{10}$$

so $\lim_{n\to\infty} x_n = 0$. If not, suppose that $\lim_{n\to\infty} x_n = l > 0$, then there exist $n_5 > n_1,\ x_n \geq l/2 > 0$ for $n > n_5$. Now substitute $x_n \geq l/2 > 0$ for x_n in (6), we obtain a contrary. This completes the proof. □

Theorem 0.4. *If conditions (i), (ii), and (iii) hold, let*

$$w_n = \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} r_t, \quad n > 0, \tag{11}$$

and if $\{w_n\}$ *is oscillation, then* (3) *is oscillation.*

Proof. Suppose that $\{x_n\}$ is an eventually positive solution of (3), then there exist $n_1 > \mu$, $x_n > 0$, $x_{n-\tau} > 0$, and $x_{n-\sigma} > 0$ for $n \geq n_1$. From (6), we have

$$a_{n+1} \Delta z_{n+1} - a_{n_1} \Delta z_{n_1} < \sum_{s=n_1}^{n} r_s. \tag{12}$$

Letting $n \to \infty$ and making use of Lemma 0.2, we get

$$-a_{n_1} \Delta z_{n_1} < \sum_{s=n_1}^{\infty} r_s \tag{13}$$

or

$$-a_n \Delta z_n < \sum_{s=n}^{\infty} r_s \quad (n > n_1). \tag{14}$$

By summing up (14) from n_1 to n, we obtain

$$z_{n_1} - z_{n+1} < \sum_{s=n_1}^{n} \frac{1}{a_s} \sum_{t=s}^{\infty} r_t. \tag{15}$$

In view of Theorem 0.3, we know that $\lim_{n\to\infty} x_n = 0$, then there exists a sequence $\{n_k\}$, such that $\lim_{k\to\infty} n_k = \infty$, $\lim_{k\to\infty} x_{n_k-\sigma} = 0$, and $\lim_{k\to\infty} x_{n_k} = 0$; by means of (15), we have

$$z_{n_1} - z_{n_k+1} < \sum_{s=n_1}^{n_k} \frac{1}{a_s} \sum_{t=s}^{\infty} r_t, \tag{16}$$

so

$$0 < z_n < \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} r_t. \tag{17}$$

This shows that $\{w_n\}$ is nonoscillatory, which is a contradiction. This completes the proof. □

The oscillation of $\{w_n\}$ is only the sufficient condition for the oscillation of (3). The following examples will illustrate this point.

Example 0.5. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta(x_n + x_{n-1})\right) + \frac{3}{n+1}x_n = \frac{3}{(n+2)(n+1)}, \quad n \geq 1. \tag{18}$$

Here, $w_n = \sum_{s=n}^{\infty} s(1/(s-1) + 1/s + 1/(s+1)) > 0$ is nonoscillatory, and the other conditions (i), (ii), and (iii) are satisfied. Equation (18) has the nonoscillatory solution $x_n = (1/n) \to 0 \ \ (n \to \infty)$.

Example 0.6. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta(x_n + x_{n-1})\right) + \frac{(n-4)(2n+1)}{(n+2)(n-1)}x_{n-4} = (-1)^n \frac{2n+1}{n(n+1)}, \quad n \geq 1. \tag{19}$$

Here, $w_n = \sum_{s=n}^{\infty} (-1)^s$ is oscillatory, and the conditions (i), (ii), and (iii) are satisfied. Equation (19) is oscillation.

Example 0.7. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta(x_n + 2x_{n-1})\right) + \left(\frac{(-1)^n}{n(n+1)} + \frac{2}{n+1} + \frac{2}{n}\right)x_{n-4} = \frac{1}{n(n+1)}, \quad n \geq 1. \tag{20}$$

Here, $w_n = \sum_{s=n}^{\infty} 1 > 0$ is nonoscillatory, and the other conditions (i), (ii), (iii) are satisfied. But (20) has the oscillatory solution $x_n = (-1)^n$.

Remarks:

(1) When $\lambda = 0$, Theorems 0.3 and 0.4 still hold.

(2) As $a_n = 1$, Lemma 0.2, Theorems 0.3, and 0.4 still hold. In Theorem 0.4,

$$w_n = \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} r_t, \quad n > 0. \tag{21}$$

It has been discussed that $\lambda \geq 0$. We have the following conclusion as $\lambda < 0$. Set

$$z_n = x_n + \lambda x_{n-\tau}. \tag{22}$$

If $\{x_n\}$ is an eventually positive solution of (3), then there exist $T > \mu$, $z_n < x_n$ for $n > T$. Thus,

$$\Delta(a_n \Delta z_n) + q_n z_{n-\sigma} \leq r_n. \tag{23}$$

Therefore, we obtain the following

Theorem 0.8. *As $\lambda < 0$, if difference inequality* (4) *is oscillation, then difference equation* (3) *is oscillation.*

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