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Research Article

Positive Solutions for System of First-Order Dynamic Equations

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We study the existence of positive solutions to the system of nonlinear first-order periodic boundary value problems on time scales $x^{\Delta}(t) + P(t)x(\sigma(t)) = F(t, x(\sigma(t)))$, $t \in [0, T]_{\mathbf{T}}$, $x(0) = x(\sigma(T))$, by using a well-known fixed point theorem in cones. Moreover, we characterize the eigenvalue intervals for $x^{\Delta}(t) + P(t)x(\sigma(t)) = \lambda H(t)G(x(\sigma(t)))$, $t \in [0, T]_{\mathbf{T}}$, $x(0) = x(\sigma(T))$.

1. Introduction

On the one hand, periodic boundary value problems (PBVPs for short) for differential equations and difference equations have received much attention in the literature. See, for example, [1–17] and references therein. On the other hand, recently, the study of dynamic equations on time scales has became a new important branch (see, e.g., [18–22]). Naturally, some authors have focused their attention on the BVPs or PBVPs for dynamic equations on time scales [23–32], in which the works in [25, 27] concerned the singular problems on time scales (concerned the study, theory, and applications of boundary value problems involving singularities of differential equations, please see [33]). In particular, for the first-order PBVP of dynamic equations on time scales

$$\begin{gathered} x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in [0, T]_{\mathbf{T}}, \\ x(0) = x(\sigma(T)), \end{gathered} \tag{1.1}$$

the works in [24, 30] obtained the existence of at least one solution. The methods involved novel inequalities and the well-known Schaefer fixed point theorem [34].

In [31], Sun and Li obtained the some existence and multiplicity criteria of positive solutions to the following first-order PBVP on time scales

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = f(x(t)), \quad t \in [0,T]_{\mathbf{T}},$$
$$x(0) = x(\sigma(T)) \tag{1.2}$$

by using Guo-Krasnoselskii fixed point theorem [35], Schauder fixed point theorem [35], and Leggett-Williams fixed point theorem [36].

Very recently, Sun and Li [32] considered the following first-order PBVP on time scales

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = \lambda f(x(t)), \quad t \in [0,T]_{\mathbf{T}},$$
$$x(0) = x(\sigma(T)), \tag{1.3}$$

where $\lambda > 0$. Some existence, multiplicity, and nonexistence criteria of positive solutions were established. The main tool used in [32] is the fixed point index theory [37].

However, up to now, there are few works for studying systems of PBVP of dynamic equations on time scales [29]. In [29], Sun and Li considered the following system of nonlinear first-order PBVP on time scales

$$u_i^{\Delta}(t) + g_i(t, u_1(\sigma(t)), u_2(\sigma(t)), \dots, u_n(\sigma(t))) = 0, \quad t = [0,T]_{\mathbf{T}},$$
$$u_i(0) = u_i(\sigma(T)), \quad i = 1,2,\dots,n. \tag{1.4}$$

By using a fixed point theorem for completely continuous operators [35], they obtained some existence criteria of one positive solution to the system.

In this paper, we study the existence of positive solutions for the following system of first-order PBVP on time scale

$$x^{\Delta}(t) + P(t)x(\sigma(t)) = F(t, x(\sigma(t))), \quad t \in [0,T]_{\mathbf{T}},$$
$$x(0) = x(\sigma(T)), \tag{1.5}$$

where $\mathbf{T}$ is a time scale, $[0,T]_{\mathbf{T}}$ means $[0,T] \cap \mathbf{T}$ (here $T > 0$ and $0, T \in \mathbf{T}$), $x = (x_1, x_2, \dots, x_n)^{\tau}$ (τ stands for the transpose), $P(t) = \operatorname{diag}[p_1(t), p_2(t), \dots, p_n(t)]$, and $F = (f_1, f_2, \dots, f_n)^{\tau}$. For $i \in \{1,2,\dots,n\}$, $p_i : [0,T]_{\mathbf{T}} \to (0,\infty)$ is right-dense continuous and $f_i : [0,T]_{\mathbf{T}} \times [0,\infty)^n \to [0,\infty)$ is continuous.

The main results in this paper are proved by a fixed point theorem (see [37]) for compact maps on conical shells which are different from those used in [24, 29–32]. To do this, we extend the ideas introduced by Lan and Webb in [38] (see also [39]) to the general time scales. This approach was used in [5] for the continuous case and in [6] for the discrete case.

As an application, we study the following eigenvalue problem:

$$x^{\Delta}(t) + P(t)x(\sigma(t)) = \lambda H(t)G(x(\sigma(t))), \quad t \in [0,T]_{\mathbf{T}},$$
$$x(0) = x(\sigma(T)), \tag{1.6}$$

where $H(t) = \operatorname{diag}[h_1(t), h_2(t), \ldots, h_n(t)], G(x) = [g^1(x), g^2(x), \ldots, g^n(x)]^{\tau}$, and $\lambda > 0$ is a positive parameter. We prove that PBVP (1.6) has at least one positive solution for each λ in an explicit eigenvalue interval. Recently, several eigenvalue characterization for different kinds of boundary value problems have appeared and we refer the readers to [32, 40–42].

It is noticed that the results obtained in this paper generalize some results in [30–32] to some degree.

In the remainder of this section, we state a fixed point theorem for compact maps on conical shell [37].

Now we recall a completely continuous operator which transforms every bounded set into a relatively compact set. If D is a subset of X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Theorem 1.1 (see [37]). *Let X be a Banach space with a cone K. Assume that Ω^1, Ω^2 are open bounded subsets of X with $\Omega^1_K \neq \phi$, $\overline{\Omega^1}_K \subset \Omega^2_K$. Let $\Phi : \overline{\Omega^2}_K \to K$ be a continuous and compact operator such that*

(i) $\|\Phi x\| \leq \|x\|$ *for* $x \in \partial_K \Omega^1$ *(or* $x \in \partial_K \Omega^2$*);*

(ii) *there exists* $e \in K \setminus \{0\}$ *such that* $x \neq \Phi x + \lambda e$ *for* $x \in \partial_K \Omega^2$ *(or* $x \in \partial_K \Omega^1$*) and* $\lambda > 0$.

Then Φ has a fixed point in $(\overline{\Omega^2}_K \setminus \overline{\Omega^1}_K)$.

Remark 1.2. In Theorem 1.1, the use of (ii) gives better results than the use of the common assumption $\|\Phi x\| \geq \|x\|$ for $x \in \partial_K \Omega^2$ (or $x \in \partial_K \Omega^1$).

2. Preliminaries

Let

$$A = \{x \mid x : [0, T]_{\mathbf{T}} \longrightarrow R \text{ is continuous}\}. \tag{2.1}$$

For $h_i \in A$, we consider the following linear PBVP:

$$\begin{gathered} x_i^{\Delta}(t) + p_i(t) x_i(\sigma(t)) = h_i(t), \quad t \in [0, T]_{\mathbf{T}}, \\ x_i(0) = x_i(\sigma(T)). \end{gathered} \tag{2.2}$$

Lemma 2.1 (see [30]). *For $h_i \in A, i = 1, 2, \ldots, n$, the PBVP* (2.2) *has a unique solution, which can be written by*

$$x_i(t) = \frac{1}{e_{p_i}(t,0)} \left[\int_0^t e_{p_i}(s,0) h_i(s) \Delta s + \frac{1}{e_{p_i}(\sigma(T),0) - 1} \int_0^{\sigma(T)} e_{p_i}(s,0) h_i(s) \Delta s \right], \quad t \in [0, \sigma(T)]_{\mathbf{T}}. \tag{2.3}$$

Remark 2.2. By Lemma 2.1, for $h_i \in A, i = 1, 2, \ldots, n$, the PBVP (2.2) has a unique solution:

$$x_i(t) = \int_0^{\sigma(T)} G_i(t,s) h_i(s) \Delta s, \quad t \in [0, \sigma(T)]_{\mathbf{T}}, \tag{2.4}$$

where

$$G_i(t,s)=\begin{cases}\dfrac{e_{p_i}(s,t)e_{p_i}(\sigma(T),0)}{e_{p_i}(\sigma(T),0)-1}, & 0\le s\le t\le\sigma(T),\\ \dfrac{e_{p_i}(s,t)}{e_{p_i}(\sigma(T),0)-1}, & 0\le t< s\le\sigma(T).\end{cases} \tag{2.5}$$

Lemma 2.3. *Let $G_i(t,s)$ be defined as Remark 2.2; then*

$$A_i \triangleq \frac{1}{e_{p_i}(\sigma(T),0)-1}\le G_i(t,s)\le \frac{e_{p_i}(\sigma(T),0)}{e_{p_i}(\sigma(T),0)-1}\triangleq B_i,\quad i=1,2,\ldots,n. \tag{2.6}$$

Let

$$B=\{x\mid x:[0,\sigma(T)]_{\mathbf{T}}\longrightarrow R \text{ is continuous}\} \tag{2.7}$$

with the norm $|x|_0=\max_{t\in[0,\sigma(T)]_{\mathbf{T}}}|x(t)|$, and $X=B^n$, for any $x=(x_1,x_2,\ldots,x_n)\in X$, its norm

$$\|x\|=\max\{|x_1|_0,|x_2|_0,\ldots,|x_n|_0\}, \tag{2.8}$$

and then X is a Banach space.

Let

$$K=\{x=(x_1,x_2,\ldots,x_n)\in X: x_i(t)\ge 0,\ t\in[0,\sigma(T)]_{\mathbf{T}},\ x_i(t)\ge\delta_i|x_i|_0,\ \forall i=1,2,\ldots,n\}, \tag{2.9}$$

where $\delta_i=A_i/B_i=e_{p_i}(0,\sigma(T))\in(0,1)$. It is not difficult to verify that K is a cone in X.

We define an operator $\Phi: K\to X$ as follows:

$$(\Phi x)=(\Phi_1 x,\Phi_2 x,\ldots,\Phi_n x)^{\tau}, \tag{2.10}$$

where

$$(\Phi_i x)(t)=\int_0^{\sigma(T)} G_i(t,s)f_i(s,x(\sigma(s)))\Delta s,\quad t\in[0,\sigma(T)]_{\mathbf{T}}. \tag{2.11}$$

By Lemma 2.1 and Remark 2.2, it is easy to see that fixed points of Φ are the solutions to the PBVP (1.5).

Lemma 2.4. $\Phi: K\to K$ *is completely continuous.*

Proof. First, we assert that $\Phi : K \to X$ is completely continuous.

The proof is divided into three steps.

Step 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} x_n = x$ in K. Then for all $i \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} |(\Phi_i x_n)(t) - (\Phi_i x)(t)| &= \left| \int_0^{\sigma(T)} G_i(t,s) \left[f_i(s, x_n(\sigma(s))) - f_i(s, x(\sigma(s))) \right] \Delta s \right| \\ &\leq B_i \int_0^{\sigma(T)} \left| f_i(s, x_n(\sigma(s))) - f_i(s, x(\sigma(s))) \right| \Delta s. \end{aligned} \tag{2.12}$$

Since $f_i(t,x)$ is continuous in x, we have $|(\Phi_i x_n)(t) - (\Phi_i x)(t)| \to 0$, which leads to $|\Phi_i x_n - \Phi_i x|_0 \to 0$ $(n \to \infty)$, so we get $\|\Phi x_n - \Phi x\| \to 0$ $(n \to \infty)$. That is, $\Phi : K \to X$ is continuous.

Step 2. To show that Φ maps bounded sets into bounded sets in X,let $B \subset K$ be a bounded set. Then, for $t \in [0, \sigma(T)]_{\mathbf{T}}$ and any $x \in B$, we have

$$\begin{aligned} |(\Phi_i x)(t)| &= \left| \int_0^{\sigma(T)} G_i(t,s) f_i(s, x(\sigma(s))) \Delta s \right| \\ &\leq B_i \int_0^{\sigma(T)} \left| f_i(s, x(\sigma(s))) \right| \Delta s. \end{aligned} \tag{2.13}$$

In virtue of the continuity of $f_i(t,x)$, we can conclude that $\Phi_i x$ is bounded uniformly for all $i \in \{1, 2, \dots, n\}$, which leads to Φx being bounded uniformly, and so $\Phi(B)$ is a bounded set.

Step 3. To show that Φ maps bounded sets into equicontinuous sets of X,let $t_1, t_2 \in [0, \sigma(T)]_{\mathbf{T}}$, $x \in B$, and then for all $i \in \{1, 2, \dots, n\}$

$$|(\Phi_i x)(t_1) - (\Phi_i x)(t_2)| \leq \int_0^{\sigma(T)} |G_i(t_1, s) - G_i(t_2, s)| \left| f_i(s, x(\sigma(s))) \right| \Delta s. \tag{2.14}$$

The right-hand side tends to uniformly zero as $|t_1 - t_2| \to 0$, which imply Φ maps bounded sets into equicontinuous sets of X.

Consequently, Steps 1–3 together with the Arzela-Ascoli Theorem show that $\Phi : K \to X$ is completely continuous.

Next, to show that Φ maps K into K, let $x \in K$, by Lemma 2.3; we have

$$\begin{aligned} &(\Phi_i x)(t) \geq 0, \quad i = 1, 2, \dots, n, \\ &(\Phi_i x)(t) \leq B_i \int_0^{\sigma(T)} f_i(s, x(\sigma(s))) \Delta s, \end{aligned} \tag{2.15}$$

and this implies that

$$|\Phi_i x|_0 = \max_{t\in[0,\sigma(T)]_{\mathbf{T}}} |(\Phi_i x)(t)| \leq B_i \int_0^{\sigma(T)} f_i(s, x(\sigma(s)))\Delta s. \tag{2.16}$$

On the other hand, from Lemma 2.3 we have

$$(\Phi_i x)(t) \geq A_i \int_0^{\sigma(T)} f_i(s, x(\sigma(s)))\Delta s. \tag{2.17}$$

Therefore,

$$\begin{aligned}(\Phi_i x)(t) &\geq A_i \int_0^{\sigma(T)} f_i(s, x(\sigma(s)))\Delta s \\ &= \delta_i B_i \int_0^{\sigma(T)} f_i(s, x(\sigma(s)))\Delta s \\ &\geq \delta_i |\Phi_i x|_0.\end{aligned} \tag{2.18}$$

That is $\Phi(K) \subset K$. □

3. Existence of Positive Solutions for the PBVP (1.5)

In this section, we establish the existence of positive solutions for the PBVP (1.5). First we extend the ideas introduced by Lan and Webb in [38, 39] to the general time scale.

For $r > 0$, we define the open sets

$$\begin{aligned}\Omega^r &= \left\{x \in X \mid \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) < \delta_i r \ \forall i = 1, 2, \ldots, n\right\}, \\ B^r &= \{x \in X \mid \|x\| < r\}.\end{aligned} \tag{3.1}$$

Lemma 3.1. *Ω^r, B^r defined above have the following properties.*

(a) *Ω^r_K and B^r_K are open relative to K.*

(b) *$B^{\delta r}_K \subset \Omega^r_K \subset B^r_K$, here $\delta = \min\{\delta_i, i = 1, 2, \ldots, n\}$.*

(c) *$x \in \partial_K \Omega^r$ if and only if $x \in K$ and $\min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_j(t) = \delta_j r$ for some $j \in \{1, 2, \ldots, n\}$ and $\min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) \leq \delta_i r$ for each $i \in \{1, 2, \ldots, n\}$.*

(d) *If $x \in \partial_K \Omega^r$, then $\delta_j r \leq x_j(t) \leq r, t \in [0, \sigma(T)]_{\mathbf{T}}$ for some $j \in \{1, 2, \ldots, n\}$ and $0 \leq x_i(t) \leq r, t \in [0, \sigma(T)]_{\mathbf{T}}$ for each $i \in \{1, 2, \ldots, n\}$. Moreover, $|x_i|_0 \leq r$.*

(e) *For each $\rho > r$, the following relations hold:*

$$\Omega^r_K = (\Omega^r \cap B^\rho)_K, \qquad \overline{\Omega^r}_K = (\overline{\Omega^r \cap B^\rho})_K. \tag{3.2}$$

Proof. (a) and (c) are obvious. So, we only prove that (b), (d), and (e) hold.

First we assert (b). Let $x \in B_K^{\delta r}$; then for each $i \in \{1,2,\ldots,n\}$, we have $|x_i|_0 < \delta r$, so $\min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) < \delta r \le \delta_i r$, and $x \in \Omega_K^r$. On the other hand, if $x \in \Omega_K^r$, then for each $i \in \{1,2,\ldots,n\}$, we have $\min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) < \delta_i r$ and $x_i(t) \ge \delta_i |x_i|_0$ for $t \in [0,\sigma(T)]_{\mathbf{T}}$. So $|x_i|_0 < r$, that is, $\Omega_K^r \subset B_K^r$. Hence (b) holds.

Next, we assert (d). Let $x \in \partial_K \Omega^r$; so we have from (c) that there exists $j \in \{1,2,\ldots,n\}$ such that

$$\delta_j \left|x_j\right|_0 \le \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_j(t) = \delta_j r. \tag{3.3}$$

Thus $|x_j|_0 \le r$ and $\delta_j r \le x_j(t) \le r$, $t \in [0,\sigma(T)]_{\mathbf{T}}$. Furthermore notice for each $i \in \{1,2,\ldots,n\}$ that $\delta_i |x_i|_0 \le \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) \le \delta_i r$, so $|x_i|_0 \le r$ and $0 \le x_i(t) \le r$ for $t \in [0,\sigma(T)]_{\mathbf{T}}$; that is, (d) holds.

Finally we assert (e). From (b), the first equality is obvious. Now we prove the second equality.

Let $x \in \overline{\Omega^r}_K$; then from (c), we have that

$$\delta_i |x_i|_0 \le \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) \le \delta_i r < \delta_i \rho, \quad i = 1,2,\ldots,n. \tag{3.4}$$

So $|x_i|_0 < \rho, i = 1,2,\ldots,n$, and this implies that $x \in (\overline{\Omega^r} \cap B^\rho) \cap K$. Since Ω^r and B^ρ are open sets, we have $\overline{\Omega^r} \cap B^\rho \subset \overline{\Omega^r \cap B^\rho}$. Thus $x \in (\overline{\Omega^r \cap B^\rho})_K$, that is, $\overline{\Omega^r}_K \subseteq (\overline{\Omega^r \cap B^\rho})_K$. The reverse inclusion is trivial. □

Remark 3.2. It is clear that the sets Ω^r are unbounded sets for each $r > 0$; so we cannot use Theorem 1.1 with Ω^r directly. However we will be able to apply Theorem 1.1 with Ω_K^r since (e) holds.

Theorem 3.3. *Suppose the following.*

(H_1) *For each $i = 1,2,\ldots,n$, there exist a constant $\alpha > 0$ and a continuous function ψ_i : $[0,T]_{\mathbf{T}} \to (0,\infty)$ such that*

$$\begin{gathered} f_i(t,x) \ge \delta_i \alpha \psi_i(t), \quad \forall t \in [0,T]_{\mathbf{T}},\ 0 \le x_l \le \alpha\ (l \in \{1,2,\ldots,n\} \setminus \{i\}),\ \delta_i \alpha \le x_i \le \alpha, \\ \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G_i(t,s)\psi_i(s)\Delta s \ge 1. \end{gathered} \tag{3.5}$$

(H_2) *For each $i = 1,2,\ldots,n$, there exist a constant $\beta > 0$ and a continuous function χ_i : $[0,T]_{\mathbf{T}} \to (0,\infty)$ such that*

$$\begin{gathered} f_i(t,x) \le \beta \chi_i(t), \quad \forall t \in [0,T]_{\mathbf{T}},\ 0 \le x_i \le \beta, \\ \max_{t\in[0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G_i(t,s)\chi_i(s)\Delta s \le 1. \end{gathered} \tag{3.6}$$

Then, the following results hold:

(a) *if* $\beta < \delta\alpha$, *then the PBVP* (1.5) *has at least one positive solution* x *satisfying*

$$\beta \le \|x\| \le \alpha; \tag{3.7}$$

(b) *if* $\beta > \alpha$, *then the PBVP* (1.5) *has at least one positive solution* x *satisfying*

$$\delta\alpha \le \|x\| \le \beta. \tag{3.8}$$

Proof. Now we assert that the conditions of Theorem 1.1 are satisfied.

First, we assert that $\|\Phi x\| \le \|x\|$ for $x \in \partial_K B^{\beta}$.

For any $x \in \partial_K B^{\beta}$, we have $|x_i|_0 \le \beta$ for each $i \in \{1,2,\dots,n\}$. Fix $i \in \{1,2,\dots,n\}$. Then from (H_2) we obtain, for each $t \in [0,\sigma(T)]_{\mathbb{T}}$,

$$\begin{aligned}
(\Phi_i x)(t) &= \int_0^{\sigma(T)} G_i(t,s) f_i(s, x(\sigma(s)))\Delta s \\
&\le \beta \int_0^{\sigma(T)} G_i(t,s)\chi_i(s)\Delta s \\
&\le \beta \max_{t\in[0,\sigma(T)]_{\mathbb{T}}} \int_0^{\sigma(T)} G_i(t,s)\chi_i(s)\Delta s \\
&\le \beta.
\end{aligned} \tag{3.9}$$

Hence, $|\Phi_i x|_0 \le \|x\|$ for each $i \in \{1,2,\dots,n\}$. This implies $\|\Phi x\| \le \|x\|$ for $x \in \partial_K B^{\beta}$.

Next, we assert that there exists $e \in K \setminus \{0\}$ such that $x \ne \Phi x + \lambda e$, for all $x \in \partial_K \Omega^{\alpha}$ and all $\lambda > 0$.

Let $e(t) \equiv 1$; so $e \in K \setminus \{0\}$. Suppose that there exist $x \in \partial_K \Omega^{\alpha}$ and $\lambda > 0$ such that $x = \Phi x + \lambda e$. Since $x \in \partial_K \Omega^{\alpha}$, then from Lemma 3.1(d) there exists $j \in \{1,2,\dots,n\}$ with $\delta_j \alpha \le x_j(t) \le \alpha$, $t \in [0,\sigma(T)]_{\mathbb{T}}$, and $0 \le x_i(t) \le \alpha$ for $t \in [0,\sigma(T)]_{\mathbb{T}}$ and $i \in \{1,2,\dots,n\} \setminus \{j\}$.

Hence, from (H_1) we have

$$\begin{aligned}
x_j(t) &= (\Phi_j x)(t) + \lambda \\
&= \int_0^{\sigma(T)} G_j(t,s) f_j(s, x(\sigma(s)))\Delta s + \lambda \\
&\ge \alpha\delta_j \int_0^{\sigma(T)} G_j(t,s)\psi_j(s)\Delta s + \lambda \\
&\ge \alpha\delta_j \min_{t\in[0,\sigma(T)]_{\mathbb{T}}} \int_0^{\sigma(T)} G_j(t,s)\psi_j(s)\Delta s + \lambda \\
&\ge \alpha\delta_j + \lambda.
\end{aligned} \tag{3.10}$$

Thus, $\min_{t\in[0,\sigma(T)]_{\mathbb{T}}} x_j(t) \ge \alpha\delta_j + \lambda > \alpha\delta_j$, contradicting the statement of Lemma 3.1(c). That is, there exists $e \in K \setminus \{0\}$ such that $x \ne \Phi x + \lambda e$, for all $x \in \partial_K \Omega^{\alpha}$ and all $\lambda > 0$.

If $\beta < \delta\alpha$, then from Lemma 3.1 we have that $\overline{B^{\beta}}_K \subset B_K^{\delta\alpha} \subset \Omega_K^{\alpha}$, and therefore it follows from Theorem 1.1 that Φ has at least one fixed point $x \in \overline{\Omega_K^{\alpha}} \setminus B_K^{\beta}$. Hence $\|x\| \geq \beta$ and $\min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) \leq \delta_i\alpha$ for each $i \in \{1,2,\dots,n\}$. On the other hand, $\delta_i|x_i|_0 \leq \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x_i(t) \leq \delta_i\alpha$ and therefore $|x_i|_0 \leq \alpha$ for each $i \in \{1,2,\dots,n\}$. This implies that $\|x\| \leq \alpha$.

If $\beta > \alpha$, then we have that $\overline{\Omega^{\alpha}}_K \subset B_K^{\beta}$, and therefore Theorem 1.1 guarantees the existence of at least one fixed point $x \in \overline{B^{\beta}}_K \setminus \Omega_K^{\alpha}$ of Φ. So, we obtain $\delta\alpha \leq \|x\| \leq \beta$. □

4. Eigenvalue Interval of PBVP (1.6)

In this section, we characterize the eigenvalue intervals of system (1.6) by employing Theorem 3.3.

First we establish one existence result for the following system:

$$x^{\Delta}(t) + P(t)x(\sigma(t)) = H(t)G(x(\sigma(t))), \quad t \in [0,T]_{\mathbf{T}},$$
$$x(0) = x(\sigma(T)), \tag{4.1}$$

where $H(t) = \operatorname{diag}[h_1(t), h_2(t), \dots, h_n(t)], G(x) = [g^1(x), g^2(x), \dots g^n(x)]^{\tau}$.

For each $i = 1,2,\dots,n$, we assume the following.

(H_3) $g^i : [0,\infty)^n \to [0,\infty)$ is continuous with $g^i(x) > 0$ for $\|x\| > 0$.

(H_4) $h_i : [0,T]_{\mathbf{T}} \to [0,\infty)$ is continuous and $\int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s > 0$.

Theorem 4.1. *Suppose that conditions (H_3) and (H_4) hold. Then the PBVP* (4.1) *has at least one positive solution x with x not identically vanishing on* $[0,\sigma(T)]_{\mathbf{T}}$ *if one of the following conditions holds:*

(H_5) $0 \leq g_0^i < C_i^{-1}$ *and* $D_i^{-1} < g_{\infty}^i \leq \infty$, $i = 1,2,\dots,n$;

(H_6) $0 \leq g_{\infty}^i < C_i^{-1}$ *and* $D_i^{-1} < g_0^i \leq \infty$, $i = 1,2,\dots,n$;

where $g_0^i = \lim_{x\to 0^+}(g^i(x)/\|x\|)$, $g_{\infty}^i = \lim_{x\to\infty}(g^i(x)/\|x\|)$, $i = 1,2,\dots,n$, *and*

$$C_i = \max_{t\in[0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s, \qquad D_i = \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s. \tag{4.2}$$

Proof. To see this, we will apply Theorem 3.3 with $f_i(t,x) = h_i(t)g^i(x)$, $i = 1,2,\dots,n$. Suppose that (H_5) holds; then there exists $\beta > 0$ such that $g^i(x) \leq C_i^{-1}\beta$ for $0 < \|x\| \leq \beta$.

Choose $\chi_i(t) = C_i^{-1} h_i(t)$ for $i = 1, 2, \ldots, n$. Fix $i \in \{1, 2, \ldots, n\}$. Then $f_i(t, x) = h_i(t) g^i(x) \leq C_i^{-1} \beta h_i(t) = \beta \chi_i(t)$ if $t \in [0, \sigma(T)]_{\mathbf{T}}$ and $0 < x_i \leq \beta$ and

$$\begin{aligned}\int_0^{\sigma(T)} G_i(t,s)\chi_i(s)\Delta s &= C_i^{-1}\int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s \\ &\leq C_i^{-1}\max_{t\in[0,\sigma(T)]_{\mathbf{T}}}\int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s \\ &= 1.\end{aligned} \tag{4.3}$$

Thus hypothesis (H_2) holds.

From the second part of (H_5), there exists $\alpha > 0$ such that $\delta_i \alpha > \beta$ and $g^i(x) \geq D_i^{-1} \delta_i \alpha$ for $x_i \geq \delta_i \alpha$, $i = 1, 2, \ldots, n$.

Choose $\psi_i(t) = D_i^{-1} h_i(t)$; then

$$f_i(t,x) = h_i(t)g^i(x) \geq D_i^{-1}\delta_i\alpha h_i(t) = \delta_i\alpha\psi_i(t), \quad \text{if } t\in[0,\sigma(T)]_{\mathbf{T}},\ x_i \geq \delta_i\alpha \tag{4.4}$$

(so in particular for $\delta_i \alpha \leq x_i \leq \alpha$) and

$$\begin{aligned}\int_0^{\sigma(T)} G_i(t,s)\psi_i(s)\Delta s &= D_i^{-1}\int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s \\ &\geq D_i^{-1}\min_{t\in[0,\sigma(T)]_{\mathbf{T}}}\int_0^{\sigma(T)} G_i(t,s)h_i(s)\Delta s \\ &= 1.\end{aligned} \tag{4.5}$$

This implies that hypothesis (H_1) holds. The result now follows from Theorem 3.3.

The case when (H_6) holds is similar. So we omit here. □

Remark 4.2. By the proof of Theorem 4.1, we emphasize that Theorem 3.3 is very easy to apply; roughly speaking, it only requires an integral representation of the considered equation and some bounds for the kernel of the equivalent integral equation. So, in this way, the corresponding existence result, that is, [31, Theorem 4.1], is improved.

Theorem 4.3. *Suppose that conditions (H_3) and (H_4) hold. Then the PBVP* (1.6) *has at least one positive solution for each*

$$\lambda \in \left(\frac{1}{D\min_{i=1,2,\ldots,n}\left\{g_\infty^i\right\}}, \frac{1}{C\max_{i=1,2,\ldots,n}\left\{g_0^i\right\}}\right) \tag{4.6}$$

if $1/D\min_{i=1,2,\ldots,n}\{g_\infty^i\} < 1/C\max_{i=1,2,\ldots,n}\{g_0^i\}$. *The same result remains valid for each*

$$\lambda \in \left(\frac{1}{D\min_{i=1,2,\ldots,n}\left\{g_0^i\right\}}, \frac{1}{C\max_{i=1,2,\ldots,n}\left\{g_\infty^i\right\}}\right) \tag{4.7}$$

if $1/D \min_{i=1,2,\dots,n}\{g_0^i\} < 1/C \max_{i=1,2,\dots,n}\{g_\infty^i\}$, *where*

$$C = \max\{C_i,\ i = 1,2,\dots,n\}, \qquad D = \min\{D_i,\ i = 1,2,\dots,n\}, \tag{4.8}$$

and one writes $1/g_\alpha^i = 0$ *if* $g_\alpha^i = \infty$ *and* $1/g_\alpha^i = \infty$ *if* $g_\alpha^i = 0$, *here* $\alpha = 0,\infty$.

Proof. We consider the case (4.6). The case (4.7) is similar.

If (4.6) holds, then

$$\begin{aligned} \lambda g_0^i &\le \lambda \max_{i=1,2,\dots,n}\left\{g_0^i\right\} < \frac{1}{C} \le \frac{1}{C_i}, \quad i = 1,2,\dots,n, \\ \lambda g_\infty^i &\ge \lambda \min_{i=1,2,\dots,n}\left\{g_\infty^i\right\} > \frac{1}{D} \ge \frac{1}{D_i}, \quad i = 1,2,\dots,n. \end{aligned} \tag{4.9}$$

Thus Theorem 4.1 applies directly. □

Remark 4.4. By Theorem 4.3, the corresponding existence results in [32] are improved.

5. Example

For convenience, the example is given here when $n = 1$.

Example 5.1. Let $\mathbf{T} = [0,1] \cup [2,3]$. We consider the following problem:

$$\begin{aligned} x^\Delta(t) + p(t)x(\sigma(t)) &= \lambda h(t) g(x(\sigma(t))), \quad t \in [0,3]_{\mathbf{T}}, \\ x(0) &= x(3), \end{aligned} \tag{5.1}$$

where $p(t) \equiv 1, T = 3, h(t) \equiv 1$, and $g(x) = x^2$; it is easy to see that h and g satisfy the conditions (H_3) and (H_4).

Then we get

$$g_0 = \lim_{x\to 0^+} \frac{g(x)}{x} = 0, \qquad g_\infty = \lim_{x\to\infty} \frac{g(x)}{x} = \infty. \tag{5.2}$$

By $\int_0^{\sigma(T)} G(t,s)h(s)\Delta s \equiv 1$, we have

$$C = \max_{t\in[0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G(t,s)h(s)\Delta s = \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G(t,s)h(s)\Delta s = D = 1. \tag{5.3}$$

Then we get $g_0 = 0 < 1 = C^-$ and $D^- = 1 < g_\infty$; that is, condition (H_5) is satisfied. So, by Theorem 4.1 the problem (5.1) has at least one positive solution when $\lambda = 1$. Furthermore, for all $\lambda \in (0,\infty)$, the problem (5.1) has at least one positive solution from Theorem 4.3.

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