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Research Article

Infinite Boundary Value Problems for Second-Order Nonlinear Impulsive Differential Equations with Supremum

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We investigate the infinite boundary value problems for second-order impulsive differential equations with supremum by establishing a new comparison result and using the lower and upper solution method, and obtain the existence results for their maximal and minimal solutions.

1. Introduction

Differential equations with supremum are used modelling different real processes, and have been receiving much attention in recent years (see [1, 2]). In the theory of automatic regulation, for example, they are used in describing the system for regulation of the voltage of generator with constant current: $T_0 u'(t) + u(t) + q \max_{s\in[t-h,t]} u(s) = f(t)$ (see [1]). If the equation is impulsive, periodic boundary value problem for first-order differential equation with supremum on finite domain was studied in [2], and on infinite domain, infinite boundary value problem for the same equation was investigated in [3]. Such equations with supremum are about first-order in the previous literature [1–3], but little is about second-order. Motivated by [2–5], we discuss in this paper the existence of maximal and minimal solutions of the system (IBVP):

$$\begin{gathered}
x''(t) = f\left(t, x(t), x'(t), \sup_{s\in[t-h,t]} x'(s)\right), \quad t \neq t_k,\ t \in J,\ k = 1, 2, \dots, \\
\Delta x|_{t=t_k} = a_k x'(t_k), \quad k = 1, 2, \dots, \\
\Delta x'\big|_{t=t_k} = \widetilde{I}_k\left(x(t_k), x'(t_k)\right), \quad k = 1, 2, \dots, \\
x(0) = x_0, \quad x'(0) = x'(\infty), \quad x'(t) = x'(0), \quad t \in [-h, 0],
\end{gathered} \tag{1.1}$$

where $J = [0, +\infty)$; $f \in C[J \times R \times R \times R, R]$; $a_k, h \in R_+$, $\widetilde{I}_k \in C[R \times R, R]$, $0 < t_1 < t_2 < \cdots < t_k < \cdots < +\infty$, and $t_k \to +\infty$ as $k \to +\infty$, $k = 1, 2, \ldots$; $x_0 \in R$ and $\sum_{k=1}^{\infty} a_k$ is convergent, $x'(\infty) = \lim_{t \to +\infty} x'(t)$; $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$, where $x(t_k^+)$ and $x(t_k^-)$ resent the right-hand and left-hand limit of $x(t)$ at $t = t_k$, respectively. $\Delta x'|_{t=t_k}$ has similar meaning for $x'(t)$. Denote $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_k = (t_k, t_{k+1}] \ldots$, $J' = J \backslash \{t_1, t_2, \ldots, t_k \ldots\}$, $J_h = [-h, +\infty)$.

Let $\mathrm{PC}[J_h, R] = \{x : J_h \to R \mid x(t) = x(0)$, for $t \in [-h, 0]$; $x(t)$ is continuous at $t \in J'$, left continuous at $t = t_k$, and each $x(t_k^+)$ exists, for $k = 1, 2, \ldots\}$, $\mathrm{PC}^1[J_h, R] = \{x : J_h \to R \mid x'(t) = x'(0)$, for $t \in [-h, 0]$; $x'(t)$ is continuous at $t \neq t_k$, $x(t_k^-)$, $x(t_k^+)$, $x'(t_k^+)$ and $x'(t_k^-)$ exist, and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots\}$, $\mathrm{BPC}[J_h, R] = \{x \in \mathrm{PC}[J_h, R] | \sup_{t \in J} \|x(t)\| < +\infty\}$, $\mathrm{TPC}[J_h, R] = \{x \in \mathrm{BPC}[J_h, R] \mid \lim_{t \to +\infty} x(t) = x(\infty)$ exists$\}$, $\mathrm{BPC}^1[J_h, R] = \{x \in \mathrm{PC}^1[J_h, R] | \sup_{t \in J} \|x'(t)\| < +\infty\}$, and $\mathrm{TPC}^1[J_h, R] = \{x \in \mathrm{BPC}^1[J_h, R] \mid \lim_{t \to +\infty} x'(t) = x'(\infty)$ exists$\}$.

We get from [4] that $x'_-(t_k) = x'(t_k^-)$. In the following, $x'(t_k)$ is understood as $x'_-(t_k)$. Evidently, $\mathrm{BPC}[J_h, R]$ equipped with the norm $\|x\|_B = \sup_{t \in J} \|x(t)\|$ is a Banach space and $\mathrm{TPC}[J_h, R] \subset \mathrm{BPC}[J_h, R]$.

We say $x \in \mathrm{TPC}^1[J_h, R] \cap C^2[J', R]$ is a solution of IBVP(1.1), if it is satisfies (1.1).

In Section 2, we prove the existence result of minimal and maximal solutions for first-order impulsive differential equations which nonlinearly involve the operator B, that is, Theorem 2.5. In special case of IBVP(2.1) where $f = f(t, u(t), \sup_{s \in [t-h, t]} u(s))$ and $\widetilde{I}_k = \widetilde{I}_k(u(t_k))$, the infinite boundary value problems for first-order impulsive differential equations were studied in [3]. In Section 3, by applying Theorem 2.5, the main result (Theorem 3.1) of this paper is obtained, that is the existence theorem of minimal and maximal solutions of IBVP(1.1).

2. Result for First-Order Impulsive Differential Equation with Nonlinear Operator Terms

Consider the existence of solutions for the following first-order impulsive differential equations:

$$
\begin{gathered}
y'(t) = f\left(t, (By)(t), y(t), \sup_{s \in [t-h, t]} y(s)\right), \quad t \in J,\ t \neq t_k, \\
\Delta y\big|_{t=t_k} = \widetilde{I}_k\big((By)(t_k), y(t_k)\big), \quad k = 1, 2, \ldots, \\
y(0) = y(\infty), \quad y(t) = y(0), \quad t \in [-h, 0],
\end{gathered}
\tag{2.1}
$$

where $f, \widetilde{I}_k$ $(k = 1, 2, \ldots)$ are the same as IBVP (1.1), and $(By)(t) = x_0 + \int_0^t y(s) ds + \sum_{0 < t_k < t} a_k y(t_k)$.

Lemma 2.1 (Comparison Result). *Let $x \in TPC[J_h, R] \cap C^1[J', R]$. Assume that there exist $a, b, c \in C[J, R_+] \cap L^1(J)$, $ta \in L^1(J)$, $b \not\equiv 0$, constants $L_k \geq 0, k = 1, 2, \ldots$, and $\sum_{k=1}^{\infty} L_k < \infty$ such that*

$$\begin{gathered}
x'(t) \geq -a(t)(Dx)(t) - b(t)x(t) - c(t) \sup_{s\in[t-h,t]} x(s), \quad t \in J,\ t \neq t_k, \\
\Delta x|_{t=t_k} \geq -L_k x(t_k), \quad k = 1, 2, \ldots, \\
x(0) \geq x(\infty), \quad x(t) = x(0), \quad t \in [-h, 0].
\end{gathered} \tag{2.2}$$

Then $x(t) \geq 0$ for $t \in J_h$ provided that

$$e^{\int_0^{\infty} b(\tau)d\tau} \left\{ \int_0^{\infty} e^{\int_0^t b(\tau)d\tau} \left[c(t) + a(t) \left(t + \sum_{k=1}^{\infty} a_k \right) \right] dt + \sum_{k=1}^{\infty} L_k \right\} \leq 1, \tag{2.3}$$

where $(Dx)(t) = \int_0^t x(s)ds + \sum_{0<t_k<t} a_k x(t_k)$.

Proof. Set $m(t) = x(t)e^{\int_0^t b(\tau)d\tau}$, then we have from (2.2) that

$$\begin{gathered}
m'(t) \geq -a(t)e^{\int_0^t b(\tau)d\tau} \left[\int_0^t m(s)e^{-\int_0^s b(\tau)d\tau} ds + \sum_{0<t_k<t} a_k m(t_k) e^{-\int_0^{t_k} b(\tau)d\tau} \right] \\
- c(t)e^{\int_0^t b(\tau)d\tau} \sup_{s\in[t-h,t]} m(s)e^{-\int_0^s b(\tau)d\tau}, \quad t \in J,\ t \neq t_k, \\
\Delta m|_{t=t_k} \geq -L_k m(t_k), \quad k = 1, 2, \ldots, \\
m(0) \geq m(\infty)e^{-\int_0^{\infty} b(\tau)d\tau}, \quad m(t) = m(0)e^{\int_0^t b(\tau)d\tau}, \quad t \in [-h, 0].
\end{gathered} \tag{2.4}$$

We claim that $m(t) \geq 0$ for $t \in J$, moreover $m(t) \geq 0$ for $t \in J_h$. Otherwise, we will consider two cases.

Case 1. $m(t) \leq 0$ for $t \in J$, and there exists $t_1^* \in J$ such that $m(t_1^*) < 0$.

Case 2. there exist $t_1^*, t_2^* \in J$ such that $m(t_1^*) < 0$, $m(t_2^*) > 0$.

In Case 1, we see from (2.4) that $m'(t) \geq 0$ for $t \in J$, $t \neq t_k$. On the other hand $m(t_k^+) = m(t_k) + \Delta m|_{t=t_k} \geq m(t_k)$, thus $m(t)$ is increasing on J, and $m(0) \leq m(t_1^*) < 0, m(0) \leq m(\infty) \leq m(0)e^{\int_0^{\infty} b(\tau)d\tau}$. Hence $e^{\int_0^{\infty} b(\tau)d\tau} \leq 1$, which is a contradiction.

In Case 2, denote $\sup_{t\in J} m(t) = \lambda$, then $\lambda > 0$, and it is clear that $\sup_{t\in J_h} m(t) = \sup_{t\in J} m(t) = \lambda$. Then we have either that (a): there exists some J_i such that $m(t_0^*) = \lambda$ for some $t_0^* \in J_i$, or $m(t_i^+) = \lambda$, or that $(b) : m(\infty) = \lambda$.

In subcase (a), we only discuss the case of $m(t_0^*) = \lambda$ for $t_0^* \in J_i$, since the discussion of the case of $m(t_i^+) = \lambda$ is similar.

If there exists some J_i such that $m(t_0^*) = \lambda$ for $t_0^* \in J_i$, then we have from (2.4) that for $t \in J$, $t \neq t_k$,

$$m'(t) \geq \lambda e^{\int_0^t b(\tau)d\tau}\left[-c(t) - a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]. \tag{2.5}$$

For any integer $l \geq i$, the calculus fundamental principle implies that

$$\begin{aligned} m(t_{l+1}) - m(t_0^*) &= \int_{t_0^*}^{t_{l+1}} m'(t)dt + \sum_{k=i+1}^{l} \Delta m|_{t=t_k} \\ &\geq -\lambda \int_{t_0^*}^{t_{l+1}} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt - \lambda \sum_{k=i+1}^{l} L_k. \end{aligned} \tag{2.6}$$

Let $l \to +\infty$, we have

$$m(\infty) - \lambda \geq -\lambda \int_{t_0^*}^{\infty} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt - \lambda \sum_{k=i+1}^{\infty} L_k. \tag{2.7}$$

This means that

$$m(\infty) \geq \lambda\left\{1 - \int_{t_0^*}^{\infty} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt - \sum_{k=i+1}^{\infty} L_k\right\}. \tag{2.8}$$

From (2.3), we have $m(\infty) \geq 0$ and $m(0) \geq m(\infty)e^{-\int_0^{\infty} b(\tau)d\tau} \geq 0$. Therefore $0 < t_1^* < +\infty$. Without loss of generality, we assume that $t_1^* \in J_j$.

If $t_0^* < t_1^*$, then $i \le j$. Hence using the same method as is used above, we have

$$\begin{aligned}
-\lambda > m(t_1^*) - m(t_0^*) &\ge -\lambda\left\{\int_{t_0^*}^{t_1^*} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=i+1}^{j} L_k\right\} \\
&\ge -\lambda\left\{\int_0^{\infty} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{\infty} L_k\right\},
\end{aligned} \tag{2.9}$$

hence,

$$\int_0^{\infty} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{\infty} L_k > 1, \tag{2.10}$$

which is a contradiction to (2.3).

If $t_0^* > t_1^*$, then $i \ge j$. Similar argument shows that

$$-m(0) > m(t_1^*) - m(0) \ge -\lambda\left\{\int_0^{t_1^*} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{j} L_k\right\}, \tag{2.11}$$

which, noticing $m(0) \ge m(\infty)e^{-\int_0^{\infty} b(\tau)d\tau}$, implies that

$$-m(\infty) > -\lambda e^{\int_0^{\infty} b(\tau)d\tau}\left\{\int_0^{t_1^*} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{j} L_k\right\}. \tag{2.12}$$

Adding (2.8) and (2.12), we show that

$$\begin{aligned}
1 &< \int_{t_0^*}^{\infty} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt \\
&\quad + \sum_{k=i+1}^{\infty} L_k + e^{\int_0^{\infty} b(\tau)d\tau}\left\{\int_0^{t_1^*} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{j} L_k\right\} \\
&< e^{\int_0^{\infty} b(\tau)d\tau}\left\{\int_0^{\infty} e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{\infty} L_k\right\},
\end{aligned} \tag{2.13}$$

which also contradicts (2.3).

In subcase (b), $m(\infty) = \lambda$, then it follows from (2.12) that

$$-\lambda > -\lambda e^{\int_0^\infty b(\tau)d\tau}\left\{\int_0^\infty e^{\int_0^t b(\tau)d\tau}\left[c(t) + a(t)\left(t + \sum_{k=1}^{\infty} a_k\right)\right]dt + \sum_{k=1}^{\infty} L_k\right\}. \tag{2.14}$$

This also leads to a contradiction with (2.3).

Therefore, the Case 2 is also impossible. Then, we conclude that $m(t) \geq 0$ on J_h, and hence $x(t) \geq 0$ on J_h. The proof is complete. □

We first consider the following linear impulsive differential equations:

$$\begin{aligned}
y'(t) &= f\left(t, B\eta(t), \eta(t), \sup_{s\in[t-h,t]} \eta(s)\right) - a(t)\big((By)(t) - (B\eta)(t)\big) \\
&\quad - b(t)\big(y(t) - \eta(t)\big) - c(t)\left(\sup_{s\in[t-h,t]} y(s) - \sup_{s\in[t-h,t]} \eta(s)\right), \quad t \in J,\ t \neq t_k, \\
\Delta y\big|_{t=t_k} &= \widetilde{I}_k\big((B\eta)(t_k), \eta(t_k)\big) - L_k\big(y(t_k) - \eta(t_k)\big), \quad k = 1, 2, \ldots, \\
y(0) &= y(\infty), \quad y(t) = y(0), \quad t \in [-h, 0].
\end{aligned} \tag{2.15}$$

Let us list some conditions for convenience.

(H_1) There exist $p, q, l \in C[J, R_+] \cap L^1(J), tp \in L^1(J)$, such that

$$\big|f(t, u, v, w)\big| \leq p(t)|u| + q(t)|v| + l(t)|w|, \quad u, v, w \in R,\ t \in J. \tag{2.16}$$

(H_2) There exist $\widetilde{L}_k \geq 0, k = 1, 2, \ldots$, such that $\sum_{k=1}^{\infty} \widetilde{L}_k$ is convergent and

$$|I_k(u, v)| \leq \widetilde{L}_k|v|, u, v \in R, \quad \forall t \in J,\ k = 1, 2, \ldots. \tag{2.17}$$

Lemma 2.2. *Let $a, b, c, ta \in C[J, R_+] \cap L^1(J)$, $b \not\equiv 0$, $L_k \geq 0$, $k = 1, 2, \ldots$, with $\sum_{k=1}^{\infty} L_k < \infty$, and assume also that conditions (H_1) and (H_2) hold. Then for any $\eta \in$* BPC$[J_h, R]$,

$y \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$ is a solution of the linear impulsive differential equations (2.15) *if and only if $y \in \mathrm{BPC}[J_h, R]$ is a solution of the following impulsive integral equation:*

$$\begin{aligned}
y(t) &= e^{-\int_0^t b(\tau)d\tau} \\
&\quad \times \Bigg\{ \left(e^{\int_0^\infty b(\tau)d\tau} - 1\right)^{-1} \Bigg[\int_0^\infty e^{\int_0^s b(\tau)d\tau} \Bigg[f\left(s, B\eta(s), \eta(s), \sup_{r\in[s-h,s]} \eta(r)\right) \\
&\qquad - a(s)\left((By)(s) - (B\eta)(s)\right) + b(s)\eta(s) \\
&\qquad - c(s)\left(\sup_{r\in[s-h,s]} y(r) - \sup_{r\in[s-h,s]} \eta(r)\right)\Bigg] ds \\
&\qquad + \sum_{k=1}^{\infty} e^{\int_0^{t_k} b(\tau)d\tau}\left[\widetilde{I}_k\left((B\eta)(t_k), \eta(t_k)\right) - L_k\left(y(t_k) - \eta(t_k)\right)\right]\Bigg] \\
&\quad + \int_0^t e^{\int_0^s b(\tau)d\tau}\Bigg[f\left(s, B\eta(s), \eta(s), \sup_{r\in[s-h,s]} \eta(r)\right) - a(s)\left((By)(r) - (B\eta)(r)\right) \\
&\qquad + b(s)\eta(s) - c(s)\left(\sup_{r\in[s-h,s]} y(r) - \sup_{r\in[s-h,s]} \eta(r)\right)\Bigg] ds \Bigg\} \\
&\quad + \sum_{0<t_k<t} e^{\int_t^{t_k} b(\tau)d\tau}\left[\widetilde{I}_k\left((B\eta)(t_k), \eta(t_k)\right) - L_k\left(y(t_k) - \eta(t_k)\right)\right], \quad \forall t \in J,
\end{aligned} \tag{2.18}$$

with the initial condition $y(t) = y(0)$, for $t \in [-h, 0]$.

Proof. By the definition of B, we have $|B\eta(s)| \le |x_0| + (s + \sum_{k=1}^{\infty} a_k)\|\eta\|_B$. Together with (H_1), (H_2), we have

$$\begin{aligned}
&\Bigg| f\left(s, B\eta(s), \eta(s), \sup_{r\in[s-h,s]} \eta(r)\right) - a(s)\left((By)(s) - (B\eta)(s)\right) \\
&\quad + b(s)\eta(s) - c(s)\left(\sup_{r\in[s-h,s]} y(r) - \sup_{r\in[s-h,s]} \eta(r)\right)\Bigg| \\
&\le \left[p(s)\left(s + \sum_{k=1}^{\infty} a_k\right) + q(s) + l(s) + a(s)\left(s + \sum_{k=1}^{\infty} a_k\right) + b(s) + c(s)\right]\|\eta\|_B \\
&\qquad + \left[c(s) + a(s)\left(s + \sum_{k=1}^{\infty} a_k\right)\right]\|y\|_B + \left(p(s) + 2a(s)\right)|x_0| =: M(s), \\
&\left|\widetilde{I}_k\left((B\eta)(t_k), \eta(t_k)\right) - L_k\left(y(t_k) - \eta(t_k)\right)\right| \le \left(\widetilde{L}_k + L_k\right)\|\eta\|_B + L_k\|y\|_B =: N_k,
\end{aligned} \tag{2.19}$$

which, noticing $M(s) \in L^1(J)$ and $\sum_{k=1}^{\infty} N_k < \infty$, implies that the right hand of (2.18) is well defined. Moreover, we show by direct computation that $y \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$ is a solution of (2.15).

We next prove the uniqueness of solution. Let y_1, y_2 be any two solutions of (2.15), and $y = y_1 - y_2$, then we have

$$y'(t) = y_1'(t) - y_2'(t) = -a(t)(Dy)(t) - b(t)y(t) - c(t) \sup_{s\in[t-h,t]} y(s), \quad t \in J,\ t \neq t_k,$$

$$\Delta y\big|_{t=t_k} = -L_k y(t_k), \quad k = 1, 2, \ldots, \tag{2.20}$$

$$y(0) = y(\infty), \quad y(t) = y(0), \quad t \in [-h, 0].$$

Hence Lemma 2.1 implies that $y \geq 0$, that is, $y_1 \geq y_2$. Similar argument shows that $y_1 \leq y_2$. Therefore $y_1 = y_2$. We complete the proof. □

Lemma 2.3. *Let* (H_1) *and* (H_2) *be satisfied. Assume further that*

$$g = \frac{1}{e^{\int_0^\infty b(\tau)d\tau} - 1}\left[\left(2e^{\int_0^\infty b(\tau)d\tau} - 1\right)\int_0^\infty \left[c(s) + a(s)\left(s + \sum_{k=1}^{\infty} a_k\right)\right]ds\right.$$
$$\left. + \sum_{k=1}^{\infty} L_k\left(e^{\int_0^\infty b(\tau)d\tau} + e^{\int_0^{t_k} b(\tau)d\tau} - 1\right)\right] < 1, \tag{2.21}$$

then the integral equation (2.18) *possesses a unique solution* $y \in \mathrm{BPC}[J_h, R]$.

Proof. For any $\eta \in \mathrm{BPC}[J_h, R]$, we define the operator T by $(Ty)(t)$ being the right hand of (2.18) and $(Ty)(0) = (Ty)(t)$, $t \in [-h, 0]$. By virtue of (H_1), (H_2), it is obvious that $T : \mathrm{BPC}[J_h, R] \to \mathrm{BPC}[J_h, R]$. Then for any $y_1, y_2 \in \mathrm{BPC}[J_h, R]$, we have

$$\left|(By_2)(s) - (By_1)(s)\right| \leq \left(s + \sum_{k=1}^{\infty} a_k\right)\|y_1 - y_2\|_B. \tag{2.22}$$

Moreover,

$$
\begin{aligned}
&\left|(Ty_1)(t)-(Ty_2)(t)\right| \\
&\quad\le \frac{1}{e^{\int_0^\infty b(\tau)d\tau}-1}\left[\left|\int_0^\infty e^{\int_0^s b(\tau)d\tau}\left[a(s)\left(By_2(s)-By_1(s)\right)+c(s)\sup_{r\in[s-h,s]}\left(y_2(r)-y_1(r)\right)\right]ds\right|\right.\\
&\qquad\qquad\left.+\sum_{k=1}^{\infty}e^{\int_0^{t_k} b(\tau)d\tau}L_k\left|y_2(t_k)-y_1(t_k)\right|\right]\\
&\qquad+\left|\int_0^t e^{-\int_s^t b(\tau)d\tau}\left[a(s)\left((By_2)(s)-(By_1)(s)\right)+c(s)\sup_{r\in[s-h,s]}\left(y_2(r)-y_1(r)\right)\right]ds\right|\\
&\qquad+\sum_{0<t_k<t}e^{-\int_{t_k}^t b(\tau)d\tau}L_k\left|y_2(t_k)-y_1(t_k)\right|\\
&\quad\le \frac{e^{\int_0^\infty b(\tau)d\tau}}{e^{\int_0^\infty b(\tau)d\tau}-1}\int_0^\infty\left[a(s)\left|(By_2)(s)-(By_1)(s)\right|+c(s)\sup_{r\in[s-h,s]}\left|y_2(r)-y_1(r)\right|\right]ds\\
&\qquad+\frac{1}{e^{\int_0^\infty b(\tau)d\tau}-1}\sum_{k=1}^{\infty}e^{\int_0^{t_k} b(\tau)d\tau}L_k\left|y_2(t_k)-y_1(t_k)\right|\\
&\qquad+\int_0^\infty\left[a(s)\left|(By_2)(s)-(By_1)(s)\right|+c(s)\sup_{r\in[s-h,s]}\left|y_2(r)-y_1(r)\right|\right]ds\\
&\qquad+\sum_{k=1}^{\infty}L_k\left|y_2(t_k)-y_1(t_k)\right|\\
&\quad\le \frac{1}{e^{\int_0^\infty b(\tau)d\tau}-1}\left[\left(2e^{\int_0^\infty b(\tau)d\tau}-1\right)\int_0^\infty\left[c(s)+a(s)\left(s+\sum_{k=1}^{\infty}a_k\right)\right]ds\right.\\
&\qquad\qquad\left.+\sum_{k=1}^{\infty}L_k\left(e^{\int_0^\infty b(\tau)d\tau}+e^{\int_0^{t_k} b(\tau)d\tau}-1\right)\right]\left\|y_1-y_2\right\|_B.
\end{aligned}
\tag{2.23}
$$

Thus, $\|Ty_1-Ty_2\|_B\le g\|y_1-y_2\|_B$. Hence, Banach's fixed point theorem implies that T has a unique fixed point, that is, a unique solution of (2.18).

For any $\eta\in \mathrm{BPC}[J_h,R]$, define an operator A by $A:(A\eta)(t)=$ the right hand of (2.18) on J, and $(A\eta)(t)=(A\eta)(0)$ for $t\in[-h,0]$. □

Lemmas 2.2 and 2.3 immediately yield the following result.

Lemma 2.4. *$y\in \mathrm{TPC}[J_h,R]\cap C^1[J',R]$ is a solution of (2.1) if and only if $y\in\mathrm{BPC}[J_h,R]$ is a fixed point of A.*

Let us list some conditions for convenience.

(H_3) There exist the upper and lower solutions of (2.1), that is, $u_0, v_0 \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$, satisfying $u_0(t) \le v_0(t)$,

$$u_0'(t) \le f\left(t, (Bu_0)(t), u_0(t), \sup_{s\in[t-h,t]} u_0(s)\right), \quad t \in J,\ t \ne t_k,$$
$$\Delta u_0|_{t=t_k} \le \widetilde{I}_k((Bu_0)(t_k), u_0(t_k)), \quad k = 1, 2, \ldots, \tag{2.24}$$
$$u_0(0) \le u_0(\infty), \quad u_0(t) = u_0(0), \quad t \in [-h, 0],$$

and $v_0(t)$ satisfies inverse inequalities above.

Define the sets $[u_0, v_0] = \{u \in \mathrm{PC}[J_h, R] : u_0(t) \le u(t) \le v_0(t),\ t \in J_h\}$, $\Omega = \{(t, x, y, z) : t \in J, (Bu_0)(t) \le x(t) \le (Bv_0)(t),\ u_0(t) \le y(t) \le v_0(t),\ \sup_{s\in[t-h,t]} u_0(s) \le z(t) \le \sup_{s\in[t-h,t]} v_0(s)\}$.

(H_4) There exist $a, b, c, ta \in C[J, R_+] \cap L^1(J)$ with $b \not\equiv 0$, such that

$$f(t, x, y, z) - f(t, \overline{x}, \overline{y}, \overline{z}) \ge -a(t)(x - \overline{x}) - b(t)(y - \overline{y}) - c(t)(z - \overline{z}), \quad \forall t \in J,$$
$$\widetilde{I}_k(x, y) - \widetilde{I}_k(\overline{x}, \overline{y}) \ge -L_k(y - \overline{y}), \quad k = 1, 2, \ldots, \tag{2.25}$$

where $(t, x, y, z), (t, \overline{x}, \overline{y}, \overline{z}) \in \Omega, \overline{x} \le x, \overline{y} \le y, \overline{z} \le z$.

Theorem 2.5. *Assume that conditions (H_1)–(H_4), (2.3), and (2.21) hold. Then (2.1) has minimal and maximal solutions $u_*, v^* \in [u_0, v_0]$; moreover, the iterative sequences $\{v_n(t)\}$ and $\{u_n(t)\}$ converge uniformly on each J_k to $v^*(t)$ and $u_*(t)$, where*

$$u_n(t) = Au_{n-1}(t), \quad v_n(t) = Av_{n-1}(t), \quad \forall t \in J,$$
$$u_n(t) = u_n(0), \quad v_n(t) = v_n(0), \quad t \in [-h, 0],\ n = 1, 2, \ldots. \tag{2.26}$$

Proof. Firstly, the proof of Lemma 2.2 implies that the operator A is well defined. □

Next, we will show that $u_0 \le Au_0, Av_0 \le v_0$ and A is nondecreasing in $[u_0, v_0]$.

Indeed, for any $\eta \in [u_0, v_0]$, we have by Lemmas 2.2 and 2.3 that $A\eta \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$ is a unique solution of (2.15), together with (2.26), we deduce that

$$
\begin{aligned}
u_n(t) = e^{-\int_0^t b(s)ds} \Bigg\{ & \left(e^{\int_0^\infty b(\tau)d\tau} - 1\right)^{-1} \\
& \times \Bigg[\int_0^\infty e^{\int_0^s b(\tau)d\tau} \Bigg[f\Bigg(s, (Bu_{n-1})(s), u_{n-1}(s), \sup_{r\in[s-h,s]} u_{n-1}(r)\Bigg) \\
& \qquad - a(s)((Bu_n)(s) - (Bu_{n-1})(s)) + b(s)u_{n-1}(s) \\
& \qquad - c(s)\Bigg(\sup_{r\in[s-h,s]} u_n(r) - \sup_{r\in[s-h,s]} u_{n-1}(r)\Bigg)\Bigg] ds \\
& + \sum_{k=1}^{\infty} e^{\int_0^{t_k} b(\tau)d\tau}\left[\widetilde{I}_k((Bu_{n-1})(t_k), u_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k))\right]\Bigg] \\
& + \int_0^t e^{\int_0^s b(\tau)d\tau} \Bigg[f\Bigg(s, (Bu_{n-1})(s), u_{n-1}(s), \sup_{r\in[s-h,s]} u_{n-1}(r)\Bigg) \\
& \qquad - a(s)((Bu_n)(s) - (Bu_{n-1})(s)) \\
& \qquad + b(s)u_{n-1}(s) - c(s)\Bigg(\sup_{r\in[s-h,s]} u_n(r) - \sup_{r\in[s-h,s]} u_{n-1}(r)\Bigg)\Bigg] ds \Bigg\} \\
& + \sum_{0<t_k<t} e^{\int_t^{t_k} b(\tau)d\tau}\left[\widetilde{I}_k((Bu_{n-1})(t_k), u_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k))\right].
\end{aligned}
\tag{2.27}
$$

Let $u_1 - u_0 = u$, then by (H_3), (2.15), and the definition of A, we have

$$
\begin{aligned}
u'(t) = u_1'(t) - u_0'(t) &\geq -a(t)(Du)(t) - b(t)u(t) \\
&\quad -c(t) \sup_{s\in[t-h,t]} u(s), \quad t \in J,\ t \neq t_k, \\
\Delta u|_{t=t_k} &\geq -L_k u(t_k), \quad k = 1, 2, \ldots, \\
u(0) \geq u(\infty), &\quad u(t) = u(0), \quad t \in [-h, 0].
\end{aligned}
\tag{2.28}
$$

This implies by Lemma 2.1 that $u(t) \geq 0$, that is, $Au_0 = u_1 \geq u_0$. Analogously, we get $Av_0 \leq v_0$. Similar argument by the facts that $A\eta$ is a solution of (2.15) and (H_4), shows that A is nondecreasing. Moreover, together with (2.26), we have

$$
u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \leq \cdots \leq v_n(t) \leq \cdots \leq v_1(t) \leq v_0(t), \quad t \in J_h. \tag{2.29}
$$

Therefore it follows from (2.29) that

$$\lim_{n\to\infty} u_n(t) = u_*(t), \quad t \in J_h, \tag{2.30}$$

and then there exists a constant $L^* > 0$, such that $\|u_n\|_B \leq L^*, n = 1, 2, \ldots$. Hence, for $s \in J$, by (H_1), (H_2), we have

$$\begin{aligned}
&\left| f\left(s, (Bu_{n-1})(s), u_{n-1}(s), \sup_{r\in[s-h,s]} u_{n-1}(r) \right) - a(s)((Bu_n)(s) - (Bu_{n-1})(s)) \right. \\
&\quad \left. + \; b(s)u_{n-1}(s) - c(s)\left(\sup_{r\in[s-h,s]} u_n(r) - \sup_{r\in[s-h,s]} u_{n-1}(r) \right) \right| \\
&\qquad \leq \left[(p(s) + 2a(s))\left(s + \sum_{k=1}^{\infty} a_k \right) + q(s) + l(s) + b(s) + 2c(s) \right] L^* + \left(2a(s) + p(s)\right)|x_0|, \\
&\left| \widetilde{I}_k((Bu_{n-1})(t_k), u_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)) \right| \leq L^*\left(\widetilde{L}_k + 2L_k \right), \quad n, k = 1, 2, \ldots.
\end{aligned} \tag{2.31}$$

Hence it follows from (2.26), (2.31) that $\{u_n(t)\}$ is equicontinuous on each J_k. So in view of (2.30), an application of *Arzela-Ascoli's* theorem and diagonal method implies that there exists a subsequence $\{u_{n_i}(t)\} \subset \{u_n(t)\}$ such that $\{u_{n_i}(t)\}$ converges uniformly on each J_k to $u_*(t)$. Then the whole sequence $\{u_n(t)\}$ converges uniformly on each J_k to $u_*(t)$. Thus $u_*(t) \in \mathrm{PC}[J_h, R]$, and the fact that $\|u_n\|_B \leq L^*$ implies $\|u_*\|_B \leq L^*$. Hence $u_* \in \mathrm{BPC}[J_h, R]$. In view of (2.30), the continuity of f and $\widetilde{I}_k$ gives that

$$\begin{aligned}
&f\left(s, (Bu_{n-1})(s), u_{n-1}(s), \sup_{r\in[s-h,s]} u_{n-1}(r) \right) - a(s)(u_n(s) - u_{n-1}(s)) \\
&\qquad + b(s)u_{n-1}(s) - c(s)\left(\sup_{r\in[s-h,s]} u_n(r) - \sup_{r\in[s-h,s]} u_{n-1}(r) \right) \\
&\qquad \longrightarrow f\left(s, (Bu_*)(s), u_*(s), \sup_{r\in[s-h,s]} u_*(r) \right) + b(s)u_*(s), \quad s \in J, \; n \longrightarrow \infty, \\
&\widetilde{I}_k((Bu_{n-1})(t_k), u_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)) \\
&\qquad \longrightarrow \widetilde{I}_k((Bu_*)(t_k), u_*(t_k)), \quad n \longrightarrow \infty, \; k = 1, 2, \ldots.
\end{aligned} \tag{2.32}$$

By the facts that $a(s), b(s), c(s), sa(s), p(s), q(s), l(s), sp(s) \in L^1(J)$, and $\sum_{k=1}^{\infty} L^*(\widetilde{L}_k + 2L_k)$ is convergent, observing (2.27) and taking limits as $n \to \infty$, the dominated convergence theorem yields that

$$\begin{aligned}
u_*(t) = &\left(e^{\int_0^\infty b(\tau)d\tau} - 1\right)^{-1} \\
&\times \left\{ \int_0^\infty e^{\int_t^s b(\tau)d\tau} \left[f\left(s, (Bu_*)(s), u_*(s), \sup_{r\in[t-h,t]} u_*(r)\right) + b(s)u_*(s) \right] ds \right. \\
&\quad \left. + \sum_{k=1}^{\infty} e^{\int_t^{t_k} b(\tau)d\tau} \widetilde{I}_k((Bu_*)(t_k), u_*(t_k)) \right\} \\
&+ \int_0^t e^{\int_t^s b(\tau)d\tau} \left[f\left(s, (Bu_*)(s), u_*(s), \sup_{r\in[t-h,t]} u_*(r)\right) + b(s)u_*(s) \right] ds \\
&+ \sum_{0<t_k<t} e^{\int_t^{t_k} b(\tau)d\tau} \widetilde{I}_k((Bu_*)(t_k), u_*(t_k)), \quad t \in J,
\end{aligned} \tag{2.33}$$

that is, $u_*(t) = Au_*(t), u_*(t)$ is a fixed point of A. It is easy to check that $u_*(t) \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$. Therefore we conclude by Lemma 2.4 that $u_*(t)$ is a solution of (2.1).

Similarly, we can show that $\{v_n(t)\}$ converges uniformly on each J_k to $v^*(t)$, and $v^* \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$ is also a solution of (2.1).

Clearly, $u_*, v^* \in [u_0, v_0]$. Using a standard method, we can show that u_*, v^* is the minimal and maximal solutions of (2.1) in $[u_0, v_0]$.

Remark 2.6. Theorem of [3] is a special case of Theorem 2.5 in this paper, where f and $\widetilde{I}_k$ did not involve the operator B. Hence Theorem 2.5 in this paper extends and improves the result of [3].

Remark 2.7. In system 2.1, if the interval is finite $[0, m]$, then the conditions of (H_1), (H_2) can be deleted. Thus Theorem 2.5 in this paper extends and improves the result of [2].

3. Main Result for Second-Order Impulsive Differential Equation

Let us list other conditions for convenience.

(H_3') There exist $y_0, z_0 \in \mathrm{TPC}^1[J_h, R] \cap C^2[J', R]$, and $y_0(t) \le z_0(t), y_0'(t) \le z_0'(t)$ such that

$$\begin{gathered}
z_0''(t) \ge f\left(t, z_0(t), z_0'(t), \sup_{r\in[t-h,t]} z_0'(r)\right), \quad t \in J,\ t \ne t_k, \\
\Delta z_0|_{t=t_k} = a_k z_0'(t_k), \quad k = 1, 2, \ldots, \\
\Delta z_0'|_{t=t_k} \ge \widetilde{I}_k(z_0(t_k), z_0'(t_k)), \quad k = 1, 2, \ldots, \\
z_0(0) = x_0, \quad z_0'(0) \ge z_0'(\infty), \\
z_0'(0) = z_0'(t), \quad t \in [-h, 0],
\end{gathered} \tag{3.1}$$

and $y_0(t)$ satisfies inverse inequalities above.

(H_4') There exist $a, b, c, ta \in C[J, R_+] \cap L^1(J)$ with $b \not\equiv 0$, such that

$$f(t,x,y,z) - f(t,\overline{x},\overline{y},\overline{z}) \geq -a(t)(x-\overline{x}) - b(t)(y-\overline{y}) - c(t)(z-\overline{z}), \quad \forall t \in J,$$
$$\widetilde{I}_k(x,y) \geq \widetilde{I}_k(\overline{x},\overline{y}), \quad k = 1,2,\ldots, \tag{3.2}$$

where $(t,x,y,z), (t,\overline{x},\overline{y},\overline{z}) \in \Omega'$, $\overline{x} \leq x, \overline{y} \leq y, \overline{z} \leq z$, $\Omega' = \{(t,x,y,z) : t \in J,\ y_0(t) \leq x(t) \leq z_0(t),\ y_0'(s) \leq y(t) \leq z_0'(s), \sup_{s\in[t-h,t]} y_0'(s) \leq z(t) \leq \sup_{s\in[t-h,t]} z_0'(s)\}$.

Theorem 3.1. *Assume that conditions* (H_1), (H_2), (H_3'), (H_4') *and* (2.3), (2.21) *hold. Then IBVP*(1.1) *has minimal and maximal solutions* $y_*, z^* \in TPC^1[J_h, R] \cap C^2[J', R]$.

Proof. Let $x'(t) = y(t)$. Then IBVP(1.1) is equivalent to the following system:

$$x'(t) = y(t), \quad y'(t) = f\left(t, x(t), y(t), \sup_{s\in[t-h,t]} y(s)\right), \quad t \in J,\ t \neq t_k,$$
$$\Delta x|_{t=t_k} = a_k y(t_k), \quad \Delta y\big|_{t=t_k} = \widetilde{I}_k(x(t_k), y(t_k)) \quad k = 1,2,\ldots,$$
$$x(0) = x_0, \quad y(0) = y(\infty), \quad y(t) = y(0), \quad t \in [-h,0]. \tag{3.3}$$

Clearly, the system

$$x'(t) = y(t),\ t \in J,$$
$$\Delta x|_{t=t_k} = a_k y(t_k), \quad k = 1,2,\ldots, \tag{3.4}$$
$$x(0) = x_0,$$

has a unique solution $x \in \mathrm{PC}[J_h, R] \cap C^1[J', R]$ and $x(t) = x_0 + \int_0^t y(s)ds + \sum_{0<t_k<t} a_k y(t_k)$. Let

$$(By)(t) = x_0 + \int_0^t y(s)ds + \sum_{0<t_k<t} a_k y(t_k), \tag{3.5}$$

we have $x(t) = (By)(t)$, and then IBVP (1.1) is transformed into first-order impulsive equations (2.1).

Let $y_0'(t) = u_0(t), z_0'(t) = v_0(t)$, we have $u_0 \leq v_0$. By the condition (H_3') and the definition of B, we get that $y_0(t) = (Bu_0)(t)$, $z_0(t) = (Bv_0)(t)$, and u_0, v_0 satisfy (H_3). By the condition (H_4'), it is easy to see that (H_4) holds. Hence, it follows from Theorem 2.5 that (2.1) has minimal and maximal solutions $u_*, v^* \in \mathrm{TPC}[J_h, R] \cap C^1[J', R]$.

Let $y_*(t) = (Bu_*)(t), z^*(t) = (Bv^*)(t)$, then $y_*, z^* \in \mathrm{TPC}^1[J_h, R] \cap C^2[J', R]$. It follows by simple calculation that

$$\begin{aligned} y_*'(t) &= u_*(t), \quad t \in J,\ t \neq t_k, \\ \Delta y_*\big|_{t=t_k} &= a_k u_*(t_k), \quad k = 1, 2, \ldots, \\ y_*(0) &= x_0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} (z^*)'(t) &= v^*(t), \quad t \in J,\ t \neq t_k, \\ \Delta z^*\big|_{t=t_k} &= a_k v^*(t_k), \quad k = 1, 2, \ldots, \\ z^*(0) &= x_0. \end{aligned} \tag{3.7}$$

The facts that u_*, v^* satisfies (2.1) and y_*, z^* satisfies (3.7) imply that $y_*, z^* \in \mathrm{TPC}^1[J_h, R] \cap C^2[J', R]$ are solutions of IBVP(1.1).

Finally, it is easy to show that y_*, z^* are the minimal and maximal solutions of IBVP(1.1), respectively. We complete the proof. □

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