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Research Article

Multiple Attractors and Nonlinear Dynamics in an Overlapping Generations Model with Environment

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This paper develops a one-sector productive overlapping generations model with environment where a CES technology is assumed. Relying on numerical and geometrical approaches, various dynamic properties of the proposed model are explored: the existence of the phenomenon of multistability or the coexistence of different attractors was demonstrated. Finally, we describe a nontypical global bifurcation which determines the appearance of an attracting cycle.

1. Introduction

After the seminal contribution of John and Pecchenino [1] the OLG specification has become a standard framework for the analysis of interplays between economic growth and environmental quality. In this field, during the last decade, two different lines of research have been developed: the former has focalized on the role of environmental policies (taxation schemes, patents, etc.), considering the *static* structure of the models, that is, when the variables are evaluated at the steady states (see, e.g., [2–5]); the latter has investigated the possibility of *nonlinear dynamics* out of the steady states (see, e.g., [6–8]).

From a general point of view, in contrast with the Solow or the Ramsey models, the OLG framework seems to provide a more appealing description of environmental dynamics: (a) the discrete time allows to introduce temporal lags between anthropic activities and their environmental impact; (b) the demographic structure (i.e., the finite living agents assumption) creates a decisional mechanism for which the evolution of environment is not internalized by agents and a cohort may generate environmental changes that outlive them and rebound over successive cohorts.

We stress the fact that while the papers on the policy implication of environment are stated in a really general framework, to avoid analytical complexity, most of the studies on the dynamic aspects of such OLG models are based on the Cobb-Douglas specification of technology and on simplifying assumptions (e.g., the gross substitution between environment and private consumption) on preferences.

The aim of this paper is to study the dynamics of a model in which a simple specification of utility is assumed but a less restrictive specification of production function is introduced. (In a related paper, by Antoci et al. [9], the roles of assumptions on preferences and of heterogenity of agents are investigated.)

A part from the more general assumptions on technology, the present paper stays very close to the analytical specification proposed in John and Pecchenino's work. We consider an overlapping generations model with the following characteristics: there exists a population of individuals whose welfare depends on the stock E_t of a free access environmental good and on the consumption c_t of a private good. We assume that E_t is negatively affected by private consumption, but it is improved by specific environmental expenses. Different from Zhang's model, where a social planner is introduced, according to Antoci et al. [9] we consider a decentralized solution: each economic agent can invest in environment but he considers as given the allocations of the other agents of the same generation. (This implies that the choices of each agent generate negative externalities on the others, and because the environment is a public good, its conservation is characterized by the standard free rider problem, intragenerational and intertemporal.)

In this context, we describe conditions for which interesting dynamic phenomena arise and we show that coexistence of attractive steady states or coexistence of nontrivial attractors may emerge for very different parameters configurations. According to works in continuous time (see e.g., [10, 11]), these results suggest that environmental externalities could be an engine for Poverty trap and/or complicated behaviour. (Poverty trap is a self-perpetuating condition where an economy, caught in a vicious condition, suffers from persistent underdevelopment.)

The paper is organized as follows. In the next section, we develop the basic OLG model where each generation is comprised of N identical individuals and they act strategically. In Sections 3 and 4 we study the static and dynamic behaviour.

2. Model Setup

We assume that environmental quality at time t could be described by a positive synthetic index E_t. Without human activity the dynamics are given by

$$E_{t+1} = (1 - b)E_t + b\overline{E} \tag{2.1}$$

with $b \in (0,1), \overline{E} > 0$. It follows that such index tends asymptotically to the long-run value $\overline{E}$. To introduce the influence of anthropic activity, we assume that the economy is populated by two-period lived agents. At each date t, N identical persons are born. In the first period of their life (when young), the agents supply inelastically their time-endowment, normalized to one, to productive sector. Individuals born in t have preferences defined over consumption and an environmental index in old age, respectively, c_{t+1} and E_{t+1}. (This assumption is adopted in several overlapping generations models (see, among the others, [6, 9]). It simplifies our analysis by abstracting from the consumption-saving choices of agents.) They allocate the income between current consumption and in investment to preserve or improve the environmental quality, m_t and saving s_t. (E_t could be interpreted as an index of the environmental amenity or as the stock of the free access environmental good at time t. It is common to all agents and could be regarded as a public good.)

We assume that the impact of consumption and environmental expenses is linear:

$$E_{t+1} = (1-b)E_t - \beta N c_t + \gamma N m_t + b\overline{E}. \tag{2.2}$$

$\beta > 0$ measures the degree of the impact of private consumption on environmental quality, and $\gamma > 0$ measures the efficiency of environmental expenses.

The preferences of an individual are representable by a utility function $U(c_{t+1}, E_{t+1})$. U is assumed twice continuously differentiable and such that $U_c(\cdot) > 0, U_E(\cdot) > 0$, $U_{c,c}(\cdot) < 0, U_{E,E}(\cdot) < 0$, and $U_{c,E}() > 0$. We assume also that the *Inada* condition $\lim_{c\to 0} U_c(c, E) = +\infty$ holds in order to avoid corner solutions with $c = 0$.

2.1. The Productive Sector

The economy is perfectly competitive so we can introduce the representative firm producing the private good. We consider a CES-technology

$$Y = Af(k_t) = A\left(\alpha k_t^{-\rho} + (1-\alpha)\right)^{1/-\rho}, \tag{2.3}$$

where k_t is physical capital at time t, $A > 0$ is a productive parameter, $\alpha \in (0,1)$ measures the degree of capital intensity of production, while $\theta = 1/(1+\rho)$ (with $\rho > -1, \rho \neq 0$) is the elasticity of substitution between labour and capital. We assume that capital depreciates at rate δ. From the usual optimality conditions, equilibrium expressions of the wage and of the interest rate follow that

$$\begin{aligned} w_t &= A(1-\alpha)\left(\alpha k_t^{-\rho} + (1-\alpha)\right)^{-(1+\rho)/\rho}, \\ r_t &= A\alpha k^{-1-\rho}\left(\alpha k_t^{-\rho} + (1-\alpha)\right)^{-(1+\rho)/\rho}. \end{aligned} \tag{2.4}$$

2.2. Agent's Problem

To describe the allocation problem of the resource, we consider a decentralized system of decisions where agents act strategically. Each individual takes as given the wage, w_t, the return on saving, r_{t+1}, environmental quality at the beginning of period t, E_t, and consumption of the old generation, c_t. Furthermore he *formulates expectations* on the investment in environmental maintenance and improvement of the other agents.

Assuming that agents are identical, the problem faced by a generic agent born in t is to maximize with respect to c_{t+1} and m_{t+1} the objective function

$$\max_{c_{t+1}, m_{t+1}} U\left(c_{t+1}, E^e_{t+1}\right) \tag{2.5}$$

with the constraints

$$\begin{aligned} w_t &= s_t + m_t, \\ c_{t+1} &= (1 + r_{t+1} - \delta) s_t, \end{aligned} \tag{2.6}$$

where E^e_{t+1} is the expectation on environmental quality at the period $t + 1$:

$$E^e_{t+1} = (1 - b)E_t - \beta N c_t + \gamma \left(m^i_t + (N - 1) m^e_t \right) + b\overline{E} \tag{2.7}$$

depending on m^e_t, that is, the expectations of agent i about the strategies of the other identical $N - 1$ agents.

3. Equilibrium Dynamics

We assume that the young agent, at time t, is able to perfectly foresee the values of others' protections (and consequently, the environmental index E_{t+1}). The equilibrium conditions for all t become

$$\begin{gathered} -U_{ct+1}(\cdot)(1 + r_{t+1} - \delta) + \gamma \cdot U_{E^e_{t+1}}(\cdot) = 0, \\ m^e_t = m^*_t, \\ E_{t+1} = (1 - b)E_t - \beta N c^*_t + \gamma N m^*_t + b\overline{E}, \\ k_{t+1} = s^{i*}_t, \\ \text{equation (2.4).} \end{gathered} \tag{3.1}$$

The first equation in (3.1) is the *F.O.C.* for the generic agent and implicitly defines the solution for his maximization problem: m^*_t, s^*_t, c^*_t; the second group imposes the *ex-post perfect foresight* condition on m_t; the third and the fourth equations are the equilibrium dynamics equations for environment and physical capital; (2.4) imposes the market equilibrium conditions. Notice that each path followed by the economy represents a Nash equilibrium; that is, no agent has an incentive to modify his choices if the choices of the others are fixed.

Analogously to the model of Zhang, to make the problem more handable and to reduce the dimensionality of the system, we introduce the following assumption.

Assumption 1. Let

$$\eta_{c,E} \equiv \left| \frac{\Delta c / c}{\Delta E / E} \right| = \frac{(\partial U / \partial c)/c}{(\partial U / \partial E)/E} = \frac{E}{c} \frac{U_E}{U_c} > 0 \tag{3.2}$$

be the elasticity of private consumption of agent i with respect to environment. We assume that this value is constant. (see Zhang for further comments. Nevertheless, notice that a large numbers of usual functional specifications of utility functions satisfy this property:

logarithmic, Cobb-Douglas, and Ces.) Notice that this parameter measures the reactivity of private consumption to environmental quality.

The following equilibrium relation holds:

$$k_t = \frac{E_t}{\eta_{c,E}\gamma}, \tag{3.3}$$

and the expressions of c_t^* and m_t^* in terms of E_t and E_{t+1} are

$$\begin{aligned} c_t^* &= (1 + r_t - \delta)k_t = A\alpha\left[\frac{E_t}{\gamma\eta_{c,E}}\right]^{-\rho}\left(\alpha\left[\frac{E_t}{\gamma\eta_{c,E}}\right]^{-\rho} + (1-\alpha)\right)^{-(1+\rho)/\rho} + (1-\delta)\left[\frac{E_t}{\gamma\eta_{c,E}}\right], \\ m_t^* &= w_t - s_t = A(1-\alpha)\left(\alpha\left[\frac{E_t}{\gamma\eta_{c,E}}\right]^{-\rho} + (1-\alpha)\right)^{-(1+\rho)/\rho} - \left[\frac{E_{t+1}}{\gamma\eta_{c,E}}\right]. \end{aligned} \tag{3.4}$$

Now, we can characterize the intertemporal equilibrium conditions through a single nonlinear difference equation in E_t:

$$\begin{aligned} E_{t+1} = \frac{\eta_{c,E}}{N + \eta_{c,E}}\Bigg[\left((1-b) - \frac{N\beta(1-\delta)}{\gamma\eta_{c,E}}\right)E_t + NA\left[\gamma(1-\alpha) - \beta\alpha\left[\frac{E_t}{\gamma\eta_{c,E}}\right]^{-\rho}\right] \\ \times\left\{\alpha\left(\frac{E_t}{\gamma\eta_{c,E}}\right)^{-\rho} + 1 - \alpha\right\}^{-(1+\rho)/\rho} + b\overline{E}\Bigg] \equiv G(E_t). \end{aligned} \tag{3.5}$$

The dynamics are described by a very complicated nonlinear equation and we are not able to characterize analytically the steady states of the model. Nevertheless, some partial results can be derived.

Lemma 3.1. *(a) If $\rho < 0$, then* $\lim_{E_t \to 0} G(E) = \eta_{c,E}/(N + \eta_{c,E})(NA\gamma(1-\alpha)^{-1/\rho} + b\overline{E}) > 0$.
(b) If $\rho > 0$, then $\lim_{E \to 0} G(E) = b\overline{E}\eta_{c,E}/(N + \eta_{c,E}) > 0$.

From the definition of G, it follows that the positivity of G is a necessary condition to have a well defined evolution of E. Since $\lim_{E\to 0} G(E) > 0$ for every values of parameters, then G is positive on an interval $[0, E^*)$, with E^* being possible infinite. Nevertheless, to have the dynamics defined for every initial condition on $[0, E^*)$ and for every t, more restrictive assumptions are required.

Proposition 3.2. *Let $E_{\sup}$ be the superior extremum for G on R^+ and let E^* be the first positive value of E such that $G(E^*) = 0$. The dynamics described by G are defined for every initial condition on $[0, E^*)$ and for every t if and only if $G(E_{\sup}) < E^*$. If E^* does not exist, then the map is well defined for each t and for every initial condition.*

We distinguish two cases with respect to the sign of ρ.

3.1. The Case $\rho < 0$

To investigate the dynamics of the model under this assumption the following lemma that gives some insights on the conditions in Proposition 3.2 is useful.

Lemma 3.3. *One has the following.*
Let $\rho < 0$, then

(a) *If $1 - b - N\beta(1-\delta)/\gamma\eta_{c,E} > NA\beta\alpha^2/\gamma\eta_{c,E}$, then $\lim_{E\to\infty} G(E) = \eta_{c,E}/(N + \eta_{c,E})(NA\gamma(1-\alpha)(1-\alpha)^{-(1+\rho)/\rho} + b\overline{E})$ and the dynamics are defined for every t.*

(b) *If $1 - b - N\beta(1-\delta)/\gamma\eta_{c,E} < NA\beta\alpha^2/\gamma\eta_{c,E}$, then there exist $\widehat{E} : G(\widehat{E}) = 0$.*

The current case is easy enough to study because a unique steady state exists for all the configurations of parameters, as proved by the following proposition.

Proposition 3.4. *If $\rho < 0$, then there exists a unique positive steady state.*

Proof. To study the existence and the numerosity of the interior steady states of (3.5), we introduce the functions h and r, where

$$h(E) \equiv \frac{\eta_{c,E}}{N+\eta_{c,E}} NA\left[\gamma(1-\alpha) - \beta\alpha\left(\frac{E}{\gamma\eta_{c,E}}\right)^{-\rho}\right]\left\{\alpha\left(\frac{E}{\gamma\eta_{c,E}}\right)^{-\rho} + 1 - \alpha\right\}^{-(1+\rho)/\rho} + b\overline{E},$$
$$r(E) \equiv E\left[1 - \frac{\eta_{c,E}}{N+\eta_{c,E}}\left(1 - b - \frac{N\beta(1-\delta)}{\gamma\eta_{c,E}}\right)\right]. \tag{3.6}$$

Steady states are given by the values of E such that $r(E) = h(E)$. The graph of r is a line with positive slope and

$$\operatorname{sign}(h'(E)) = \operatorname{sign}\left(\beta\rho + (1+\rho)\frac{\gamma(1-\alpha) - \beta\alpha(E_t/\gamma\eta_{c,E})^{-\rho}}{\left(\alpha(E/\gamma\eta_{c,E})^{-\rho} + 1 - \alpha\right)}\right). \tag{3.7}$$

It follows that h' becomes negative for high enough value of E. With easy but long calculations it is possible to verify that h is concave in the interval where h is increasing. It implies that h is bell shaped and a unique equilibrium (stable or unstable) exists. □

Essentially, for $\rho < 0$, we have the same qualitative results of the Zhang model: the unique steady state could be attractive or repelling and, in the second case, limit cycles or a chaotic attractor arises. However, differently from the Zhang model and Antoci et al. [9], the productivity parameter A affects the stability of the equilibrium.

Figure 1(a) shows the case in which the interior steady state is globally stable, with $\gamma = 1.56$, $\alpha = 0.831$, $\beta = 0.55$, $\eta_{c,E} = 5$, $b = 0.52$, $A = 3$, $\rho = -0.4$, $N = 2000$, $\delta = 0.001$, $\overline{E} = 1$. Considering long-run dynamics, it is plausible to assume that A may increase due to technological development. Starting from this configuration of parameters, if A increases, the equilibrium loses its stability and the map undergoes a period doubling route to chaos. (The result is quite robust and holds for many different constellations of parameters.) Figure 1(b) shows the case in which for $A = 6$, the map generates chaotic dynamics.

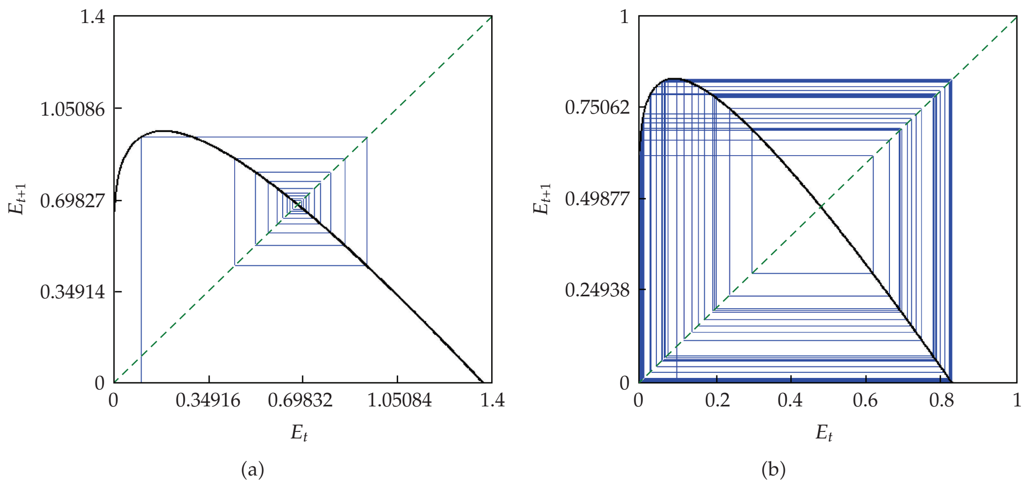

Figure 1: (a) Globally stable equilibrium. (b) Chaotic dynamics.

3.2. *The Case* $\rho > 0$

In the current case, more interesting phenomena are possible. Following the same steps as in case $\rho < 0$, we state first the following lemma.

Lemma 3.5. *Let* $\rho > 0$, *then*

(a) *if* $1 - b < N\beta(1-\delta)/\gamma\eta_{c,E}$, *then there exist* $\widehat{E} : G(\widehat{E}) = 0$;

(b) *if* $1 - b > N\beta(1-\delta)/\gamma\eta_{c,E}$, *then* $\lim_{E\to+\infty} G(E) = +\infty$ *and the dynamics are defined for every* t.

By using Lemma 3.5, it is easy to prove the next statement.

Proposition 3.6. *Let* $\rho > 0$, *then, generically, an odd number of steady states exists and the steady state with an even index is unstable.*

The main difference from the case $\rho < 0$ is that multiple equilibria may exist. If this case occurs, then we can rank the equilibria according to the associated well-being.

Proposition 3.7. *The values of the utility function* U_t *and the index* E_{t+1} *are positively correlated. This implies that if there exist two steady states* E_1^* *and* E_2^* *such that* $E_2^* > E_1^*$, *then* E_2^* *Pareto-dominates* E_1^*; *that is,* E_1^* *is a poverty trap.*

Even if analytically we are not able to prove it, numerically, we have verified that a maximum number of three steady states exist. It is interesting to note that a necessary condition to have multiple steady states is that ρ is enough far from 0. In fact if we consider $\rho \to 0^+$, then $f(k)$ tends to the Cobb-Douglas function and this case the map admits a unique steady state (see e.g., [12]). In the next section we develop the analysis when multiple steady states occur.

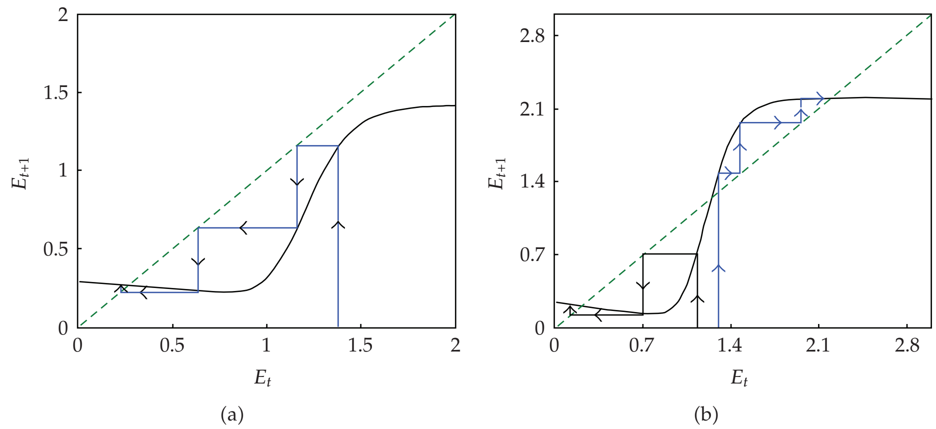

Figure 2: (a) Unique stable equilibrium. (b) Fold bifurcation and the birth of a poverty trap.

4. The Role of the Increasement of Productivity

In this section, we will concentrate on the case $\rho > 0$ and we investigate the role of productive parameter A.

Notice that a riseup of A affects in a nonunivocal way the points of the map. The points of the map on the left of $\widehat{E} \equiv (\gamma(1-\alpha)/\beta\alpha)^{-1/\rho}$ are translated upwards, the reverse occurs for the points on the right of $\widehat{E}$, and this stretch makes the two "humps" of the map more pronounced.

It follows that an increasement of A could be the engine of a poverty trap. To understand the phenomenon, we provide a numerical example. Let, $\gamma = 1.12$, $\alpha = .621$, $\beta = .1$, $\overline{E} = 33$, $\eta_{c,E} = 1.03$, $b = .7$, $A = 0.8$, $\rho = 10.5$, $N = 100$, and $\delta = 0.8$. As shown in Figure 2(a), a unique attractive steady state E_1^* exists. If the value of productivity A grows until $A = 0.88$, two new steady states, E_2^*, E_3^*, are born via a *fold bifurcation* , where $E_2^* < E_3^*$ is repelling and separates the basin of attractions of E_1^* and E_3^*. In Figure 2(b), we have considered $A = 0.92$, for which we have $E_1^*= 0.22$, $E_2^* = 1.25$, and $E_3^* = 2.23$. Notice that the growth of productivity does not imply for an economy the convergence to the more developed state E_3^*: because of the presence of a threshold, economies with a low E at the moment of the bifurcation continue to converge to E_1^*.

This result contributes to the growing literature, initiated by Azariadis and Drazen [13], on poverty traps. It provides a different explanation of their emergence that resides in the existence of a threshold in the relationship among productivity and environmental quality.

Differently from what we have seen previously, we present an experiment that shows how changes in technology may be an engine of complex dynamics and bistable regime. To give some insights on the stability of equilibria, we rely on numerical analysis. In particular, apart from the graphical analysis, we will use the so-called bifurcation diagrams, showing the possible long-term values (steady states, periodic or chaotic orbits) of the system as a function of a parameter. In what follows, we fix $\gamma = 0.04$, $\alpha = 0.07$, $\beta = 0.1$, $\overline{E} = 22$, $\eta_{c,E} = 11$, $b = 0.58$, $\rho = 12$, $N = 100$, and $\delta = 0.4$ and we use A as the bifurcation parameter. Starting from $A = 0.179$ a unique unstable steady state E^1 exists, and this is enclosed by an attractive 2-period limit cycle. By increasing the value of A the trajectories near to equilibrium follow

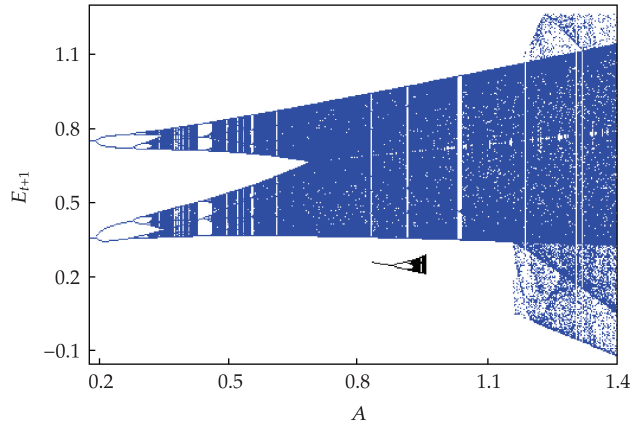

Figure 3: Overlapping bifurcation diagrams varying A.

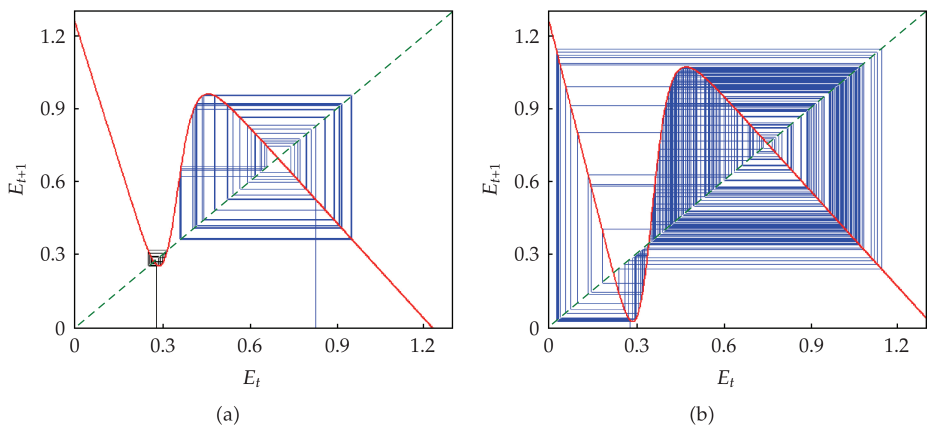

Figure 4: (a) Two chaotic trajectories starting from 0.83 to 0.28 ($A = 0.86$). (b) Enlargement of the attractor ($A = 1.29$).

to the classical period doubling route to chaos. For $A = 0.86$, via a fold bifurcation two other steady states E^3, E^2 arise with $E^3 < E^2 < E^1$ (the bifurcation value is $A = 0.82$); E^3 is attracting while E^2 is repelling and separates the basins of attraction of E^3 and the basin of the chaotic attractor. If we let A increase, then the point E_3 loses its stability through cascades of flip bifurcations (see Figure 3) and there exists a space of parameters for which the coexistence of two attractors occurs (see Figure 4(a)). If we get A higher, the attractor dies ($A = 0.89$) and its points are attracted by the remaining attractor that lives at the right of the repelling steady state E_2^*. A further increase of A ($A > 1.18$) causes an enlargement of the attractor that invades the space before occupied by the other attractor (see Figure 4(b)).

4.1. *The Emergence of a Nonlocal Bifurcation*

In the present paragraph we describe an uncommon bifurcation arising for a particular set of parameters. To illustrate such scenario we fix $\gamma = 0.04$, $\alpha = 0.07$, $\beta = 0.1$, $\overline{E} = 22$, $\eta_{c,E} = 11$,

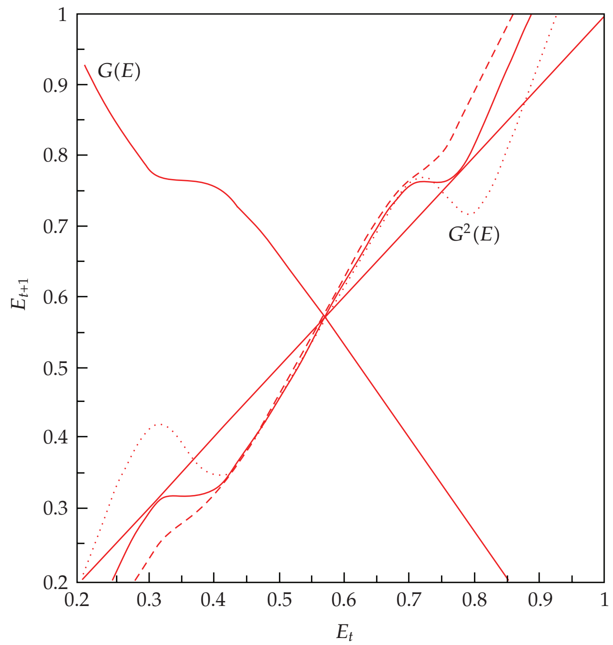

Figure 5: Evolution of the second iterate (G is drawn at the bifurcation value $A = 0.132$).

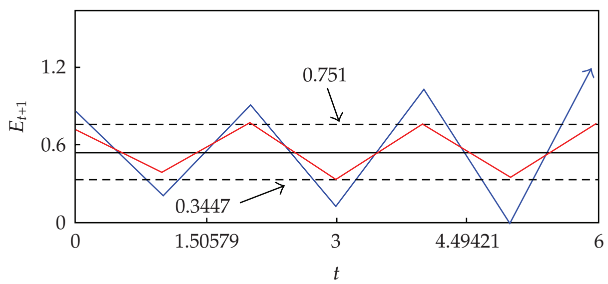

Figure 6: Time evolution of E with three different initial conditions: $E(0) = 0.532$ (the unstable steady state), $E(0) = 0.63$ (trajectory converging to the 2-period stable limit cycle), and $E(0) = 0.862$ (trajectory diverging from the unstable 2 period limit cycle).

$b = 0.58$, $\rho = 12$, $N = 100$, and $\delta = 0.4$ and we let A vary. In Figure 5 we show the evolution of the second iterate of the map at the increase of A.

We can note that second iterate is characterized by 2 humps or more exactly by a maximum and a minimum point. If we let A increase, such humps become more pronounced: starting from $A = 0.10$, G^2 has no intersection point with the 45°-*line* (the dashed line). For $A = 0.132$ the second iterate creates 2 tangent points (the solid line), and for $A = .16$ two fixed points of the second iterate are born (the dotted line). In this process, driven by the increase of A we assist to a *fold* bifurcation of G^2 inducing a 2-period stable cycle and a 2-period unstable cycle. We underline that this is a global bifurcation and not a local one: it arises "far" from the fixed point that holds its instability for the whole process (in Figure 5 we have drawn G when the bifurcation occurs). In particular, as shown by the simulations in Figure 6, the interior 2-period cycle is stable, while the external is unstable.

5. Conclusions

We have investigated the local and global dynamics of an overlapping generations model with environment when a CES production function is assumed. Despite its simplicity, this model displays very rich dynamics. First of all, differently from other works inspired by the paper of John and Pecchenino [1], multistability could occur even if a strictly positive environmental maintenance is assumed. Moreover, if we let productivity increase, we find the emergence of complex dynamics: apart from the classical period-doubling cascades to chaos, nonlocal bifurcations may occur and drastically change the global behaviour of the model.

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