Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 862538, 18 pages
doi:10.1155/2010/862538

Research Article

On the Stability of Numerical Methods for Nonlinear Volterra Integral Equations

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Received 3 March 2010; Accepted 12 May 2010

Academic Editor: Manuel De la Sen

Here we investigate the behavior of the analytical and numerical solution of a nonlinear second kind Volterra integral equation where the linear part of the kernel has a constant sign and we provide conditions for the boundedness or decay of solutions and approximate solutions obtained by Volterra Runge-Kutta and Direct Quadrature methods.

1. Introduction

In this paper we consider Volterra integral equations (VIEs) of Hammerstein type, that is,

$$y(t) = g(t) + \int_0^t k(t,s) f(y(s))\, ds, \tag{1.1}$$

where g, k, and f are given real-valued functions. This type of equation arises in problems related to evolutionary processes in ecology, in nuclear reactors and in control theory (see, e.g., [1], [2, Chapter 1], [3, Chapter 2] and references therein). There has been interest in the literature over many years in studying the asymptotic behavior of the solution of (1.1) and, in particular, many results appeared on the existence of solutions that decay to zero as $t \to \infty$ for the convolution version of (1.1) ($k(t,s) = k(t-s)$, see, e.g., [4, 5]). It is well known that in the numerical solution of this type of equations, the discrete approximation should emulate the features of the analytical solution, hence an analysis is due which correlates the behavior of the continuous and the discrete problem. There is a large literature in this sense, however the most of the results refer to the linear version of (1.1) and many others to the convolution case [6–9] which arises commonly in models of phenomena where no ageing or seasoning and therefore no explicit dependence upon time are present. In this paper the kernel k in (1.1) is not necessarily of convolution type and we only assume that the function f is regular enough to guarantee the existence and the uniqueness of the solution and that g and k are continuous functions (see, e.g., [10], [11, Chapter 4]). Our aim is to give sufficient conditions under which the analytical solution of (1.1) and the numerical one provided by the "most popular" linear (Volterra Direct Quadrature (VDQ)) and nonlinear (Volterra Runge-Kutta (VRK)) methods have the same behavior. This of course in order to assure that the numerical solution well emulates the analytical one. The starting idea for this kind of analysis comes out from an investigation that the authors carried out in 2008 on a nonlinear Volterra discrete equation [12] which turns out to be a discrete analogue of (1.1). The paper is organized as follows. Section 2 is devoted to establishing bounds on the analytical solution of (1.1), on $[0,T]$ or on $[0,\infty)$, under certain conditions on $k(\cdot,\cdot)$, $f(\cdot)$, and $g(\cdot)$. Section 3 is devoted to obtaining similar bounds for approximate solutions obtained by the two types of methods mentioned above. In Section 3 some numerical experiments are reported.

2. The Behavior of the Analytical Solution

In this section we prove some results on the behavior of the solution $y(t)$ of (1.1). It is assumed that (1.1) is satisfied for $t \in [0,T]$, where T is a positive constant of for $t \in [0,\infty)$. The following theorem gives the conditions for $y(t)$ to be bounded on $[0,T]$, which is the extended nonnegative real line in the latter case.

Theorem 2.1. *Suppose that the following set of hypotheses hold:*

(i) $\exists \tilde{f} \geq 0 : f(x) \geq -\tilde{f}$, $\forall x \in \mathbb{R}$,

(ii) $\limsup_{x\to-\infty} f(x) = \overline{f} < +\infty$,

(iii) $k(t,s) \leq 0$, $\forall s \leq t$, $t \in [0,T]$,

(iv) $\exists \tilde{K} > 0 : \sup_{t\in[0,T]} \int_0^t |k(t,s)|ds \leq \tilde{K}$, $\forall t \in [0,T]$,

(v) $\exists G_1, G_2 : G_1 \leq g(t) \leq G_2$, $\forall t \in [0,T]$,

then $y(t)$ is bounded for $t \in [0,T]$ and it is

$$G_1 - \tilde{K}F^* \leq y(t) \leq G_2 + \tilde{f}\tilde{K}, \tag{2.1}$$

where F^* is a given positive constant.

Proof. From (ii) we have that

$$\exists \delta : f(x) \leq \overline{f}, \quad \forall x \in (-\infty, \delta). \tag{2.2}$$

From (i) and (iii) there results $k(t,s)f(y(s)) \leq -\tilde{f}k(t,s)$, hence

$$\int_0^t k(t,s)f(y(s))ds \leq \tilde{f}\int_0^t |k(t,s)|ds \leq \tilde{f}\tilde{K}, \quad \forall t \in [0,T]. \tag{2.3}$$

From here and (v) there results

$$y(t) \le G_2 + \tilde{f}\tilde{K}. \tag{2.4}$$

Hence,

$$f(y) \le \max\left\{\overline{f}, \max_{x\in\left[\delta, G_2+\tilde{f}\tilde{K}\right]} f(x)\right\}, \tag{2.5}$$

where δ is defined in (2.2). This assures that the positive constant F^* given by

$$F^* = \max\left\{0, \max\left\{\overline{f}, \max_{x\in[\delta,\, G_2+\tilde{f}\tilde{K}]} f(x)\right\}\right\} \tag{2.6}$$

is such that $\forall t \in [0, T]$, $f(y) \le F^*$. Therefore,

$$\int_0^t k(t,s) f(y(s))ds \ge F^* \int_0^t k(t,s)ds \ge -\tilde{K}F^*. \tag{2.7}$$

Taking into account (v) we obtain

$$y(t) \ge G_1 - \tilde{K}F^*, \tag{2.8}$$

which, together with (2.4) prove the result. □

Remark 2.2. The essence of the hypotheses of Theorem 2.1 are the following: the integrand $k(t,s)f(y(s))$ has a negative linear part $(k(t,s))$, hence, when $f(y)$ is allowed to become positive and large, it is multiplied by a negative quantity so that the positive growth of y is avoided. On the other hand, when $f(y)$ is negative, it is also bounded from below by hypothesis (i).

In the examples below we see some applications of the results proved in Theorem 2.1. Of course, the bound (2.1) is not necessarily sharp (it depends on the shape of the function g) and this is clear in Example 2.3, nevertheless the purpose of the previous result is to give some information about the qualitative behavior of the solution also on an unbounded interval.

Example 2.3. Consider (1.1) with $t \in [0,1]$, $k(t,s) = -e^{-s}/(1+t^2)$, $f(y) = (e^{3y}-1)/3$ and $g(t)$ such that $y(t) = t - 0.5$. By Theorem 2.1 we obtain $-1.12 \le y(t) \le 1.85$, $\forall t \in [0,1]$.

Finer bounds are obtained in the following examples.

Example 2.4. Consider (1.1) with $T \in [0,1]$, $k(t,s) = -e^{-s}/(1+t^2)$, $f(y) = -\sin^2 y$, and $g(t)$ such that $y(t) = \pi/2$. By (2.1) we obtain $1.23 \le y(t) \le 1.98$, $\forall t \in [0,1]$.

Example 2.5. Consider (1.1) with $T \in [0,1]$, $k(t,s) = -e^{-s}/(1+t^2)$, $f(y) = e^y$, and $g(t)$ such that $y(t) = t^2 - 0.5$. By (2.1) we obtain $-1.22 \leq y(t) \leq 0.75$, $\forall t \in [0,1]$.

In order to prove that the solution $y(t)$ of (1.1) tends to zero as $t \to \infty$, we need the following lemma.

Lemma 2.6. *Assume that*

(i) $\exists \tilde{f} \geq 0 : f(x) \geq -\tilde{f}$, $\forall x \in \mathbb{R}$,

(ii) $xf(x) \geq 0$, $\forall x \in \mathbb{R}$,

(iii) $k(t,s) \leq 0$, $\forall s \leq t$, $t \geq 0$,

(iv) $\lim_{t\to\infty} k(t,s) = 0$,

(v) $\lim_{t\to\infty} g(t) = 0$,

(vi) $y(t)$ *ultimately nonnegative for* $t \geq 0$,

then $\lim_{t\to\infty} y(t) = 0$.

Proof. From hypothesis (vi) there exists $\bar{t}$ such that $y(t) \geq 0$, for $t > \bar{t}$. Assume $t > \bar{t}$ and write

$$y(t) = g(t) + \int_0^{\bar{t}} k(t,s) f(y(s)) ds + \int_{\bar{t}}^{t} k(t,s) f(y(s)) ds, \tag{2.9}$$

in view of (ii) and (iii), we have

$$0 \leq y(t) \leq g(t) + \int_0^{\bar{t}} k(t,s) f(y(s)) ds, \tag{2.10}$$

which, taking into account (i), gives

$$0 \leq y(t) \leq g(t) - \tilde{f} \int_0^{\bar{t}} k(t,s) ds. \tag{2.11}$$

Finally, hypotheses (iv) and (v) and the Squeeze theorem prove the result stated in the theorem. □

Now the following theorem, which is the continuous analogue of Theorem 3.4 in [12] can be proved.

Theorem 2.7. *Assume that*

(i) $\exists \tilde{f} \geq 0 : f(x) \geq -\tilde{f}$, $\forall x \in \mathbb{R}$,

(ii) f *is nondecreasing*, $xf(x) \geq 0$, $\forall x \in \mathbb{R}$,

(iii) $k(t,s) \leq 0$ *for* $s \leq t$, $t \geq 0$,

(iv) $\exists \tilde{K} > 0 : \sup_{t\geq 0} \int_0^t |k(t,s)| ds \leq \tilde{K}$, $\forall t \geq 0$,

(v) $\lim_{t\to\infty} g(t) = 0$,

(vi) $\lim_{t\to\infty} k(t,s) = 0$,

(vii) $-\widetilde{K} f(-\widetilde{K} f(-x)) > -x$, $\forall x > 0$,

then $\lim_{t\to\infty} y(t) = 0$.

Proof. From Lemma 2.6 it is obvious that if $y(t)$ is ultimately nonnegative the desired result is true. So, let us proceed by contradiction and assume that

$$\liminf_{t\to+\infty} y(t) = \overline{y} < 0, \tag{2.12}$$

which leads to

$$\forall \epsilon > 0, \exists \delta : t > \delta \Longrightarrow y(t) > \overline{y} - \epsilon. \tag{2.13}$$

Let us denote by $\Lambda(x)$ the continuous function

$$\Lambda(x) = -\widetilde{K} f\left(x - \widetilde{K} f(\overline{y} - x)\right) - 2x, \tag{2.14}$$

for which, by virtue of (vii), $\Lambda(0) = -\widetilde{K} f(-\widetilde{K} f(\overline{y})) > \overline{y}$. Since Λ is a continuous function, there exists $\epsilon_0 > 0$ such that $\Lambda(\epsilon_0) > \overline{y}$ or equivalently

$$\exists \epsilon_0 : -\epsilon_0 - \widetilde{K} f\left(\epsilon_0 - \widetilde{K} f(\overline{y} - \epsilon_0)\right) > \overline{y} + \epsilon_0. \tag{2.15}$$

Moreover, by the definition (2.13) of the inferior limit, for the same ϵ_0 there exists δ_0 such that $t > \delta_0 \Rightarrow y(t) > \overline{y} - \epsilon_0$. Since f is nondecreasing (ii)), then

$$f(y(t)) \ge f(\overline{y} - \epsilon_0), \quad t > \delta_0, \tag{2.16}$$

with $f(\overline{y} - \epsilon_0) < 0$ in view of (ii) (recall that $\overline{y} < 0$).

Now, assume that $t > \delta_0$ and rewrite (1.1) in the following form:

$$y(t) = g(t) + \int_0^{\delta_0} k(t,s) f(y(s))ds + \int_{\delta_0}^{t} k(t,s) f(y(s))ds. \tag{2.17}$$

Then, from (2.16) and (iii) we obtain

$$y(t) \le g(t) + \int_0^{\delta_0} k(t,s) f(y(s))ds - \widetilde{K} f(\overline{y} - \epsilon_0), \quad t > \delta_0. \tag{2.18}$$

Since the hypotheses (v) and (vi) assure that, for any fixed T

$$\lim_{t\to\infty}\left(g(t) + \int_0^{T} k(t,s) f(y(s))ds\right) = 0, \tag{2.19}$$

we have that $\exists\delta_1 : \forall t > \delta_1, -\epsilon_0 < g(t) + \int_0^{\delta_0} k(t,s)f(y(s))ds < \epsilon_0$ and hence, from (2.18) $\forall t > \delta_2 = \max\{\delta_1, \delta_0\}$ there results $y(t) < \epsilon_0 - \widetilde{K}f(\overline{y} - \epsilon_0)$. Since f is nondecreasing, we have

$$f(y(t)) < f\left(\epsilon_0 - \widetilde{K}f(\overline{y} - \epsilon_0)\right), \quad \forall t > \delta_2, \tag{2.20}$$

and hence

$$\int_{\delta_2}^{t} k(t,s)f(y(s))ds \geq -f\left(\epsilon_0 - \widetilde{K}f(\overline{y} - \epsilon_0)\right)\widetilde{K}. \tag{2.21}$$

From (2.21) it is

$$y(t) \geq g(t) + \int_0^{\delta_2} k(t,s)f(y(s))ds - f\left(\epsilon_0 - \widetilde{K}f(\overline{y} - \epsilon_0)\right)\widetilde{K}, \quad t > \delta_2, \tag{2.22}$$

and, once again, from (2.19) we get that there exists ϵ_0 and δ_3 such that for all $t > \delta_3$

$$y(t) \geq -\epsilon_0 - f\left(\epsilon_0 - \widetilde{K}f(\overline{y} - \epsilon_0)\right)\widetilde{K}, \tag{2.23}$$

and therefore, for (2.15), $y(t) > \overline{y} + \epsilon_0$. This contradicts the assumption (2.12), therefore $y(t)$ is ultimately nonnegative and the desired result comes out. □

Of course hypothesis (vii) may appear quite restrictive, however, the following remark is useful to find examples of functions satisfying it.

Remark 2.8. It can be easily seen that Theorem 2.7 remains valid if instead of (i), (ii), and (vii) we assume that there exists a function ϕ, satisfying (i), (ii), and (vii) such that

$$\begin{aligned} &0 \leq f(y) \leq \phi(y), \quad y \geq 0, \\ &0 \geq f(y) \geq \phi(y), \quad y < 0. \end{aligned} \tag{2.24}$$

This, of course, enlarge the class of problems to which Theorem 2.7 can be applied. Namely, the function $\phi(y) = (e^{\alpha y} - 1)/\alpha$ satisfies (i), (ii), and (vii) for any $\alpha > 0$ and $\widetilde{K} < 1$. For different α we can construct a variety of functions f such that (2.24) holds (see [12, Remark 3.7] for some plots of admissible functions f). Moreover, we note that functions of the type ϕ, which satisfies (2.24), directly arise in many applications concerning the spread of an epidemic in a population (see, e.g., [2, Example 2.3]).

3. The Behavior of the Numerical Solution

In this section we want to prove that under some additional mild hypotheses on the kernel k the most popular numerical methods applied to (1.1) inherit the asymptotic behavior of the analytical solution. We consider here the partition

$$\Pi_N = \{t_n : 0 = t_0 < t_1 < \cdots t_N = T\} \tag{3.1}$$

of the time interval $[0, T]$ and we assume that the stepsize $h = t_{n+1} - t_n$, $n = 0, \ldots, N-1$ is constant. In agreement with the continuous case we prove the following theorems in $[0, T]$ and the results obtained can be easily generalized to the case $t \in [0, +\infty)$. So, N can be the natural infinity if T is the real point at plus infinity.

3.1. Volterra Runge-Kutta Methods

Let us consider the classical m-stage Volterra Runge-Kutta methods of extended type [13, Chapter 4]:

$$y_{n+1} = F_n(t_{n+1}) + h\sum_{j=1}^{m} b_j k\big(t_{n+1} + (\gamma_j - 1)h, t_n + c_j h\big) f\big(Y_{nj}\big), \quad n = 0, \ldots, N-1, \tag{3.2}$$

$$Y_{ni} = F_n(t_n + \theta_i h) + h\sum_{j=1}^{m} a_{ij} k\big(t_n + d_{ij} h, t_n + c_j h\big) f\big(Y_{nj}\big), \quad i = 1, \ldots, m, \tag{3.3}$$

with

$$F_n(t) = g(t) + h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j k\big(t + (\gamma_j - 1)h, t_l + c_j h\big) f\big(Y_{lj}\big), \quad n = 0, 1, \ldots, N-1, \tag{3.4}$$

where $y_n \approx y(t_n)$ and $b = (b_1, \ldots, b_m)^T$, $\gamma = (\gamma_1, \ldots, \gamma_m)^T$, $\theta = (\theta_1, \ldots, \theta_m)^T$, $A = (a_{ij})_{i,j=1,\ldots,m}$, and $D = (d_{ij})_{i,j=1,\ldots,m}$ are the given coefficients of the method [13, page 170].

The following lemmas are useful to prove the main result of this section.

Lemma 3.1. *Assume that*

(i) $\exists \widetilde{K} > 0 : \sup_{t \geq 0} \int_0^t |k(t, s)| ds \leq \widetilde{K}$, $t \in [0, T]$,

(ii) $|k(t, s)|$ *is ultimately nonincreasing with respect to* s, $\forall t \in [0, T]$.

Then $\forall \overline{s} \leq t$, $\exists K^*(\overline{s})$ *such that* $|k(t, \overline{s})| \leq K^*(\overline{s})$, $\forall t \in [0, T]$.

Proof. The proof is obvious and therefore is omitted. □

Lemma 3.2. *Assume that the following hypotheses on the problem* (1.1) *and on the method coefficients in* (3.1), (3.4) *hold:*

(i) $\exists \widetilde{f} > 0 : f(x) \geq -\widetilde{f}$, $\forall x \in \mathbb{R}$,

(ii) $\limsup_{x \to -\infty} f(x) = \overline{f} < +\infty$,

(iii) $k(t,s) \leq 0$, $\forall s \leq t$, $t \in [0,T]$,

(iv) $\exists \widetilde{K} > 0 : \sup_{t\geq 0} \int_0^t |k(t,s)|ds \leq \widetilde{K}$, $\forall t \in [0,T]$,

(v) $\exists G_1, G_2 : G_1 \leq g(t) \leq G_2$, $t \in [0,T]$,

(vi) $|k(t,s)|$ *ultimately nonincreasing with respect to* s, $\forall t \in [0,T]$,

(vii) $b_j \geq 0$, $j = 1,\ldots,m$.

Then, $\forall t \in [0,T]$ *and* $n = 0,1,\ldots,N$ *the following upper bound for the lag term* $F_n(t)$ *holds:*

$$F_n(t) \leq G_2 + \widetilde{f}\widetilde{K} + hC_3, \tag{3.5}$$

where C_3 *is a positive constant.*

Proof. For (i) it is $f(Y_{lj}) > -\widetilde{f}$, then for (iii) and (vii)

$$F_n(t) \leq g(t) + h\widetilde{f}\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j \left|k\left(t + \left(\gamma_j - 1\right)h, t_l + c_j h\right)\right|. \tag{3.6}$$

According to (vi), there exists $\overline{n}$ such that $\forall s \geq t_{\overline{n}}$, $|k(t + (\gamma_j - 1)h, s)|$ is nonincreasing with respect to s. Hence, for $n = \overline{n} + 1,\ldots,N$ let us rewrite (3.4)

$$\begin{aligned} F_n(t) \leq{}& g(t) + h\widetilde{f}\sum_{l=0}^{\overline{n}-1}\sum_{j=1}^{m} b_j \left|k\left(t + \left(\gamma_j - 1\right)h, t_l + c_j h\right)\right| \\ &+ h\widetilde{f}\sum_{j=1}^{m} b_j \sum_{l=\overline{n}}^{n-1}\left|k\left(t + \left(\gamma_j - 1\right)h, t_l + c_j h\right)\right|. \end{aligned} \tag{3.7}$$

and recall that for any nonincreasing function $\eta(t) \geq 0$ in $[t_1, t_{n-1}]$, we have $h\sum_{j=1}^{n} \eta(t_j) \leq \int_{t_0}^{t_n} \eta(t)dt$. Then, for $n = \overline{n} + 1,\ldots,N$

$$\begin{aligned} F_n(t) \leq{}& g(t) + h\widetilde{f}\sum_{l=0}^{\overline{n}-1}\sum_{j=1}^{m} b_j \left|k\left(t + \left(\gamma_j - 1\right)h, t_l + c_j h\right)\right| \\ &+ h\widetilde{f}\sum_{j=1}^{m} b_j \left|k\left(t + \left(\gamma_j - 1\right)h, t_{\overline{n}} + c_j h\right)\right| + \widetilde{f}\sum_{j=1}^{m} b_j \int_{t_{\overline{n}}}^{t_n}\left|k\left(t + \left(\gamma_j - 1\right)h, s\right)\right|ds. \end{aligned} \tag{3.8}$$

Now, recalling that $\sum_{j=1}^{m} b_j = 1$, according to Lemma 3.1, and using hypothesis (iv) we have that there exists a constant $\widetilde{C} > 0$ such that

$$F_n(t) \leq g(t) + h\widetilde{f}\sum_{l=0}^{\overline{n}-1}\sum_{j=1}^{m} b_j \left|k\left(t + \left(\gamma_j - 1\right)h, t_l + c_j h\right)\right| + h\widetilde{C} + \widetilde{f}\widetilde{K}, \quad n = \overline{n} + 1,\ldots,N. \tag{3.9}$$

Since the summation in (3.9) is finite, by hypothesis (v) and, once again, using the result in Lemma 3.1, we get

$$F_n(t) \leq G_2 + \widetilde{f}\widetilde{K} + hC_1, \quad n = \overline{n} + 1, \ldots, N. \tag{3.10}$$

For $n \leq \overline{n}$ the whole summation appearing in (3.4) is finite and so there exists a constant C_2 such that

$$F_n(t) \leq G_2 + hC_2, \quad n \leq \overline{n}. \tag{3.11}$$

The result stated in the theorem follows with $C_3 = \min\{C_1, C_2\}$. □

Lemma 3.3. *Assume that hypotheses (i)–(vii) of Lemma 3.2 hold, moreover let*

(viii) $a_{ij} \geq 0$, *for* $i \geq j$.

Then there exists a positive constant C such that

$$Y_{nj} \leq G_2 + \widetilde{f}\widetilde{K} + hC, \quad n = 0, \ldots, N, \ j = 1, \ldots, m. \tag{3.12}$$

Proof. Let us consider the last stage

$$Y_{n,m} = F_n(t_n + \theta_m h) + h\sum_{j=1}^{m} a_{mj} k(t_n + d_{mj}h, t_n + c_j h) f(Y_{n,j}), \tag{3.13}$$

it can be easily seen that, because of (i), (iii), and (viii) it is

$$Y_{n,m} \leq F_n(t_n + \theta_m h) + h\widetilde{f}\sum_{j=1}^{m} a_{mj} \left| k(t_n + d_{mj}h, t_n + c_j h) \right|. \tag{3.14}$$

Moreover, because of (vi) and Lemma 3.1 there exists K^* and α_m such that

$$\left| k(t_n + d_{mj}h, t_n + c_j h) \right| \leq K^*, \quad \forall n, \tag{3.15}$$

$$Y_{n,m} \leq G_2 + \widetilde{f}\widetilde{K} + h\alpha_m. \tag{3.16}$$

Proceeding as in the proof of Theorem 2.1, it follows that

$$f(Y_{n,m}) \leq F_m(h), \quad n = 0 \ldots, N, \tag{3.17}$$

where $F_m(h) = \max\{0, \max\{\overline{f}, \max_{x\in[\delta, G+\tilde{f}\tilde{K}+h\alpha_m]} f(x)\}\}$ and δ is defined in the proof of Theorem 2.1. Now, rewrite $Y_{n,m-1}$ in the following way:

$$Y_{n,m-1} = F_n(t_n + \theta_{m-1}h) + h\sum_{j=1}^{m-1} a_{m-1,j}k(t_n + d_{m-1,j}h, t_n + c_j h) f(Y_{n,j}) \\ + ha_{m-1,m}k(t_n + d_{m-1,m}h, t_n + c_m h) f(Y_{n,m}), \tag{3.18}$$

and observe the two possible cases that may occur.

(a) $a_{m-1,m} < 0$. Then $a_{m-1,m}k(t_n + d_{m-1,m}h, t_n + c_m h) \geq 0$ and hence, for (3.17) and (3.15),

$$ha_{m-1,m}k(t_n + d_{m-1,m}h, t_n + c_m h) f(Y_{n,m}) \\ \leq hF_m(h)|a_{m-1,m}| \cdot |k(t_n + d_{m-1,m}h, t_n + c_m h)| \leq hF_m(h)|a_{m-1,m}|K^*. \tag{3.19}$$

By recalling that $a_{m-1,j} \geq 0$, $j = 1, \ldots, m-1$, the first summation in (3.18) can be bounded by proceeding as for the previous stage, therefore there exists α_{m-1} such that $Y_{n,m-1} \leq G + \tilde{f}\tilde{K} + h\alpha_{m-1}$.

(b) $a_{m-1,m} \geq 0$. Then, the same procedure as the one used for the $Y_{n,m}$ stage can be applied.

Applying the same procedure to $Y_{n,m-2}, Y_{n,m-3}, \ldots, Y_{n,1}$, the desired result follows with $C = \max\{\alpha_1, \ldots, \alpha_m\}$. □

Lemma 3.4. *Assume that all the hypotheses of Lemma 3.3 hold. Then there exist two positive constants C_4 and C_5 such that*

$$F_n(t) \geq G_1 - \tilde{F}(h)\tilde{K} + hC_4, \quad n = 0, 1, \ldots, N,\ t \in [0, T], \\ Y_{n,j} \geq G_1 - \tilde{F}(h)\tilde{K} + hC_5, \quad n = 0, 1, \ldots, N,\ j = 0, \ldots, m. \tag{3.20}$$

Proof. In view of Lemma 3.3 and by taking into account hypothesis (ii) we can define

$$\tilde{F}(h) = \max\left\{0, \max\left\{\overline{f}, \max_{x\in[\delta, G_2+\tilde{f}\tilde{K}+hC]} f(x)\right\}\right\}, \tag{3.21}$$

with δ given in (2.2), such that $f(Y_{l,j}) \leq \tilde{F}(h)$, $l \geq 0$, $1 \leq j \leq m$. Therefore,

$$h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j k(t + (\gamma_j - 1)h, t_l + c_j h) f(Y_{l,j}) \geq \tilde{F}(h)\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j k(t + (\gamma_j - 1)h, t_l + c_j h) \\ = -\tilde{F}(h)\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j \left|k(t + (\gamma_j - 1)h, t_l + c_j h)\right|. \tag{3.22}$$

Now, proceeding as in the proof of Lemma 3.2 and because of (v), we have

$$F_n(t) \geq G_1 - \widetilde{F}(h)\left[\widetilde{K} + hC_1\right], \tag{3.23}$$

which is is equivalent to the first of (3.20). What is more, from (i) and (3.21), we have

$$-\widetilde{f} < f\left(Y_{n,j}\right) \leq \widetilde{F}(h), \quad \forall n \leq j \leq m, \tag{3.24}$$

and, taking into account (3.15), it can be easily proved that there exists a constant β such that

$$h\sum_{j=1}^{m} a_{i,j}k\left(t_n + d_{i,j}h, t_n + c_jh\right)f\left(Y_{n,j}\right) \geq h\beta, \quad \forall i = 1,\ldots,m. \tag{3.25}$$

From here and (3.6) for $Y_{n,i}$ given in (3.3) holds:

$$Y_{n,i} \geq G_1 - \widetilde{F}(h)\widetilde{K} + hC_5, \tag{3.26}$$

which corresponds to the second of (3.20). □

Now, we are ready to prove our result on the boundedness of the numerical solution y_n of (3.1)–(3.4).

Theorem 3.5. *Assume that all the hypotheses of Lemma 3.3 hold. Then there exist two positive constants $\overline{A}$ and $\overline{B}$ such that*

$$G_1 - \widetilde{F}(h)\widetilde{K} + \overline{A}h \leq y_n \leq G_2 + \widetilde{f}\widetilde{K} + \overline{B}h, \quad n = 0,1,\ldots,N. \tag{3.27}$$

Proof. From Lemma 3.2 it is

$$y_{n+1} \leq G_2 + \widetilde{f}\widetilde{K} + hC_1 + h\sum_{j=1}^{m} b_jk\left(t_{n+1} + \left(\gamma_j - 1\right)h, t_{n,j}\right)f\left(Y_{n,j}\right). \tag{3.28}$$

Since (i), (iii), (vii), and Lemma 3.1 hold, there exists a positive constant $\overline{B}$ such that

$$y_{n+1} \leq G_2 + \widetilde{f}\widetilde{K} + \overline{B}h. \tag{3.29}$$

Moreover, from (3.20) and (3.21), we immediately have

$$y_{n+1} \geq G_1 + \widetilde{F}(h)\widetilde{K} + hC_4 + hC\widetilde{F}(h), \tag{3.30}$$

that, together with (3.29), gives the result stated in the theorem. □

Observe that, in agreement with the continuous case, N is allowed to be ∞.

As we could expected because of the convergence of the method the following result can be observed.

Corollary 3.6. *The numerical and analytical solutions are bounded by the same constants as the step-size h goes to zero.*

Proof. The desired result comes immediately from (3.27) and (2.1) and observing that for $\widetilde{F}(h)$ defined in (3.21), we have

$$\lim_{h\to 0}\widetilde{F}(h) = F^*, \tag{3.31}$$

with F^* given in (2.6). □

Observe that the hypotheses (vii) and (viii) on the coefficients of the methods are not restrictive at all because they are satisfied by any VRK method.

A similar result can be easily proved on VRK methods of mixed type. Compared with (3.1)–(3.4), such methods present a different expression of the lag-term formula, that is,

$$F_n(t) = g(t) + h\sum_{l=0}^{n} w_{n,l}k(t,t_l)f(y_l), \tag{3.32}$$

where $w_{n,l}$ are given weights of a quadrature formula.

The proof of the following theorem has the same plot as the one of Theorem 3.5 and therefore, it is omitted.

Theorem 3.7. *Assume that*

(i) $\exists\widetilde{f}\ge 0 : f(x)\ge -\widetilde{f}$, $\forall x\in\mathbb{R}$,

(ii) $\limsup_{x\to-\infty} f(x) = \overline{f} < +\infty$,

(iii) $k(t,s)\le 0$, $\forall s\le t$, $t\in[0,T]$,

(iv) $\exists\widetilde{K}>0 : \sup_{t\ge 0}\int_0^t |k(t,s)|ds\le\widetilde{K}$, $\forall t\in[0,T]$,

(v) $\exists G_1, G_2 : G_1\le g(t)\le G_2$, $\forall t\in[0,T]$,

(vi) $|k(t,s)|$ *nonincreasing with respect to* s, $t\in[0,T]$,

(vii) $w_{n,j}\ge 0$, $n=0,\ldots,N$, $j=0,\ldots,n$,

(viii) $a_{i,j}\ge 0$, $i\ge j$,

then, for y_n solution of (3.1)–(3.3) *with* (3.32), *we have the following bounds for $n=0,\ldots,N$:*

$$G_1 - \widetilde{F}(h)\widetilde{K} + A_1 h \le y_n \le G_2 + \widetilde{K}\widetilde{f} + B_1 h, \tag{3.33}$$

where A_1 and B_1 are two positive constants.

Remark 3.8. If the hypotheses of Theorem 2.1, which assure the boundedness of $y(t)$, are compared with those of Theorems 3.5 and 3.7 for the boundedness of y_n, we note the following. Besides the natural assumption on the coefficients of the numerical methods, only the hypothesis (vi) on the monotonicity of $|k(t,s)|$ has been added. In this sense we can claim that under mild additional conditions on the kernel k, the behavior of the analytical solution is inherited by the numerical one.

3.2. Volterra Direct Quadrature Methods

A general VDQ method for solving VIEs of the type (1.1), defined on the mesh (3.1), reads

$$y_n = g(t_n) + h\sum_{l=0}^{n} w_{n,l}k(t_n,t_l)f(y_l), \quad n = 0,1,\ldots,N, \tag{3.34}$$

where $w_{n,l}$ are given weights that satisfy

$$0 < w_{n,l} < 1, \quad l = 0,1,\ldots,n,\ n = 0,1,\ldots,N,. \tag{3.35}$$

The following theorems represent the discrete analogues of Theorems 2.1 and 2.7, where, due to the linearity of the method, the Volterra discrete equation (3.34) can be interpreted as the discrete counterpart of (1.1).

Theorem 3.9. *Assume that*

(i) $\exists\widetilde{f} \geq 0 : f(x) \geq -\widetilde{f}$, $\forall x \in \mathbb{R}$,

(ii) $\limsup_{x\to-\infty} f(x) = \overline{f} < +\infty$,

(iii) $k(t,s) \leq 0$, $\forall s \leq t$, $t \in [0,T]$,

(iv) $\exists\widetilde{K} > 0 : \sup_{t\geq 0}\int_0^t |k(t,s)|ds \leq \widetilde{K}$, $\forall t \in [0,T]$,

(v) $\exists G_1, G_2 : G_1 \leq g(t) \leq G_2$, $\forall t \in [0,T]$,

(vi) $|k(t,s)|$ *nonincreasing with respect to* s, $t \in [0,T]$,

(vii) $w_{n,l} \geq 0$, $n = 0,1,\ldots,N$, $l = 0,\ldots,n$,

then y_n *is bounded and it is, for* $n = 0,\ldots,N$

$$G_1 - \widetilde{F}(h)\widetilde{K} + A_2h \leq y_n \leq G_2 + \widetilde{f}\widetilde{K} + B_2h, \tag{3.36}$$

where A_2 *and* B_2 *are positive constants and* $\lim_{h\to 0}\widetilde{F}(h) = F^*$.

Proof. From (i) and (iii) there results $k(t_n,t_l)f(y_l) < -\widetilde{f}k(t_n,t_l)$ hence, by taking into account (vi), we get

$$h\sum_{l=0}^{n} w_{nl}|k(t_n,t_l)|f(y_l) \leq \widetilde{f}\left[h|k(t_n,0)| + \int_0^{t_n}|k(t_n,s)|ds\right]. \tag{3.37}$$

Thus, taking into account Lemma 3.1 and the hypothesis (v) of the theorem, we have

$$y_n \leq G_2 + \widetilde{f}\widetilde{K} + C_6h. \tag{3.38}$$

Once again

$$f(y_n) \leq \max\left\{\overline{f}, \max_{y\in[\delta, G_2+\widetilde{f}\widetilde{K}+C_6h]} f(y)\right\}, \tag{3.39}$$

with δ given in (2.2). If we define

$$\widetilde{F}(h) = \max\left\{0, \max\left\{\overline{f}, \max_{y\in[\delta, G_2+\widetilde{f}\widetilde{K}+C_6h]} f(y)\right\}\right\}, \tag{3.40}$$

then

$$f(y_n) \leq \widetilde{F}(h), \quad n = 0\ldots, N, \lim_{h\to 0} \widetilde{F}(h) = F^*, \tag{3.41}$$

where F^* is given in (2.6). Proceeding as in the proof of Theorem 2.1, we get

$$y_n \geq G_1 + \sum_{l=0}^{n} k(t_n, t_l) f(y_l) \geq G_1 - \widetilde{F}(h) \sum_{l=0}^{n} |k(t_n, t_l)| \geq G_1 - \widetilde{F}(h)\left(\widetilde{K} + C_1 h\right). \tag{3.42}$$

This last inequality, together with (3.38) and the second of (3.41) gives the desired result. □

In the case of VDQ methods we are also able to prove that, under the hypothesis of Theorem 2.7, also the numerical solution vanishes. This, of course, assures that the asymptotic behavior of the exact and the approximate solutions coincide. First of all, we enunciate the following Lemma which represents the discrete analogue of Lemma 2.6 and whose proof is then omitted.

Lemma 3.10. *Assume that*

(i) $\exists \widetilde{f} \geq 0 : f(x) \geq -\widetilde{f}, \forall x \in \mathbb{R}$,

(ii) $xf(x) \geq 0, \forall x \in \mathbb{R}$,

(iii) $k(t, s) \leq 0, \forall s \leq t, t \geq 0$,

(iv) $\lim_{t\to\infty} k(t, s) = 0$,

(v) $\lim_{t\to\infty} g(t) = 0$,

(vi) y_n *ultimately nonnegative,*

then $\lim_{n\to\infty} y_n = 0$, $n = 0, \ldots, N$.

The following result is the discrete counterpart of Theorem 2.7 and could be derived from the proof of Theorem 3.4 in [12]. We report here the proof for the particular form of the Volterra discrete equation (3.34).

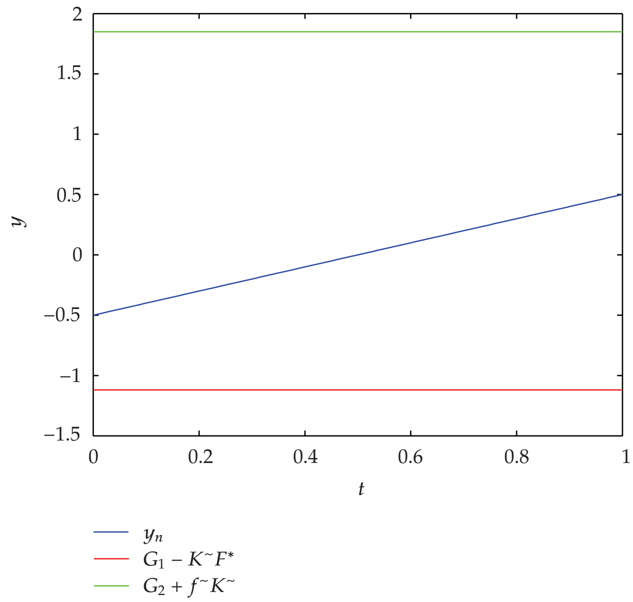

Figure 1: Numerical solution for the problem (1.1) described in Example 2.3.

Theorem 3.11. *Assume that*

(i) $\exists \tilde{f} \geq 0 : f(x) \geq -\tilde{f}$, $\forall x \in \mathbb{R}$,

(ii) f *is nondecreasing,* $xf(x) \geq 0$, $\forall x \in \mathbb{R}$,

(iii) $k(t,s) \leq 0$ *for* $s \leq t$ *and* $t \geq 0$,

(iv) $\exists \tilde{K} > 0 : \sup_{t \geq 0} \int_0^t |k(t,s)| ds \leq \tilde{K}$, $t \geq 0$,

(v) $\lim_{t \to \infty} g(t) = 0$,

(vi) $\lim_{t \to \infty} k(t,s) = 0$,

(vii) $-\tilde{K} f(-\tilde{K} f(-x)) > -x$, $\forall x > 0$,

(viii) $|k(t,s)|$ *ultimately nonincreasing with respect to* s*, for* $t \geq 0$,

(ix) $w_{n,l} \geq 0$, $n = 0, \ldots, N$, $l = 0, \ldots, n$,

then $\lim_{n \to \infty} y_n = 0$.

Proof. By absurd, assume that the sequence y_n given by (3.34) is not ultimately nonnegative, that is,

$$\liminf_{n \to +\infty} y_n = \underline{\gamma} < 0, \tag{3.43}$$

which leads to

$$\forall \epsilon > 0,\ \exists \rho > 0 : n > \rho \Longrightarrow y_n > \underline{\gamma} - \epsilon. \tag{3.44}$$

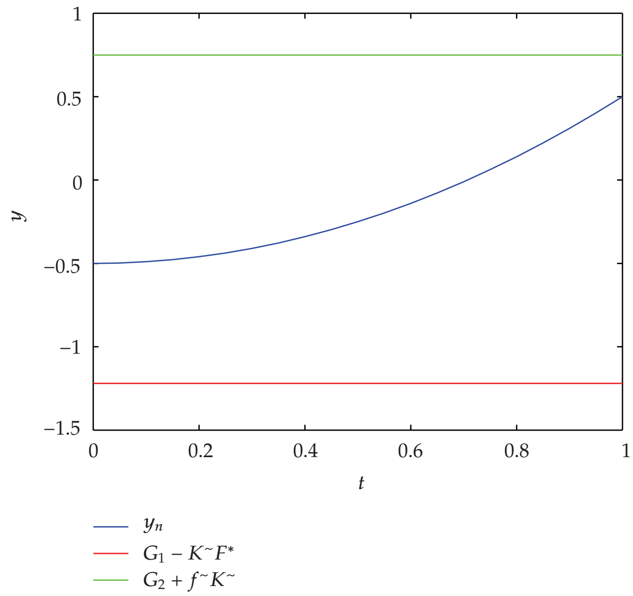

Figure 2: Numerical solution for the problem (1.1) described in Example 2.5.

Denote by $\overline{\Lambda}(x)$ the function

$$\overline{\Lambda}(x) = -\tilde{K} f\left(x - \tilde{K} f\left(\underline{\gamma} - x\right)\right) - 2x. \tag{3.45}$$

Now, proceeding as in Theorem 2.7, we can prove that $\exists \epsilon_0 > 0$, and $\exists n_3 > 0$ such that $\forall n > n_3$

$$y_n \geq -\epsilon_0 - f\left(\epsilon_0 - \tilde{K} f\left(\underline{\gamma} - \epsilon_0\right)\right)\tilde{K}, \tag{3.46}$$

and hence, $y_n \geq \underline{\gamma} + \epsilon_0$. This contradicts (3.43), which is therefore absurd. Hence, y_n is ultimately nonnegative and Lemma 3.10 holds. □

3.3. Numerical Experiments

In this section, we report some numerical experiments that show the application of our results in the study of the behavior of the numerical solutions provided by VRK and VDQ methods for some nonlinear problems. First we consider VRK methods and numerically solve, by a Radau IIa type formula of order 3 (see [13]) the problems described in Examples 2.3 and 2.5 of Section 2. In Figures 1 and 2 we observe that the behavior of the numerical solution y_n with respect to the analytical bounds given in Theorem 2.1 is consistent with the theory developed in Section 3.

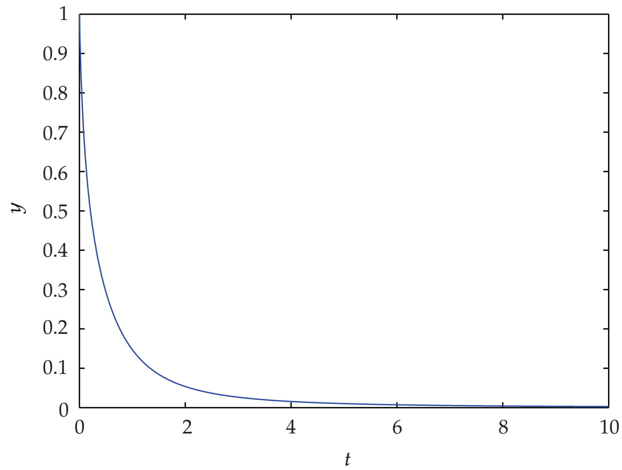

Figure 3: Numerical solution for the problem (1.1) described in (3.47).

In Figure 3 we report the result of an experiment on (1.1) with

$$\begin{aligned} k(t,s) &= \frac{-e^{-s}}{(1+t^2)}, \\ f(y) &= -\sin^2 y, \\ g(t) &= \frac{1}{(1+t^2)}, \end{aligned} \tag{3.47}$$

by a DQ method based on the trapezoidal rule of order 2. We do not know the solution, but thanks to Theorems 3.11 and 2.7, we can predict that the numerical solution y_n tends to zero as $n \to +\infty$ just like the analytical one, and this is exactly the behavior we observe in the plot.

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