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Research Article

Complex Behavior in a Fish Algae Consumption Model with Impulsive Control Strategy

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This paper investigates a dynamic mathematical model of fish algae consumption with an impulsive control strategy analytically. It is proved that the system has a globally asymptotically stable algae-eradication periodic solution and is permanent by using the theory of impulsive equations and small-amplitude perturbation techniques. Numerical results for impulsive perturbations demonstrate the rich dynamic behavior of the system. Further, we have also compared biological control with chemical control. All these results may be useful in controlling eutrophication.

1. Introduction

Controlling algae (in particular the deterioration of water caused by algae) has become an increasingly complex issue over the past two decades because economic loss will be enormous once the population of algae is out of control. At present, many of our lakes and large areas of algae bloom outbreaks per year [1, 2]; in these lakes, ecological balance is broken, the water quality is deteriorated, and human health is threatened. So research on how to control the population of algae is of great important theoretic and practical significance. Many methods have been used to control algal blooms.

Biological control is the practice of using natural enemies such as predators to suppress a prey population, as has already been done for pest control [3, 4]. In addition to the classical biological control based on predator-prey interaction, recently another form of biological control based on fish-algae interaction is extensively used. Many reservoirs have used the biological control methods to control algal blooms (they control algal blooms by stocking fish in the reservoir to graze algae directly), which has been proved to be effective in preventing the outbreak of algal blooms in East Lake in Wu Han province. However, many researchers doubt that this method is not only costly, but also cannot be effective in a few days. Another commonly used method is chemical control (usually dilution of copper sulfate), and this method can quickly kill a significant portion of the algae population, but it brings many negative impacts. Wherever possible, different methods should work together rather than against each other. In some cases, this can lead to synergy where the combined effect of different methods is greater than would be expected from simply adding the individual effects together [5]. Therefore, if we wish to eradicate the algae population, we should implement an impulsive control strategy which includes chemical control and biological control.

With the advance of the theory of impulsive differential equations [6, 7], impulsive differential equations are used to describe the evolving process and the control process of species [8–11], which make the models more reasonable [12–14]. Moreover, the theory of impulsive differential equations is being recognized not only to be richer than the corresponding theory of differential equations without impulses, but also to represent a more natural framework for the mathematical modeling of real-world phenomenon [15, 16]. In this paper, we construct a mathematical model combining the fact of period biological control with chemical control; we first introduce a proportion periodic impulsive harvesting (fish) and chemical poisoning for the algae at time $t = (n+L-1)T$, and then we introduce a constant periodic releasing for natural enemies (fish) at time $t = nT$; the system can be described as follows:

$$
\begin{aligned}
&\left.\begin{aligned}
\frac{dx(t)}{dt} &= u_c x(t)\left(\frac{(1-(x(t)/x_m))}{(1-(x(t)/x_h))}\right) - c_1 x^2(t) - \frac{u_1 x(t) y(t)}{(x(t)+k_1)},\\
\frac{dy(t)}{dt} &= -u_3 y(t) + \frac{u_2 x(t) y(t)}{(x(t)+k_1)},
\end{aligned}\right\} \quad t \neq nT,\ t \neq (n+L-1)T,\\
&\left.\begin{aligned}
\Delta x(t) &= -\delta_1 x(t),\\
\Delta y(t) &= -\delta_2 y(t),
\end{aligned}\right\} \quad t = (n+L-1)T,\\
&\left.\begin{aligned}
\Delta x(t) &= 0,\\
\Delta y(t) &= p,
\end{aligned}\right\} \quad t = nT,
\end{aligned}
\tag{1.1}
$$

where $x(t)$, $y(t)$ are the densities of the algae and fish at time t, $\Delta x(t) = x(t^+) - x(t)$, and $\Delta y(t) = y(t^+) - y(t)$; $dx(t)/dt = u_c x(t)((1-(x(t)/x_m))/(1-(x(t)/x_h)))$ is a mathematical model for a single population [17] and is established by Cui and Lawson; u_c is a growth parameter which is related to the biological characteristics of populations and the rationalization of environmental resources; $x_m (0 \leq x_m/x_h \leq 1)$ is the maximum density of the algae population (i.e., environmental carrying capacity); x_h is a nutritional parameter which is related to the resource conditions of the environment; $u_1 x(t)/(x(t)+k_1)$ is one of the most well-known functional responses describing a prey-predator interaction, called Holling-Type II functional response; c_1 is the intraspecific competition rate of the algae; u_3 is the average mortality rate for fish; $0 \leq \delta_1, \delta_2 \leq 1$ represent the fraction of the algae and fish which die due to the harvesting or chemical poisoning at $t = (n+L-1)T$; $p > 0$ is the number of fish released at time $t = nT$; T is the period of the impulsive effect; n is the set of all nonnegative integers.

With model (1.1), we can take into account the effects in the external which can rapidly change the population densities. Impulsive reduction of the algae population density is possible after its partial destruction by poisoning with chemicals, and also impulsive increase of the fish population density is possible by artificial breeding or releasing the fish population; therefore, we can use impulsive control strategy to eradicate the algae population.

2. Preliminaries and Mathematical Analysis

Let $R_+ = [0, \infty)$, $R_+^2 = \{X \in R^2 \mid X > 0\}$. Denote that $f = (f_1, f_2)$ is the map defined by the right-hand sides of the first and second equations of system (1.1). Let $V : R_+ \times R_+^2 \to R_+$, then V is said to belong to class V_0 if

(1) V is continuous in $((n-1)T, (n+l-1)T] \times R_+^2$, $((n+l-1)T, nT] \times R_+^2$, and for each $X \in R_+^2, n \in N$, $\lim_{(t,y)\to((n+l-1)T^+, X)} V(t,y) = V((n+l-1)T^+, X)$ and $\lim_{(t,y)\to(nT^+,X)} V(t,y) = V(nT^+, X)$ exist;

(2) V is locally Lipschitzian in X.

Definition 2.1. Let $V \in V_0$; for $(t,x) \in ((n-1)T, (n+l-1)T] \times R_+^2$ and $((n+l-1)T, nT] \times R_+^2$, the upper right derivative of $V(t,X)$ with respect to the impulsive differential system (1.1) is defined as

$$D^+V(t,X) = \lim_{h\to 0^+} \sup \frac{1}{h}\left[V(t+h, X+hf(t,X)) - V(t,X)\right]. \tag{2.1}$$

Remark 2.2. (1) The solution of system (1.1) is a piecewise continuous function with $X : R_+ \to R_+^2$, then $X(t)$ is continuous on $((n-1)T, (n+l-1)T)$, and $((n+l-1)T, nT)$. (2) The smoothness properties of f guarantee the global existence and uniqueness of solution of system (1.1) (for details, see book [6, 7]).

Lemma 2.3. *Assume that $X(t)$ is a solution of system* (1.1) *such that*

(1) *if $X(0^+) \geq 0$, then $X(t) \geq 0$ for all $t \geq 0$,*

(2) *if $X(0^+) > 0$, then $X(t) > 0$, for all $t > 0$.*

Lemma 2.4. *There exists a positive constant $M > 0$ such that $x(t) \leq M$ and $y(t) \leq M$ for each solution of system* (1.1) *with all t large enough.*

If the algae population is eradicated, then system (1.1) will reduce to the following system:

$$\begin{aligned}
\frac{dy(t)}{dt} &= -u_3 y(t), \quad t \neq (n+l-1)T,\ t \neq nT, \\
y(t^+) &= (1-\delta_2)y(t), \quad t = (n+l-1)T, \\
y(t^+) &= y(t) + p, \quad t = nT, \\
y(0^+) &= y_0.
\end{aligned} \tag{2.2}$$

System (2.2) is a periodically forced linear system, then we get that

$$y^*(t) = \begin{cases} \dfrac{p\exp(-u_3(t-(n-1)T))}{1-(1-\delta_2)\exp(-u_3T)}, & (n-1)T < t \le (n+l-1)T, \\ \dfrac{p(1-\delta_2)\exp(-u_3(t-(n-1)T))}{1-(1-\delta_2)\exp(-u_3T)}, & (n+l-1)T < t \le nT, \end{cases} \tag{2.3}$$

is a positive periodic solution of system (2.2) with the initial values

$$\left(y^*(0^+) = y^*(nT^+) = \frac{p}{1-(1-\delta_2)\exp(-u_3T)}, y^*(lT^+) = \frac{p(1-\delta_2)\exp(-u_3lT)}{1-(1-\delta_2)\exp(-u_3T)}\right), \tag{2.4}$$

since the general solution of (2.2) is

$$y(t) = \begin{cases} (1-\delta_2)^{n-1}\left(y(0^+) - \dfrac{p}{1-(1-\delta_2)\exp(-u_3T)}\right)\exp(-u_3T) + y^*(t), \\ \qquad\qquad (n-1)T < t \le (n+l-1)T, \\ (1-\delta_2)^{n}\left(y(0^+) - \dfrac{p}{1-(1-\delta_2)\exp(-u_3T)}\right)\exp(-u_3T) + y^*(t), \\ \qquad\qquad (n+l-1)T < t \le nT. \end{cases} \tag{2.5}$$

Then the following results can be got easily.

Lemma 2.5. *$y^*(t)$ is a positive periodic solution of system* (2.2), *and for every solution $y(t)$ of system* (2.2), *one has $y(t) \to y^*(t)$ as $t \to \infty$.*

Therefore, system (2.2) has an algae-eradication periodic solution $(0, y^*(t))$.

After the preliminaries, it is necessary to give the main theorems of this paper. Now, the conditions which assure the globally asymptotical stability of the an lgae-eradication periodic solution $(0, y^*(t))$ are given.

Theorem 2.6. *If*

$$u_cT - \frac{u_1p(1-\delta_2\exp(-u_3lT) - (1-\delta_2)\exp(-u_3T))}{u_3k_1(1-(1-\delta_2)\exp(-u_3T))} < \ln\left(\frac{1}{1-\delta_1}\right), \tag{2.6}$$

then the algae-eradication periodic solution $(0, y^*(t))$ is said to be globally asymptotically stable.

Proof. The local stability of the periodic solution $(0, y^*(t))$ may be determined by considering the behavior of small-amplitude perturbations of the solution. Define $x(t) = u(t)$, $y(t) = v(t) + y^*(t)$, then the Linearization of system (1.1) becomes

$$\begin{aligned}
&\left.\begin{aligned}
\frac{du(t)}{dt} &= \left(u_c - \frac{u_1 y^*(t)}{k_1}\right)u(t),\\
\frac{dv(t)}{dt} &= -u_3 v(t) + \frac{u_2 u(t) y^*(t)}{k_1},
\end{aligned}\right\} \quad t \neq nT,\ t \neq (n+L-1)T,\\
&\left.\begin{aligned}
\Delta u(t) &= -\delta_1 u(t),\\
\Delta v(t) &= -\delta_2 v(t),
\end{aligned}\right\} \quad t = (n+L-1)T,\\
&\left.\begin{aligned}
\Delta u(t) &= 0,\\
\Delta v(t) &= 0,
\end{aligned}\right\} \quad t = nT,
\end{aligned} \tag{2.7}$$

and as a result,

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \quad 0 \leq t < T, \tag{2.8}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} u_c - \dfrac{u_1 y^*(t)}{k_1} & 0 \\ \dfrac{u_2 y^*(t)}{k_1} & -u_3 \end{pmatrix} \Phi(t), \tag{2.9}$$

and $\Phi(0) = I$, the identity matrix. The linearization of the third and fourth equations of (2.2) becomes

$$\begin{pmatrix} u((n+l-1)T^+) \\ v((n+l-1)T^+) \end{pmatrix} = \begin{pmatrix} 1-\delta_1 & 0 \\ 0 & 1-\delta_2 \end{pmatrix} \begin{pmatrix} u((n+l-1)T) \\ v((n+l-1)T) \end{pmatrix}. \tag{2.10}$$

The linearization of fifth and sixth equations of (2.2) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \tag{2.11}$$

The stability of the periodic solution $(0, y^*(t))$ is determined by the eigenvalues of

$$\theta = \begin{pmatrix} 1-\delta_1 & 0 \\ 0 & 1-\delta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Phi(t). \tag{2.12}$$

Therefore, all eigenvalues of θ are given by

$$\lambda_1 = (1-\delta_1)\exp\left(\int_0^T \left(u_c - \frac{u_1 y^*(t)}{k_1}\right)dt\right), \qquad \lambda_2 = (1-\delta_2)\exp(-u_3 T) < 1. \tag{2.13}$$

According to Floquet theory, $(0, y^*(t))$ is locally asymptotically stable if $\lambda_1 < 1$, that is to say,

$$u_c T - \frac{u_1 p(1-\delta_2\exp(-u_3 lT) - (1-\delta_2)\exp(-u_3 T))}{u_3 k_1 (1-(1-\delta_2)\exp(-u_3 T))} < \ln\frac{1}{1-\delta_1}. \tag{2.14}$$

In the following, we prove the global attractivity. Choose a $\varepsilon > 0$ such that

$$\xi_1 \approx (1-\delta_1)\exp\left(\int_0^T \left(u_c - \frac{u_1}{k_1}(y^*(t)-\varepsilon)\right)dt\right) < 1, \tag{2.15}$$

and note that $dy(t)/dt \geq -u_3 y(t)$; from Lemma 2.5 and comparison theorem of impulsive equation, we get

$$y(t) > y^*(t) - \varepsilon, \tag{2.16}$$

for all sufficiently large t. For simplification, assuming (2.16) holds for all $t \geq 0$. From (1.1) and (2.16),

$$\begin{gathered} \frac{dx(t)}{dt} \leq x(t)\left(u_c - \frac{u_1}{k_1}(y^*(t)-\varepsilon)\right), \quad t \neq (n+l-1)T, \\ x(t^+) = (1-\delta_1)x(t), \quad t = (n+l-1)T, \end{gathered} \tag{2.17}$$

which leads to

$$\begin{aligned} x((n+l)T) &\leq x((n+l-1)T^+)\exp\left(\int_{(n+l-1)T}^{(n+l)T}\left(u_c - \frac{u_1}{k_1}(y^*(t)-\varepsilon)\right)dt\right) \\ &= x((n+l-1)T)(1-\delta_1)\exp\left(\int_{(n+l-1)T}^{(n+l)T}\left(u_c - \frac{u_1}{k_1}(y^*(t)-\varepsilon)\right)dt\right) \\ &= x((n+l-1)T)\xi_1. \end{aligned} \tag{2.18}$$

Hence, $x((n+l)T) \leq x(lT)\xi_1^n$ and $x((n+l)T) \to 0$ as $n \to \infty$. Therefore, $x(t) \to 0$ when $n \to \infty$, because $0 < x(t) < x((n+l-1)T)(1-\delta_1)\exp(u_c T)$ for $(n+l-1)T < t \leq (n+l)T$.

Next, we prove that $y(t) \to y^*(t)$ as $t \to \infty$. For $0 < \varepsilon < (u_3 k_1/u_2)$, there must exist a $T' > 0$ such that $0 < x(t) < \varepsilon$, $t \geq T'$. Without any loss of generality, we assume that $0 < x(t) < \varepsilon$ for all $t \geq 0$, then from system (1.1),

$$-u_3 y(t) \leq \frac{dy(t)}{dt} \leq \left(-u_3 + \frac{u_3 k_1}{u_2}\varepsilon\right)y(t). \tag{2.19}$$

From Lemma 2.5 and comparison theorem of impulsive equation, $z_1(t) \le y(t) \le z_2(t)$ and $z_1(t) \to y^*(t)$, $z_2(t) \to y^*(t)$ as $t \to \infty$, where $z_1(t)$ and $z_2(t)$ are solutions of

$$\begin{gathered}
\frac{dz_1(t)}{dt} = -u_3 z_1(t), \quad t \neq (n+l-1)T,\ t \neq nT, \\
z_1(t^+) = (1-\delta_2) z_1(t), \quad t = (n+l-1)T, \\
z_1(t^+) = z_1(t) + p, \quad t = nT, \\
z_1(0^+) = y(0^+),
\end{gathered} \tag{2.20}$$

$$\begin{gathered}
\frac{dz_2(t)}{dt} = \left(-u_3 + \frac{u_3 k_1}{u_2}\varepsilon\right) z_2(t), \quad t \neq (n+l-1)T,\ t \neq nT, \\
z_2(t^+) = (1-\delta_2) z_2(t), \quad t = (n+l-1)T, \\
z_2(t^+) = z_2(t) + p, \quad t = nT, \\
z_2(0^+) = y(0^+),
\end{gathered} \tag{2.21}$$

respectively,

$$z_2{}^*(t) = \begin{cases} \dfrac{p\exp((-u_3 + (u_3k_1/u_2)\varepsilon)(t-(n-1)T))}{1-(1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\varepsilon)T)}, \\ \qquad (n-1)T < t \le (n+l-1)T, \\ \dfrac{p(1-\delta_2)\exp((-u_3 + (u_3k_1/u_2)\varepsilon)(t-(n-1)T))}{1-(1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\varepsilon)T)}, \\ \qquad (n+l-1)T < t \le nT. \end{cases} \tag{2.22}$$

Therefore, for any $\varepsilon_1 > 0$, there exists a $T_1 > 0$ such that

$$z_1^*(t) - \varepsilon_1 \le y(t) \le z_2^*(t) + \varepsilon_1, \quad \text{for } t > T_1. \tag{2.23}$$

Let $\varepsilon \to 0$ such that

$$y^*(t) - \varepsilon_1 \le y(t) \le y^*(t) + \varepsilon_1, \tag{2.24}$$

for t large enough, which implies $y(t) \to y^*(t)$ as $t \to \infty$. This completes the proof. □

Now, we investigate the permanence of system (1.1).

Theorem 2.7. *System* (1.1) *is permanent provided*

$$u_c T - \frac{u_1 p(1-\delta_2\exp(-u_3 lT) - (1-\delta_2)\exp(-u_3 T))}{u_3 k_1 (1-(1-\delta_2)\exp(-u_3 T))} > \ln\left(\frac{1}{1-\delta_1}\right) \tag{2.25}$$

holds true.

Proof. Let $X(t) = (x(t), y(t))$ be any solution of system (1.1) with $X(0) > 0$. From Lemma 2.4, there exists a positive constant M such that $x(t) \le M$ and $y(t) \le M$ for t large enough. From (2.16), we have $y(t) > y^*(t) - \varepsilon$ for all sufficiently large t and some ε such that $y(t) \ge p(1-\delta_2)\exp(-u_3 T)/(1-(1-\delta_2)\exp(-u_3 T)) - \varepsilon \approx \zeta_2$ for t large enough. Therefore, it is only necessary to find an $\zeta_1 > 0$ such that $x(t) \ge \zeta_1$ for t large enough. We prove this in the following two steps.

Step 1. Let $0 < \zeta_3 < u_3 k_1/u_2$, $\varepsilon_1 > 0$ be small enough such that

$$\begin{aligned}\varphi \approx\ & u_c\left(1-\frac{\zeta_3}{x_m}\right)T - c_1\zeta_3 T - \frac{u_1\varepsilon_1}{k_1}T \\ & - \frac{u_1}{k_1}\left(\frac{u_1 p(1-\delta_2\exp((-u_3+(u_3k_1/u_2)\zeta_3)lT) - (1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\zeta_3)T))}{(u_3-(u_3k_1/u_2)\zeta_3)k_1(1-(1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\zeta_3)T))}\right) \\ & > 1,\end{aligned} \tag{2.26}$$

then it is easy to prove that $x(t) < \zeta_3$ cannot hold for all t. Otherwise,

$$\begin{gathered}\frac{dy(t)}{dt} \le \left(-u_3 + \frac{u_3k_1}{u_2}\zeta_3\right)y(t), \quad t \ne (n+l-1)T,\ t \ne nT, \\ y(t^+) = (1-\delta_2)y(t), \quad t = (n+l-1)T, \\ y(t^+) = y(t) + p, \quad t = nT, \\ y(0^+) = y_0.\end{gathered} \tag{2.27}$$

Then, $y(t) \le z(t)$ and $z(t) \to z^*(t)\ (t \to \infty)$, where $z(t)$ is the solution of

$$\begin{gathered}\frac{dz(t)}{dt} = \left(-u_3 + \frac{u_3k_1}{u_2}\zeta_3\right)z(t), \quad t \ne (n+l-1)T,\ t \ne nT, \\ z(t^+) = (1-\delta_2)z(t), \quad t = (n+l-1)T, \\ z(t^+) = z(t) + p, \quad t = nT, \\ z(0^+) = y(0^+),\end{gathered} \tag{2.28}$$

$$z^*(t) = \begin{cases} \dfrac{p\exp((-u_3+(u_3k_1/u_2)\varepsilon)(t-(n-1)T))}{1-(1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\varepsilon)T)}, \\ \qquad (n-1)T < t \le (n+l-1)T, \\ \dfrac{p(1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\varepsilon)(t-(n-1)T))}{1-(1-\delta_2)\exp((-u_3+(u_3k_1/u_2)\varepsilon)T)}, \\ \qquad (n+l-1)T < t \le nT.\end{cases} \tag{2.29}$$

Therefore, there exists a $T_1 > 0$ such that

$$y(t) \le z(t) \le z^*(t) + \varepsilon_1, \tag{2.30}$$

and it follows that

$$\begin{aligned}
&\frac{dx(t)}{dt} \ge x(t)\left(u_c\left(1 - \frac{\zeta_3}{x_m}\right) - c_1\zeta_3 - \frac{u_1}{k_1}(z^*(t) + \varepsilon_1)\right), \quad t \ne (n + l - 1)T, \\
&x(t^+) = (1 - \delta_1)x(t), \quad t = (n + l - 1)T,
\end{aligned} \tag{2.31}$$

for $t \ge T_1$. Let $(N + l - 1)T \ge T_1$, integrating (2.31) on $((n + l - 1)T, (n + l)T],\ n \ge N$, so

$$\begin{aligned}
x((n + l)T) &\ge x((n + l - 1)T)(1 - \delta_1) \exp\left(\int_{(n+l-1)T}^{(n+l)T} \left(u_c\left(1 - \frac{\zeta_3}{x_m}\right) - c_1\zeta_3 - \frac{u_1}{k_1}(z^*(t) + \varepsilon_1)\right)dt\right) \\
&= x((n + l - 1)T)\psi,
\end{aligned} \tag{2.32}$$

then $x((N + n + l)T) \ge x((N + l)T)\psi^n \to \infty$ when $n \to \infty$; it is a contradiction because $x(t)$ is ultimately bounded. Therefore, there exists a $t_1 > 0$ such that $x(t_1) \ge \zeta_3$.

Step 2. If $x(t) \ge \zeta_3$ for all $t > t_1$, then the proof will be complete. Otherwise, let $t^* = \inf_{t>t_1}\{x(t) < \zeta_3\}$, then there are two possible cases for t^*.

Case 1. If $t^* = (n_1 + l - 1)\mathrm{T}$, $n_1 \in N$, then $x(t) \ge \zeta_3$ for $t \in [t_1, t^*]$ and $(1 - \delta_1)\zeta_3 \le x(t^{*+}) = (1 - \delta_1)x(t^*) < \zeta_3$, and select $n_2, n_3 \in N$ such that

$$\begin{aligned}
&(n_2 - 1)T > \frac{\ln(\varepsilon_1/M + p)}{(-u_3 + (u_3k_1/u_2)\zeta_3)}, \\
&(1 - \delta_1)^{n_2} \exp(n_2\psi_1 T)\psi^{n_3} > (1 - \delta_1)^{n_2} \exp((n_2 + 1)\psi_1 T)\psi_1^{n_3} > 1,
\end{aligned} \tag{2.33}$$

where $\psi_1 = u_c(1 - (\zeta_3/x_m)) - c_1\zeta_3 - (u_1/k_1)M < 0$. Let $T' = n_2T + n_3T$; it is claimed here that there must be a $t_2 \in (t^*, t^* + T']$ such that $x(t_2) > \zeta_3$. Otherwise, considering (2.28) with $z(t^{*+}) = y(t^{*+})$, it follows that

$$z(t) = \begin{cases}
(1 - \delta_2)^{n-(n_1+1)}\left(z(n_1T^+) - \dfrac{p}{1 - (1 - \delta_2)\exp((-u_3 + (u_3k_1/u_2)\zeta_3)T)}\right) \\
\quad \times \exp\left(-u_3 + \dfrac{u_3k_1}{u_2}\zeta_3\right)(t - n_1T) + z^*(t), \quad (n - 1)T < t \le (n + l - 1)T, \\
(1 - \delta_2)^{n-n_1}\left(z(n_1T^+) - \dfrac{p}{1 - (1 - \delta_2)\exp((-u_3 + (u_3k_1/u_2)\zeta_3)T)}\right) \\
\quad \times \exp\left(-u_3 + \dfrac{u_3k_1}{u_2}\zeta_3\right)(t - n_1T) + z^*(t), \quad (n + l - 1)T < t \le nT,
\end{cases} \tag{2.34}$$

and $n_1+1 \le n \le n_1+n_2+n_3$. Therefore, $|z(t)-z^*(t)| < (M+p)\exp((-u_3+(u_3k_1/u_2)\zeta_3)(t-n_1T)) < \varepsilon_1$ and $y(t) \le z(t) \le z^*(t) + \varepsilon_1$ for $n_1T + (n_2-1)T \le t \le t^* + T'$ which implies that (2.31) holds for $t^* + n_2T \le t \le t^* + T'$. So as in Step 1,

$$x(t^* + T) \ge x(t^* + n_2T)\psi^{n_3}. \tag{2.35}$$

From system (1.1)

$$\begin{aligned} \frac{dx(t)}{dt} &\ge x(t)\left(u_c\left(1-\frac{\zeta_3}{x_m}\right) - c_1\zeta_3 - \frac{u_1}{k_1}M\right), \quad t \ne (n+l-1)T, \\ x(t^+) &= (1-\delta_1)x(t), \quad t = (n+l-1)T, \end{aligned} \tag{2.36}$$

for $t \in [t^*, t^* + n_2T]$. Integrating (2.36) on $t \in [t^*, t^* + n_2T]$ such that

$$x(t^* + n_2T) \ge \zeta_3(1-\delta_1)^{n_2}\exp(n_2\psi_1 T), \tag{2.37}$$

thus, $x(t^* + T') \ge \zeta_3(1-\delta_1)^{n_2}\exp(n_2\psi_1 T)\psi^{n_3} > \zeta_3$, which is a contraction.

Let $t_2 = \inf_{t>t^*}\{x(t) > \zeta_3\}$, then $x(t) \le \zeta_3$ when $t \in (t^*, t_2)$ and $x(t_2) = \zeta_3$. For $t \in (t^*, t_2)$,

$$x(t) \ge \zeta_3(1-\delta_1)^{n_2+n_3}\exp((n_2+n_3)\psi_1 T). \tag{2.38}$$

Let $\zeta_1' = \zeta_3(1-\delta_1)^{n_2+n_3}\exp((n_2+n_3)\psi_1 T)$, then we have $x(t) \ge \zeta_1'$ for $t \in (t^*, t_2)$. For $t > t_2$, the same arguments can be continued since $x(t_2) \ge \zeta_3$.

Case 2. If $t^* \ne (n_1 + l - 1)T$, $n_1 \in N$, then $x(t) \ge \zeta_3$ for $t \in [t_1, t^*]$, and $x(t^*) = \zeta_3$; suppose that $t^* \in ((n_1' + l - 1)T, (n_1' + l)T)$, $n_1 \in N$. There are also two possible cases for $t \in (t^*, (n_1' + l)T)$.

Subcase 1. If $x(t) \le \zeta_3$ for all $t \in (t^*, (n_1' + l)T)$, as in Case 1, we can prove that there must be a $t_1' \in [(n_1' + l)T, (n_1' + l)T + T']$ such that $x(t_1') > \zeta_3$. Here, we omit it.

Let $t_3 = \inf_{t>t^*}\{x(t) > \zeta_3\}$, then $x(t) \le \zeta_3$ when $t \in (t^*, t_3)$ and $x(t_3) = \zeta_3$. For $t \in (t^*, t_3)$

$$x(t) \ge \zeta_3(1-\delta_1)^{n_2+n_3}\exp((n_2+n_3+1)\psi_1 T). \tag{2.39}$$

Let $\zeta_1 = \zeta_3(1-\delta_1)^{n_2+n_3}\exp((n_2+n_3+1)\psi_1 T) < \zeta_1'$, then $x(t) \ge \zeta_1$ for $t \in (t^*, t_3)$, and when $t > t_3$, the same arguments can be got since $x(t_3) \ge \zeta_3$.

Subcase 2. There exists a $t \in (t^*, (n_1' + l)T)$ such that $x(t) > \zeta_3$. Let $t_4 = \inf_{t>t^*}\{x(t) > \zeta_3\}$ such that $x(t) \le \zeta_3$ when $t \in (t^*, t_4)$ and $x(t_4) = \zeta_3$. When $t \in (t^*, t_4)$, the inequality (2.36) holds. Integrating (2.36) on $t \in (t^*, t_4)$, then

$$x(t) \ge x(t^*)\exp(\psi_1(t-t^*)) \ge \zeta_3\exp(\psi_1 T) > \zeta_1. \tag{2.40}$$

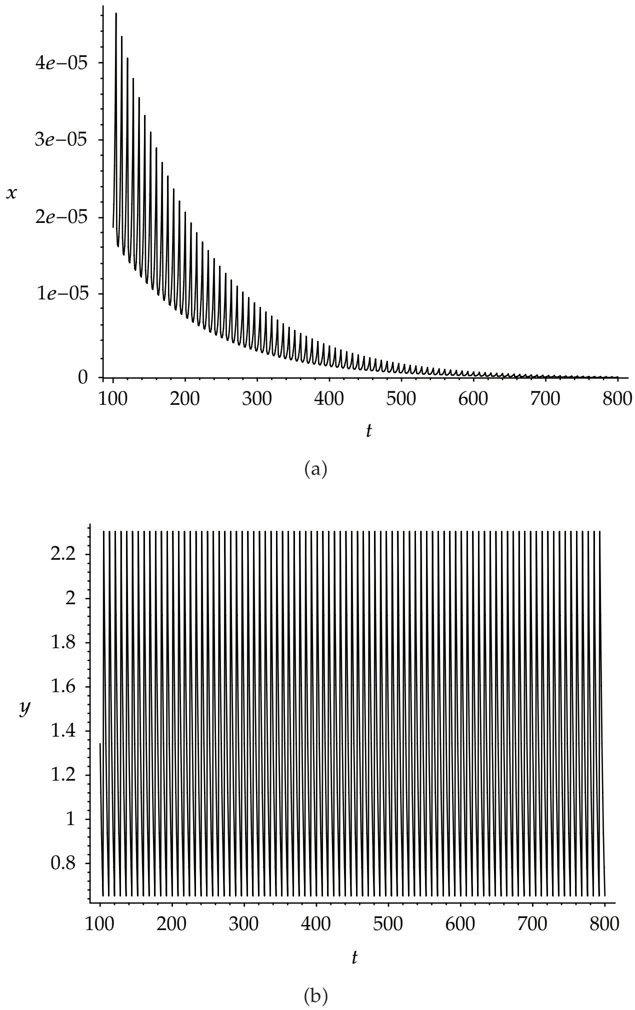


Figure 1: Dynamic behavior of system (1.1). When $T < T_{\max} \approx 9$, the algae will be eradicated. Time series evolving according to biological control system (1.1) of (a) the algae population x, (b) the fish population y.

Since $x(t_4) \geq \zeta_3$ for $t > t_4$, the same arguments can be continued. Therefore, $x(t) > \zeta_1$ for $t > t_1$, so system (1.1) is permanent. The proof is complete. □

3. Numerical Analysis

3.1. Bifurcation Analysis

The global dynamical behavior and the permanence of system (1.1) are investigated using numerical simulations; the following parameters and initial values were considered to substantiate our theoretical results: $u_1 = 0.175$, $u_2 = 0.3$, $u_3 = 0.18$, $u_c = 0.5$, $x_m = 15$, $x_h = 20$, $k_1 = 0.6$, $c_1 = 0.05$, $\delta_1 = 0.4$, $\delta_2 = 0.4$, $p = 4$, $L = 0.07$, $x_0 = 0.5$, and $y_0 = 0.5$.

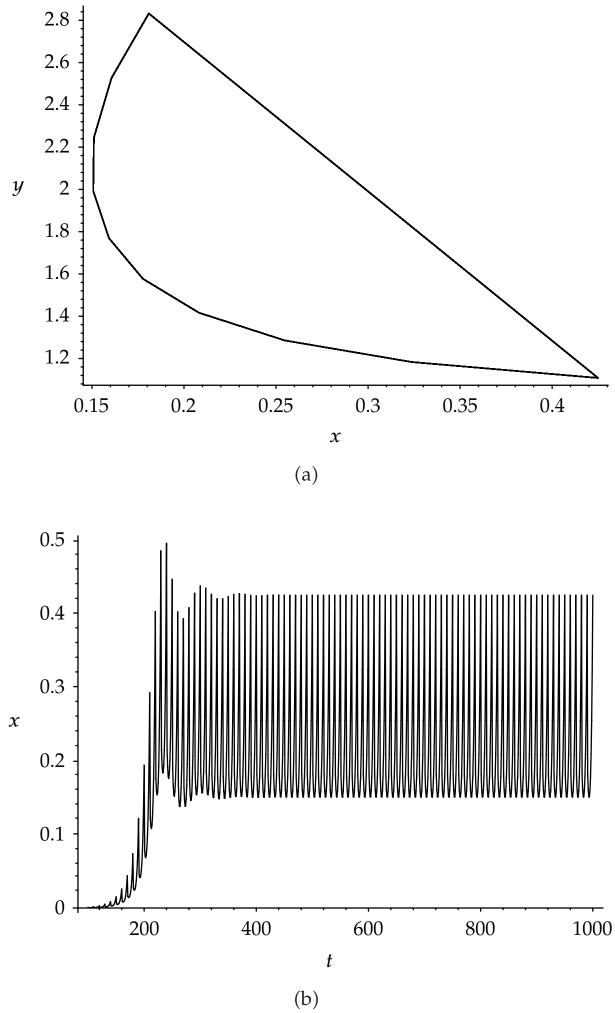
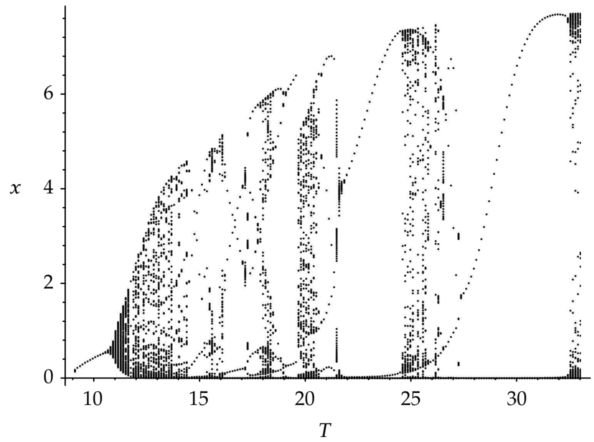
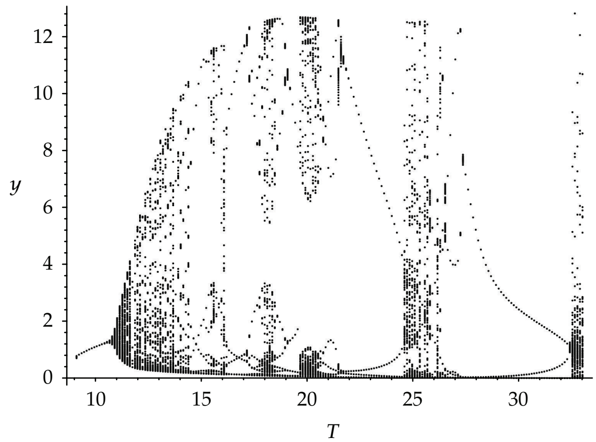


Figure 2: (a) A period T attractor when $T = 10$, (b) time series algae population when $T = 10$.

From Theorem 2.6, it is known that algae-eradication periodic solution is globally asymptotically stable when $T < T_{\max}$; this algae-eradication periodic solution $(0, y^*(t))$ is shown in Figure 1. It is clear that the variable predator y oscillates in a stable cycle, but the algae x rapidly decrease to zero, and $T_{\max} \approx 9$. If the period of the pulses T is larger than $T_{\max}$, then the algae-eradication periodic solution becomes unstable, and it is possible that the algae and the fish population can coexist on a limit cycle when $T > T_{\max}$ (Figure 2), so system (1.1) can be permanent from Theorem 2.7. As the period of pulses increases, system (1.1) exhibits rich dynamic behaviors. In Figure 3, the typical bifurcation diagrams for system (1.1) were displayed with respect to T in the range $T \in [9, 33]$. When $9 < T < 10.6$, we can see T-period solution of system (1.1), and T-period solution is stable. When $T > 12.2$, system (1.1) becomes unstable, and there is a cascade of period-doubling bifurcations leading to chaos (Figure 4). As T further increases, the bifurcation diagrams show that system (1.1) exhibits rich dynamics including period-halving bifurcation, symmetry breaking pitchfork



(a)



(b)

Figure 3: Bifurcation diagrams for system (1.1) showing the effect of T with $p = 4$; we keep other parameters the same. (a) x versus T, (b) y versus T.

bifurcation, period-doubling bifurcation, quasiperiod oscillations, narrow or wide periodic windows, and crisis.

Then, we investigate the effect of the number of fish released p to vary for system (1.1). Figure 5 shows the typical bifurcation diagrams of p for $0 < p < 9$; it is clear that with the increasing number of fish released, system (1.1) shows complex behaviors including period-doubling bifurcations, chaotic band with wide or narrow periodic windows, crisis, tangent bifurcations, and period-halving bifurcation. When $p > 8.55$, the algae will be eradicated, and the algae-eradication periodic solution occurs.

3.2. *The Largest Lyapunov Exponent*

Convincing evidence for deterministic chaos has come from several recent experiments [18, 19]. From these results, the problem of detecting and quantifying chaos has become an

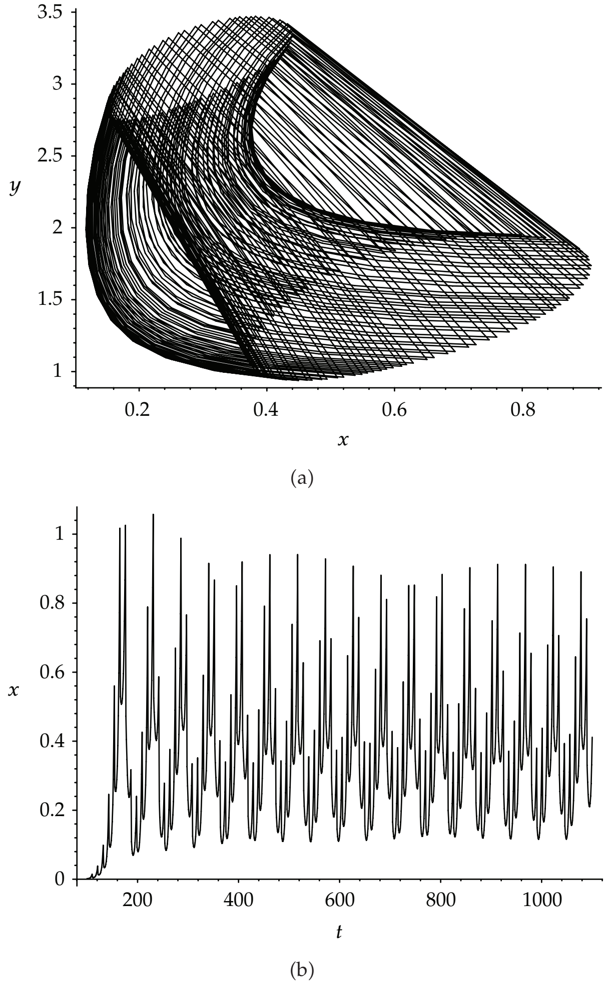
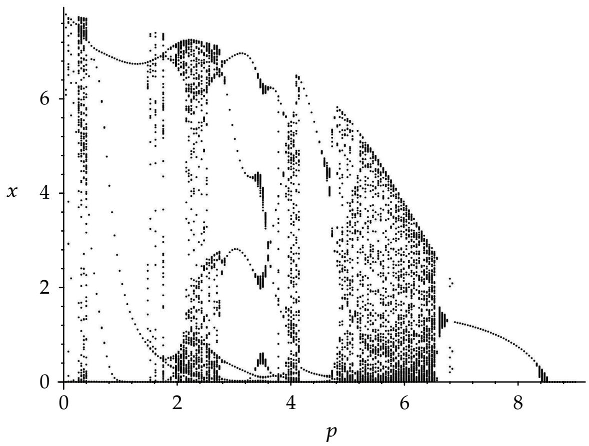


Figure 4: Strange attractor. (a) Chaotic attractor when $T = 11$, (b) time series of algae population when $T = 11$.

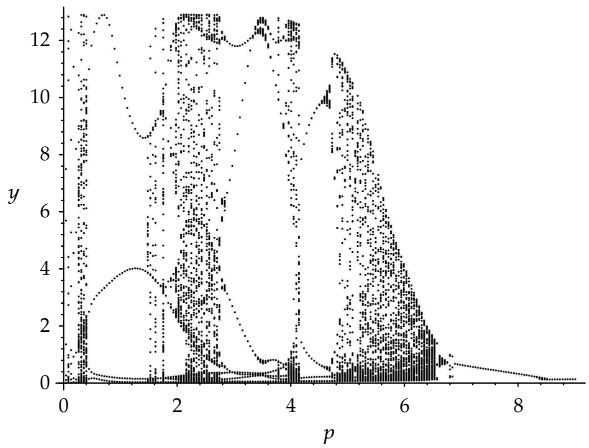
important one; it is clear that chaos plays a very significant role in these studies [20, 21]. Therefore, the largest Lyapunov exponent is considered to be the most useful diagnostic tool for chaotic systems [22–25]. The largest Lyapunov exponent λ must be positive for a chaotic attractor; otherwise, if λ is negative, the system will enter a stable state or become a periodic attractor. Reviewing the bifurcation diagram in Figures 3 and 5, we can calculate the corresponding largest Lyapunov exponent (T ranging from 9 to 23, p ranging from 0 to 6.2) for system (1.1). The output is shown in Figure 6.

4. Conclusions

In this paper, the effects of impulsive perturbations on a algae-fish consumption model have been investigated. The local and global stability of the algae-eradication periodic solution have been proved when the period of the pulses is less than critical values. In addition,



(a)

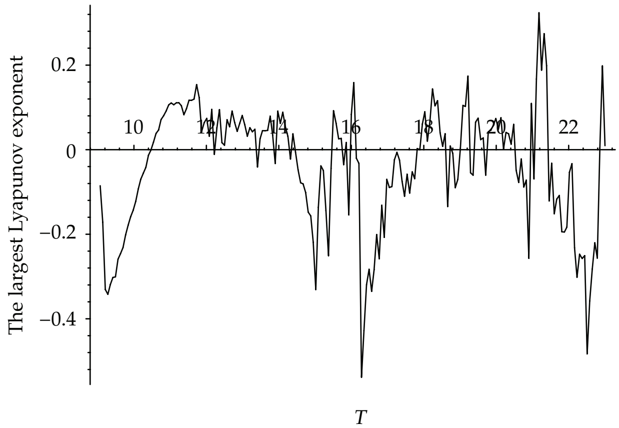


(b)

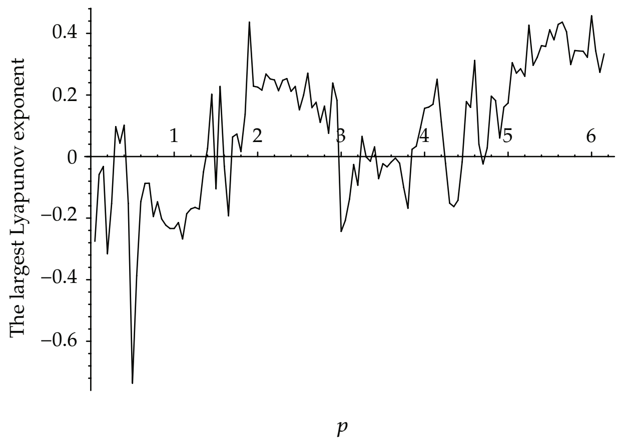
Figure 5: Bifurcation diagrams for system (1.1) showing the effect of p with $T = 20$; we keep other parameters the same. (a) x versus p, (b) y versus p.

conditions for the permanence of the system have been given by comparison theorem when the period of the pulse is larger than critical values. The largest Lyapunov exponent has been used to confirm the existence of chaotic dynamics.

From Theorem 2.6, the algae-eradication periodic solution is globally asymptotically stable when $T < T_{\max}$. Therefore, in order to eradicate the algae population, we can take impulsive control strategy considering the effect caused by the chemical control to the environment and the cost of biological control when $T < T_{\max}$. If we drive the fish population in a small pool or harvest the fish, then chemical poisoning will kill the algae population in large quantities, and the damage to fish population will be very small. If we only choose chemical control strategy ($p = 0$), from Theorem 2.6, the algae population and the fish population will be eradicated when $T < (1/u_c)\ln(1/(1-\delta_1)) \approx 1.03$; in this case, chemical control will not only destroy the biodiversity, but also cause damage to the environment, that is not desirable. If we only choose biological control strategy ($\delta_1 = 0$, $\delta_2 = 0$) and keep other parameters the same, then we have $T < 15.6$; it is clear that biological control will cost



(a)



(b)

Figure 6: (a) The largest Lyapunov exponent (T ranging from 9 to 23) for system (1.1); (b) the largest Lyapunov exponent (p ranging from 0 to 6.2) for system (1.1).

much and take a long time to eradicate the algae population. Therefore, we should combine chemical control with biological control in order to control algal blooms efficiently.

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References

[1] A. Gragnani, M. Scheffer, and S. Rinaldi, "Top-down control of cyanobacteria: a theoretical analysis," *American Naturalist*, vol. 153, no. 1, pp. 59–72, 1999.

[2] R. Semyalo, T. Rohrlack, D. Kayiira et al., “On the diet of Nile tilapia in two eutrophic tropical lakes containing toxin producing cyanobacteria,” *Limnologica*, vol. 41, no. 1, pp. 30–36, 2011.
[3] C. J. Wei and L. S. Chen, “Global dynamics behaviors of viral infection model for pest management,” *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 693472, 16 pages, 2009.
[4] R. Shi and L. Chen, “Stage-structured impulsive SI model for pest management,” *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 97608, 11 pages, 2007.
[5] B. Liu, L. Chen, and Y. Zhang, “The dynamics of a prey-dependent consumption model concerning impulsive control strategy,” *Applied Mathematics and Computation*, vol. 169, no. 1, pp. 305–320, 2005.
[6] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1989.
[7] D. Bainov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Application*, vol. 66 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Science & Technical, Harlow, UK, 1993.
[8] H. K. Baek, “Qualitative analysis of Beddington-DeAngelis type impulsive predator-prey models,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 3, pp. 1312–1322, 2010.
[9] K. Liu and L. Chen, “On a periodic time-dependent model of population dynamics with stage structure and impulsive effects,” *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 389727, 15 pages, 2008.
[10] H. Yu, S. Zhong, and R. P. Agarwal, “Mathematics and dynamic analysis of an apparent competition community model with impulsive effect,” *Mathematical and Computer Modelling*, vol. 52, no. 1-2, pp. 25–36, 2010.
[11] H. Liu and L. Li, “A class age-structured HIV/AIDS model with impulsive drug-treatment strategy,” *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 758745, 12 pages, 2010.
[12] H. Yu, S. Zhong, R. P. Agarwal, and L. Xiong, “Species permanence and dynamical behavior analysis of an impulsively controlled ecological system with distributed time delay,” *Computers and Mathematics with Applications*, vol. 59, no. 12, pp. 3824–3835, 2010.
[13] H. Yu, X. Wang, S. Zhong, and R. P. Agarwal, “Analysis of mathematics and dynamics in a food web system with impulsive perturbations and distributed time delay,” *Applied Mathematical Modelling*, vol. 34, no. 12, pp. 3850–3863, 2010.
[14] Y. Ma, B. Liu, and W. Feng, “Dynamics of a birth-pulse single-species model with restricted toxin input and pulse harvesting,” *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 142534, 20 pages, 2010.
[15] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2 of *Contemporary Mathematics and its Applications*, Hindawi, New York, NY, USA, 2006.
[16] S. T. Zavalishichin and A. N. Sesekin, *Dynamic Impulsive Systems: Theory and Applications*, vol. 394 of *Mathematics and its Applications*, Kluwer Academic, Dordrecht, Germany, 1997.
[17] Q. Cui and G. J. Lawson, “Study on models of single populations: an expansion of the logistic and exponential equations,” *Journal of Theoretical Biology*, vol. 98, no. 4, pp. 645–659, 1982.
[18] F. Grond, H. H. Diebner, S. Sahle, A. Mathias, S. Fischer, and O. E. Rossler, “A robust, locally interpretable algorithm for Lyapunov exponents,” *Chaos, Solitons and Fractals*, vol. 16, no. 5, pp. 841–852, 2003.
[19] M. T. Rosenstein, J. J. Collins, and C. J. De Luca, “A practical method for calculating largest Lyapunov exponents from small data sets,” *Physica D*, vol. 65, no. 1-2, pp. 117–134, 1993.
[20] S. Lv and M. Zhao, “The dynamic complexity of a host-parasitoid model with a lower bound for the host,” *Chaos, Solitons and Fractals*, vol. 36, no. 4, pp. 911–919, 2008.
[21] M. Zhao, H. Yu, and J. Zhu, “Effects of a population floor on the persistence of chaos in a mutual interference host-parasitoid model,” *Chaos, Solitons and Fractals*, vol. 42, no. 2, pp. 1245–1250, 2009.
[22] L. Zhang and M. Zhao, “Dynamic complexities in a hyperparasitic system with prolonged diapause for host,” *Chaos, Solitons and Fractals*, vol. 42, no. 2, pp. 1136–1142, 2009.
[23] S. Lv and M. Zhao, “The dynamic complexity of a three species food chain model,” *Chaos, Solitons and Fractals*, vol. 37, no. 5, pp. 1469–1480, 2008.
[24] L. Zhu and M. Zhao, “Dynamic complexity of a host-parasitoid ecological model with the Hassell growth function for the host,” *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1259–1269, 2009.
[25] M. Zhao and L. Zhang, “Permanence and chaos in a host-parasitoid model with prolonged diapause for the host,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 12, pp. 4197–4203, 2009.