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*Research Article*

# More on Three-Dimensional Systems of Rational Difference Equations

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We are concerned with a kind of three-dimensional system of rational difference equations, given by Kurbanli (2011). A new expression of solution of the system is presented, and the asymptotical behavior is described. At the same time, we also consider a different system and obtain some results, which expand the study of such a kind of difference equations and the method can be applied to other systems.

## 1. Introduction

Difference equations is a hot topic in that they are widely used to investigate equations arising in mathematical models describing real-life situations such as population biology, probability theory, and genetics. Recently, rational difference equations have appealed more and more scholars for their wide application. For details, see [1]. However, there are few literatures on the system of two or three rational difference equations [2–8].

In [2], Kurbanli studied a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1}, \tag{1.1}$$

where the initial conditions are arbitrary real numbers. He expressed the solution of (1.1) and investigated the behavior and computed for some initial values.

The following theorem is cited from [2].

**Theorem 1.1.** *Let $y_0$, $y_{-1}$, $x_0$, $x_{-1}$, $z_0$, $z_{-1} \in \mathbb{R}$ be arbitrary real numbers and $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$, $z_0 = e$, $z_{-1} = f$, and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.1)*. Also, assume* *that* $ad \neq 1$ *and* $bc \neq 1$*; then all solutions of* (1.1) *are*

$$x_n = \begin{cases} \dfrac{d}{(ad-1)^n}, & n \text{ is odd}, \\ c\ (bc-1)^n, & n \text{ is even}, \end{cases} \tag{1.2}$$

$$y_n = \begin{cases} \dfrac{b}{(bc-1)^n}, & n \text{ is odd}, \\ a(ad-1)^n, & n \text{ is even}, \end{cases} \tag{1.3}$$

$$z_n = \begin{cases} \dfrac{f}{(-1)^0(n/0)a^n f d^{n-1} + (-1)^1(n/1)a^{n-1} f d^{n-2} + \cdots + (-1)^{n-1}(n/(n-1))a^1 f d^0 + (-1)^n (n/n)}, \\ \hfill n \text{ is odd}, \\ \dfrac{(-1)^n\ (bc-1)^n\ e}{(-1)^n(n/1)b^1c^0e + \cdots + (-1)^1(n/n)b^n c^{n-1}e + (-1)^0(n/0)b^n c^n + \cdots + (-1)^n(n/n)b^0c^0}, \\ \hfill n \text{ is even}. \end{cases} \tag{1.4}$$

From (1.4), the expression of $z_n$ is so tedious. Although the solution is given, we are so tired to compute for large $n$.

In [2], Kurbanli only considered the asymptotical behavior of $x_n$ and $y_n$, and he has no way to consider that of $z_n$ since its expression (1.4) is too difficult to deal with.

In this paper, first, we give more results of the solution of (1.1) including a new and simple expression of $z_n$ and the asymptotical behavior of the solution. Then, we consider a system similar to (1.1) and obtain some conclusions.

## 2. More Results on the System (1.1)

First, we give another form of the expression of $z_n$.

In fact, (1.4) could be rewritten as

$$z_n = \begin{cases} \dfrac{df}{(ad-1)^k f + (-1)^k (d-f)}, & n = 2k-1, \\ \dfrac{(-1)^k ce}{c - e + \left(e/(1-bc)^k\right)}, & n = 2k, \end{cases} \quad k = 1, 2, \ldots. \tag{2.1}$$

From (2.1), it is easy to check the following:

$$\begin{aligned} z_1 &= \frac{df}{(ad-1)f + (-1)(d-f)} = \frac{f}{af-1}, \\ z_2 &= \frac{(-1)ce}{c-e+(e/(1-bc))} = \frac{e(bc-1)}{be-bc+1}, \\ z_3 &= \frac{df}{(ad-1)^2 f + (-1)^2(d-f)} = \frac{f}{a^2d\ f - 2af + 1}, \end{aligned} \tag{2.2}$$

which are consistent with (1.17) in [2]. The proof is omitted here for the limited space and one could see a similar proof in the next section.

Comparing (2.1) with (1.4), we find that it is not only simple in the form, but also giving more obvious results on the asymptotical behavior of solution of (1.1).

Next, we give the following corollaries.

**Corollary 2.1.** *Suppose that the initial values satisfy $d = f$ and one of the following:*

(i) $0 < ad < 1$, $0 < bc < 1$,

(ii) $1 < ad < 2$, $1 < bc < 2$.

*Then*

$$\begin{gathered}\lim_{k\to\infty}\left(x_{2k-1}, y_{2k-1}, z_{2k-1}\right) = (\infty,\infty,\infty),\\ \lim_{k\to\infty}\left(x_{2k}, y_{2k}, z_{2k}\right) = (0,0,0).\end{gathered} \tag{2.3}$$

**Corollary 2.2.** *Suppose that the initial values satisfy $c = e$ and one of the following:*

(i) $2 < ad < +\infty$, $2 < bc < +\infty$,

(ii) $-\infty < ad < 0$, $-\infty < bc < 0$.

*Then*

$$\begin{gathered}\lim_{k\to\infty}\left(x_{2k-1}, y_{2k-1}, z_{2k-1}\right) = (0,\ 0,0),\\ \lim_{k\to\infty}\left(x_{2k},\ y_{2k},\ z_{2k}\right) = (\infty,\infty,\infty).\end{gathered} \tag{2.4}$$

**Corollary 2.3.** *Suppose that the initial values satisfy $a = e \neq 0$, $b = f \neq 0$, and $ad = bc = 2$. Then*

$$\begin{gathered}\lim_{k\to\infty}\left(x_{2k-1}, y_{2k-1}, z_{2k-1}\right) = (d,b,b),\\ \lim_{k\to\infty}\left(x_{2k}, y_{2k}, z_{2k}\right) = (c,a,a).\end{gathered} \tag{2.5}$$

Such results expand those in [2], where the behavior of $z_n$ could not be obtained from its expression. The proof is similar to that in the next section and we omit it here.

## 3. Main Results

Motivated by [2] and other references, such as [3–8] and the references cited therein, we consider the following system:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{z_{n-1}}{x_n z_{n-1} - 1}. \tag{3.1}$$

Here, the last equation is different from that of (1.1).

Through the paper, we suppose the initial values to be

$$y_0 = a, \qquad x_0 = c, \qquad z_0 = e, \qquad y_{-1} = b, \qquad x_{-1} = d, \qquad z_{-1} = f. \tag{3.2}$$

Here, $a,\ b,\ c,$ d, $e$, and $f$ are nonzero real numbers such that $ad \neq 1$ and $bc \neq 1$. We call this hypothesis $H$).

Is the solution of (3.1) similar to that of (1.1)? The following theorem confirms this.

**Theorem 3.1.** *Suppose that hypothesis* $(H)$ *holds, and let* $(x_n,\ y_n,\ z_n)$ *be a solution of the system* (3.1)*. Then all solutions of* (3.1) *are*

$$x_n = \begin{cases} \dfrac{d}{(ad-1)^n}, & n = 2k-1, \\ c(bc-1)^n, & n = 2k, \end{cases}$$
$$y_n = \begin{cases} \dfrac{b}{(bc-1)^n}, & n = 2k-1, \\ a(ad-1)^n, & n = 2k, \end{cases}$$
$$z_n = \begin{cases} \dfrac{bf}{(bc-1)^k f + (-1)^k (b-f)}, & n = 2k-1, \\ \dfrac{(-1)^k ae}{a - e + \left(e/(1-ad)^k\right)}, & n = 2k \end{cases} \tag{3.3}$$

*for* $k = 1, 2, \ldots.$

*Proof.* First, for $k = 1, 2$, from (3.1), we easily check that

$$\begin{aligned} x_1 &= \frac{d}{ad-1}, \\ y_1 &= \frac{b}{bc-1}, \\ z_1 &= \frac{z_{-1}}{x_0 z_{-1} - 1} = \frac{f}{cf-1} = \frac{bf}{(bc-1)f - (b-f)}, \\ x_2 &= c(bc-1), \\ y_2 &= a(ad-1), \\ z_2 &= \frac{z_0}{x_1 z_0 - 1} = \frac{e(ad-1)}{de - ad + 1} = \frac{-1ae}{a - e + (e/(1-ad))}, \\ x_3 &= \frac{d}{(ad-1)^2}, \end{aligned}$$

$$y_3 = \frac{b}{(bc-1)^2},$$
$$z_3 = \frac{z_1}{x_2 z_1 - 1} = \frac{f}{bc^2 f - 2cf + 1} = \frac{bf}{(bc-1)f - (b-f)},$$
$$x_4 = c(bc-1)^2,$$
$$y_4 = a(ad-1)^2,$$
$$z_4 = \frac{z_2}{x_3 z_2 - 1} = \frac{e(ad-1)^2}{(2d - a^2 d)e + \left((ad-1)^2\right.} = \frac{(-1)^2 ae}{a - e + \left(e/(1-ad)^2\right)}. \tag{3.4}$$

Next, we assume the conclusion is true for $k$, that is, (3.3) hold.

Then, for $k+1$, from (3.1) and (3.3), we have

$$x_{2(k+1)-1} = \frac{d}{(ad-1)^{k+1}},$$
$$y_{2(k+1)-1} = \frac{b}{(bc-1)^{k+1}},$$
$$z_{2(k+1)-1} = \frac{z_{2k-1}}{x_{2k} z_{2k-1} - 1} = \frac{bf}{(bc-1)^k f + (-1)^k (b-f)} \times \frac{1}{bfc(bc-1)^k / \left((bc-1)^k f + (-1)^k (b-f)\right)}$$
$$= \frac{bf}{(bc-1)^{k+1} f + (-1)^{k+1} (b-f)},$$
$$x_{2(k+1)} = c(bc-1)^{k+1},$$
$$y_{2(k+1)} = a(ad-1)^{k+1},$$
$$z_{2(k+1)} = \frac{z_{2k}}{x_{2k+1} z_{2k} - 1} = \frac{(-1)^k a e}{a - e + \left(e/(1-ad)^k\right)} \times \frac{1}{\left(d/(ad-1)^k\right)\left((-1)^k ae / \left(a - e + \left(e/(1-ad)^k\right)\right)\right) - 1}$$
$$= (-1)^{k+1} ae / \left(a - e + \left(e/(1-ad)^{k+1}\right)\right), \tag{3.5}$$

which complete the proof. □

From the above theorem, such a simple expression of the solution of (3.1) will greatly help us to investigate the behavior of the solution.

**Corollary 3.2.** *Suppose that hypothesis* $(H)$, $b = f$, *and one of the following hold:*

(i) $0 < ad < 1$, $0 < bc < 1$;

(ii) $1 < ad < 2$, $1 < bc < 2$.

*Then*

$$\lim_{k\to\infty}\left(x_{2k-1}, y_{2k-1}, z_{2k-1}\right) = (\infty,\infty,\infty),$$
$$\lim_{k\to\infty}\left(x_{2k}, y_{2k}, z_{2k}\right) = (0,0,0). \tag{3.6}$$

*Proof.* First, for $2k-1$, we consider the following two cases.

(1) Assume that (i) holds; then $-1 < ad-1 < 0$, $-1 < bc-1 < 0$.

From (3.3), we have

$$\lim_{k\to\infty} x_{2k-1} = \lim_{k\to\infty}\frac{d}{(ad-1)^k} = \begin{cases} +\infty, & d > 0,\ k \text{ is even},\\ +\infty, & d < 0,\ k \text{ is odd},\\ -\infty, & d > 0,\ k \text{ is odd},\\ -\infty, & d < 0,\ k \text{ is even},\end{cases}$$
$$\lim_{k\to\infty} y_{2k-1} = \lim_{k\to\infty}\frac{b}{(bc-1)^k} = \begin{cases} +\infty, & b > 0,\ k \text{ is even},\\ +\infty, & b < 0,\ k \text{ is odd},\\ -\infty, & b > 0,\ k \text{ is odd},\\ -\infty, & b < 0,\ k \text{ is even},\end{cases} \tag{3.7}$$
$$\lim_{k\to\infty} z_{2k-1} = \lim_{k\to\infty}\frac{bf}{(bc-1)^k f + (-1)^k (b-\mathrm{f})} = \begin{cases} +\infty, & b > 0,\ k \text{ is even},\\ +\infty, & b < 0,\ k \text{ is odd},\\ -\infty, & b > 0,\ k \text{ is odd},\\ -\infty, & b < 0,\ k \text{ is even},\end{cases}$$

where the last equation is from $b = f$.

(2) Assume that (ii) holds; then $0 < ad - 1 < 1$, $0 < bc - 1 < 1$. Similarly, we have

$$\begin{aligned}
\lim_{k\to\infty} x_{2k-1} &= \lim_{k\to\infty} \frac{d}{(ad-1)^k} = \begin{cases} +\infty, & d > 0, \\ -\infty, & d < 0, \end{cases} \\
\lim_{k\to\infty} y_{2k-1} &= \lim_{k\to\infty} \frac{b}{(bc-1)^k} = \begin{cases} +\infty, & b > 0, \\ -\infty, & b < 0, \end{cases} \\
\lim_{k\to\infty} z_{2k-1} &= \lim_{k\to\infty} \frac{bf}{(bc-1)^k f + (-1)^k (b-f)} = \begin{cases} +\infty, & b > 0, \\ -\infty, & b < 0. \end{cases}
\end{aligned} \tag{3.8}$$

Next, for $2k$, we always have

$$\begin{aligned}
\lim_{k\to\infty} x_{2k} &= \lim_{k\to\infty} c(bc-1)^k = 0, \\
\lim_{k\to\infty} y_{2k} &= \lim_{k\to\infty} a(ad-1)^k = 0, \\
\lim_{k\to\infty} z_{2k} &= \lim_{k\to\infty} \frac{(-1)^k ae}{a - e + \left(e/(1-ad)^k\right)} = 0,
\end{aligned} \tag{3.9}$$

and complete the proof. □

**Corollary 3.3.** *Suppose that hypothesis $(H)$, $a = e$, and one of the following hold:*

(i) $2 < ad < +\infty$, $2 < bc < +\infty$,

(ii) $-\infty < ad < 0$, $-\infty < bc < 0$.

*Then*

$$\begin{aligned}
\lim_{k\to\infty} \left(x_{2k-1}, y_{2k-1}, z_{2k-1}\right) &= (0,0,0), \\
\lim_{k\to\infty} \left(x_{2k}, y_{2k}, z_{2k}\right) &= (\infty,\infty,\infty).
\end{aligned} \tag{3.10}$$

*Proof.* First, for $2k-1$, in view of (i) or (ii), we have $|ad-1| > 1$, $|bc-1| > 1$ and thus

$$\begin{aligned}
\lim_{k\to\infty} x_{2k-1} &= \lim_{k\to\infty} \frac{d}{(ad-1)^k} = 0, \\
\lim_{k\to\infty} y_{2k-1} &= \lim_{k\to\infty} \frac{b}{(bc-1)^k} = 0, \\
\lim_{k\to\infty} z_{2k-1} &= \lim_{k\to\infty} \frac{bf}{(bc-1)^k f + (-1)^k (b-f)} = 0.
\end{aligned} \tag{3.11}$$

Now, for $2k$, we consider the following two cases.

(1) Assume that (i) holds; then $1 < ad - 1 < +\infty$, $1 < bc - 1 < +\infty$. From (3.3), we have

$$\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} c(bc-1)^k = \begin{cases} +\infty, & c > 0, \\ -\infty, & c < 0, \end{cases}$$
$$\lim_{k\to\infty} y_{2k} = \lim_{k\to\infty} a(ad-1)^k = \begin{cases} +\infty, & a > 0, \\ -\infty, & a < 0, \end{cases} \tag{3.12}$$
$$\lim_{k\to\infty} z_{2k} = \lim_{k\to\infty} \frac{(-1)^k\ a\ e}{a - e + \left(e/(1-ad)^k\right)} = \begin{cases} +\infty, & a > 0, \\ -\infty, & a < 0, \end{cases}$$

where the last equation is from $a = e$.

(2) Assume that (ii) holds; then $-\infty < ad - 1 < -1$, $-\infty < bc - 1 < -1$. Similarly, we have

$$\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} c(bc-1)^k = \begin{cases} +\infty, & c > 0,\ k\text{-even}, \\ +\infty, & c < 0,\ k\text{-odd}, \\ -\infty, & c > 0,\ k\text{-odd}, \\ -\infty, & c < 0,\ k\text{-even}, \end{cases}$$
$$\lim_{k\to\infty} y_{2k} = \lim_{k\to\infty} a(ad-1)^k = \begin{cases} +\infty, & a > 0,\ k\text{-even}, \\ +\infty, & a < 0,\ k\text{-odd}, \\ -\infty, & a > 0,\ k\text{-odd}, \\ -\infty, & a < 0,\ k\text{-even}, \end{cases} \tag{3.13}$$
$$\lim_{k\to\infty} z_{2k} = \lim_{k\to\infty} \frac{(-1)^k ae}{a - e + \left(e/(1-ad)^k\right)} = \begin{cases} +\infty, & a > 0,\ k\text{-even}, \\ +\infty, & a < 0,\ k\text{-odd}, \\ -\infty, & a > 0,\ k\text{-odd}, \\ -\infty, & a < 0,\ k\text{-even}, \end{cases}$$

and complete the proof. □

**Corollary 3.4.** *Suppose that hypothesis $(H)$ holds and $a = e \neq 0$, $b = f \neq 0$, and $ad = bc = 2$. Then*

$$\lim_{k\to\infty} \left(x_{2k-1}, y_{2k-1}, z_{2k-1}\right) = (d, b, b),$$
$$\lim_{k\to\infty} \left(x_{2k}, y_{2k}, z_{2k}\right) = (c, a, a). \tag{3.14}$$

The proof is simple and we omit it here. From this theorem, we can see that $y_n = z_n$ for such initial values.

## 4. Conclusion

It is popular to study kinds of difference equations. The results can be divided into two parts. On the one hand, by linear stability theorem, one could study the behavior of solutions. Such a method is widely used to deal with a single difference equation; See [1]. On the other hand, the exact expression of solutions with respect to some difference equations is given. Generally speaking, it is difficult to obtain such an expression and to apply to other systems.

On a system consisting of two or three rational difference equations, there are few literatures. For details, see [2–8] and the references cited therein. In these papers, the exact expressions of solution are given.

In this paper, we expand the results obtained by Kurbanli in [2] and also investigate the behavior of the solution. At the same time, we consider a similar system and give some related results. The method can be applied to other kinds of difference equations.

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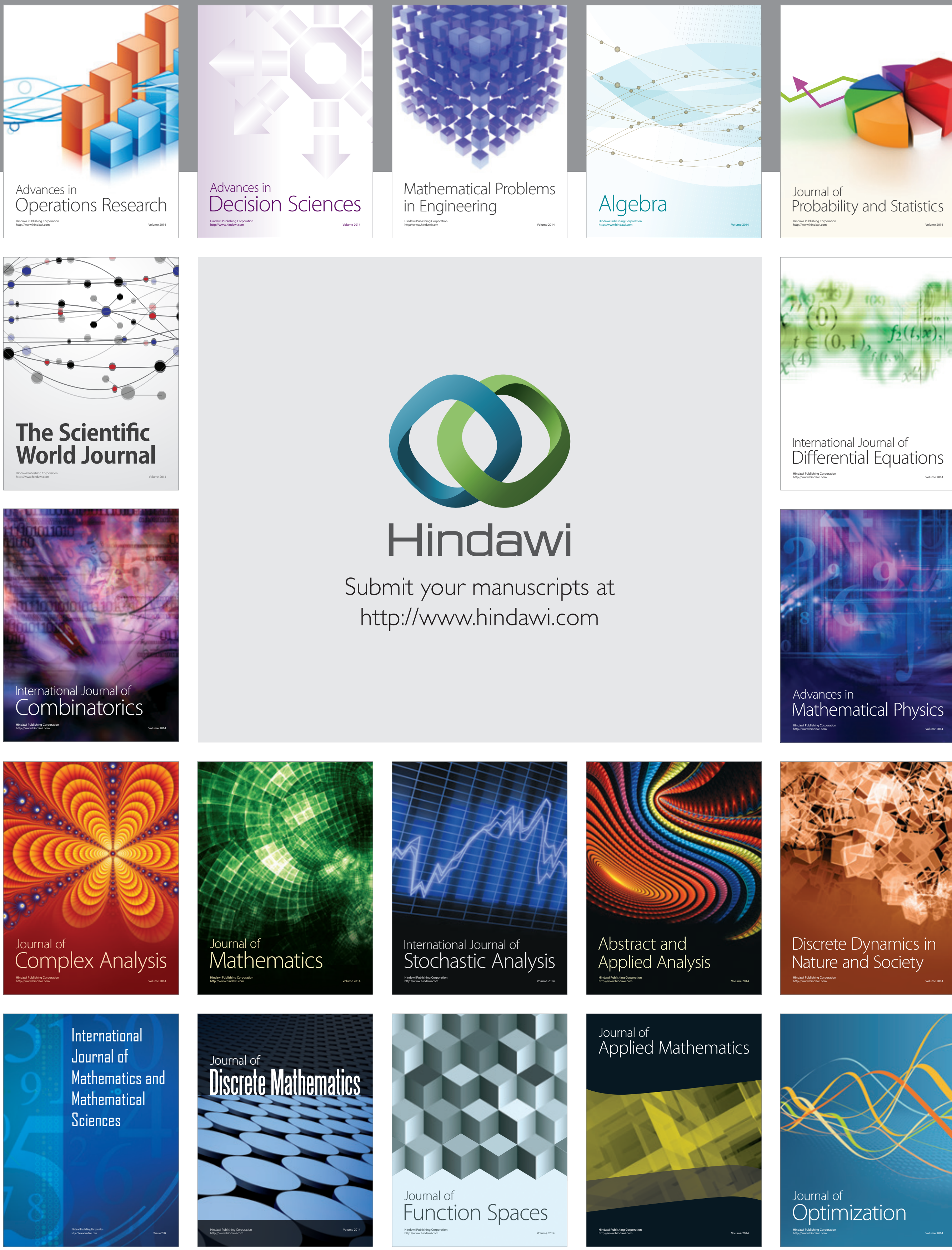

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