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Research Article

A Note on the Inverse Moments for Nonnegative ρ-Mixing Random Variables

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Wu et al. (2009) studied the asymptotic approximation of inverse moments for nonnegative independent random variables. Shen et al. (2011) extended the result of Wu et al. (2009) to the case of ρ-mixing random variables. In the paper, we will further study the asymptotic approximation of inverse moments for nonnegative ρ-mixing random variables, which improves the corresponding results of Wu et al. (2009), Wang et al. (2010), and Shen et al. (2011) under the case of identical distribution.

1. Introduction

Firstly, we will recall the definition of ρ-mixing random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ and $\mathcal{F}_S = \sigma(X_i, i \in S \subset \mathbb{N})$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|EXY - EXEY|}{\sqrt{\operatorname{Var}(X) \cdot \operatorname{Var}(Y)}}. \tag{1.1}$$

Define the ρ-mixing coefficients by

$$\rho(n) = \sup_{k \geq 1} \rho\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty\right), \quad n \geq 0. \tag{1.2}$$

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be ρ-mixing if $\rho(n) \downarrow 0$ as $n \to \infty$.

ρ-mixing sequence was introduced by Kolmogorov and Rozanov [1]. It is easily seen that ρ-mixing sequence contains independent sequence as a special case.

The main purpose of the paper is to study the asymptotic approximation of inverse moments for nonnegative ρ-mixing random variables with identical distribution.

Let $\{Z_n, n \geq 1\}$ be a sequence of independent nonnegative random variables with finite second moments. Denote

$$X_n = \frac{\sum_{i=1}^{n} Z_i}{B_n}, \qquad B_n^2 = \sum_{i=1}^{n} \operatorname{Var} Z_i. \tag{1.3}$$

It is interesting to show that under suitable conditions the following equivalence relation holds, namely,

$$E(a + X_n)^{-r} \sim (a + EX_n)^{-r}, \quad n \longrightarrow \infty, \tag{1.4}$$

where $a > 0$ and $r > 0$ are arbitrary real numbers.

Here and below, for two positive sequences $\{c_n, n \geq 1\}$ and $\{d_n, n \geq 1\}$, we write $c_n \sim d_n$ if $c_n d_n^{-1} \to 1$ as $n \to \infty$. C is a positive constant which can be different in various places.

The inverse moments can be applied in many practical applications. For example, they may be applied in Stein estimation and poststratification (see [2, 3]), evaluating risks of estimators and powers of tests (see [4, 5]). In addition, they also appear in the reliability (see [6]) and life testing (see [7]), insurance and financial mathematics (see [8]), complex systems (see [9]), and so on.

Under certain asymptotic-normality condition, relation (1.4) was established in Theorem 2.1 of Garcia and Palacios [10]. But, unfortunately, that theorem is not true under the suggested assumptions, as pointed out by Kaluszka and Okolewski [11]. The latter authors established (1.4) by modifying the assumptions as follows:

(i) $r < 3$ ($r < 4$, in the i.i.d. case);

(ii) $EX_n \to \infty$, $EZ_n^3 < \infty$;

(iii) (L_c condition) $\sum_{i=1}^{n} E|Z_i - EZ_i|^c / B_n^c \to 0$ $(c = 3)$.

Hu et al. [12] considered weaker conditions: $EZ_n^{2+\delta} < \infty$, where Z_n satisfies $L_{2+\delta}$ condition and $0 < \delta \leq 1$. Wu et al. [13] applied Bernstein's inequality and the truncated method to greatly improve the conclusion in weaker condition on moment. Wang et al. [14] extended the result for independent random variables to the case of NOD random variables. Shi et al. [15] obtained (1.4) for $B_n = 1$. Sung [16] studied the inverse moments for a class of nonnegative random variables.

Recently, Shen et al. [17] extended the result of Wu et al. [13] to the case of ρ-mixing random variables and obtained the following result.

Theorem A. *Let $\{Z_n, n \geq 1\}$ be a nonnegative ρ-mixing sequence with $\sum_{n=1}^{\infty} \rho(n) < \infty$. Suppose that*

(i) $EZ_n^2 < \infty$, *for all* $n \geq 1$;

(ii) $EX_n \to \infty$, *where* X_n *is defined by* (1.3);

(iii) *for some* $\eta > 0$,

$$R_n(\eta) := B_n^{-2}\sum_{i=1}^{n} EZ_i^2 I(Z_i > \eta B_n) \longrightarrow 0, \quad n \longrightarrow \infty; \tag{1.5}$$

(iv) *for some* $t \in (0,1)$ *and any positive constants* a, r, C,

$$\lim_{n\to\infty} (a + EX_n)^r \cdot \exp\left\{-C \cdot \frac{(EX_n)^t}{n}\right\} = 0. \tag{1.6}$$

Then for any $a > 0$ *and* $r > 0$, (1.4) *holds.*

In this paper, we will further study the asymptotic approximation of inverse moments for nonnegative ρ-mixing random variables with identical distribution. We will show that (1.4) holds under very mild conditions and the condition (iv) in Theorem A can be deleted. In place of the Bernstein type inequality used by Shen et al. [17], we make the use of Rosenthal type inequality of ρ-mixing random variables. Our main results are as follows.

Theorem 1.2. *Let* $\{Z_n, n \geq 1\}$ *be a sequence of nonnegative* ρ*-mixing random variables with identical distribution and let* $\{B_n, n \geq 1\}$ *be a sequence of positive constants. Let* $a > 0$ *and* $\alpha > 0$ *be real numbers.* $p > \max\{2, 2\alpha, \alpha + 1\}$. *Assume that* $\sum_{n=1}^{\infty} \rho^{2/p}(2^n) < \infty$. *Suppose that*

(i) $0 < EZ_n < \infty$, *for all* $n \geq 1$;

(ii) $\mu_n \doteq EX_n \to \infty$ *as* $n \to \infty$, *where* $X_n = B_n^{-1}\sum_{k=1}^{n} Z_k$;

(iii) *for all* $0 < \varepsilon < 1$, *there exist* $b > 0$ *and* $n_0 > 0$ *such that*

$$EZ_1 I(Z_1 > bB_n) \leq \varepsilon EZ_1, \quad n \geq n_0. \tag{1.7}$$

Then (1.4) *holds.*

Corollary 1.3. *Let* $\{Z_n, n \geq 1\}$ *be a sequence of nonnegative* ρ*-mixing random variables with identical distribution and* $0 < EZ_1 < \infty$. *Let* $\{B_n, n \geq 1\}$ *be a sequence of positive constants satisfying* $B_n = O(n^{\delta})$ *for some* $0 < \delta < 1$ *and* $B_n \to \infty$ *as* $n \to \infty$. *Let* $a > 0$ *and* $\alpha > 0$ *be real numbers.* $p > \max\{2, 2\alpha, \alpha + 1\}$. *Assume that* $\sum_{n=1}^{\infty} \rho^{2/p}(2^n) < \infty$. *Then* (1.4) *holds.*

By Theorem 1.2, we can get the following convergence rate of relative error in the relation (1.4).

Theorem 1.4. *Assume that conditions of Theorem* 1.2 *are satisfied and* $0 < EZ_n^2 < \infty$. $p > \max\{2, 4(\alpha + 1), 2\alpha + 3\}$. *If* $B_n \geq Cn^{1/2}$ *for all* n *large enough, where* C *is a positive constant, then*

$$\left|(a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} - 1\right| = O\left((a + EX_n)^{-1}\right). \tag{1.8}$$

Theorem 1.5. *Assume that conditions of Theorem* 1.2 *are satisfied and* $0 < EZ_n^2 < \infty$. $p > \max\{2, 4(\alpha + 1), 2\alpha + 3\}$. *Then*

$$\left|(a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} - 1\right| = O\left(n^{-1/2}\right). \tag{1.9}$$

Taking $B_n \equiv 1$ in Theorem 1.2, we have the following asymptotic approximation of inverse moments for the partial sums of nonnegative ρ-mixing random variables with identical distribution.

Theorem 1.6. *Let $\{Z_n, n \geq 1\}$ be a sequence of nonnegative ρ-mixing random variables with identical distribution. Let $a > 0$ and $\alpha > 0$ be real numbers. $p > \max\{2, 2\alpha, \alpha + 1\}$. Assume that $\sum_{n=1}^{\infty} \rho^{2/p}(2^n) < \infty$. Suppose that*

(i) $0 < EZ_n < \infty, \forall\, n \geq 1$;

(ii) $\nu_n \doteq EY_n \to \infty$ *as* $n \to \infty$, *where* $Y_n = \sum_{k=1}^{n} Z_k$;

(iii) *for all* $0 < \varepsilon < 1$, *there exist* $b > 0$ *and* $n_0 > 0$ *such that*

$$EZ_1 I(Z_1 > b) \leq \varepsilon E Z_1, \quad n \geq n_0. \tag{1.10}$$

Then $E(a + Y_n)^{-\alpha} \sim (a + EY_n)^{-\alpha}$.

Remark 1.7. Theorem 1.2 in this paper improves the corresponding results of Wu et al. [13], Wang et al. [14], and Shen et al. [17]. Firstly, Theorem 1.4 in this paper is based on the condition $EZ_n < \infty$, for all $n \geq 1$, which is weaker than the condition $EZ_n^2 < \infty$, for all $n \geq 1$ in the above cited references. Secondly, $\{B_n,\ n \geq 1\}$ is an arbitrary sequence of positive constants in Theorem 1.2, while $B_n^2 = \sum_{i=1}^{n} \operatorname{Var} Z_i$ in the above cited references. Thirdly, the condition (iv) in Theorem A is not needed in Theorem 1.2. Finally, (1.7) is weaker than (1.5) under the case of identical distribution. Actually, by the condition (1.5), we can see that

$$B_n^{-1} \sum_{i=1}^{n} EZ_i I(Z_i > \eta B_n) \leq \eta^{-1} B_n^{-2} \sum_{i=1}^{n} EZ_i^2 I(Z_i > \eta B_n) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{1.11}$$

which implies that for all $0 < \varepsilon < 1$, there exists a positive integer n_0 such that

$$B_n^{-1} \sum_{i=1}^{n} EZ_i I(Z_i > \eta B_n) \leq \varepsilon \mu_n = \varepsilon B_n^{-1} \sum_{i=1}^{n} EZ_i, \quad n \geq n_0, \tag{1.12}$$

that is, (1.7) holds.

2. Proof of the Main Results

In order to prove the main results of the paper, we need the following important moment inequality for ρ-mixing random variables.

Lemma 2.1 (c.f. Shao [18, Corollary 1.1]). *Let $q \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of ρ-mixing random variables. Assume that $EX_n = 0$, $E|X_n|^q < \infty$ and*

$$\sum_{n=1}^{\infty} \rho^{2/q}(2^n) < \infty. \tag{2.1}$$

Then there exists a positive constant $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that for any $k \geq 0$ and $n \geq 1$,

$$E\left(\max_{1\leq i\leq n}|S_k(i)|^q\right) \leq K\left[\left(n \max_{k<i\leq k+n} EX_i^2\right)^{q/2} + n \max_{k<i\leq k+n} E|X_i|^q\right], \tag{2.2}$$

where $S_k(i) = \sum_{j=k+1}^{k+i} X_j$, $k \geq 0$ and $i \geq 1$.

Remark 2.2. We point out that if $\{X_n, n \geq 1\}$ is a sequence of ρ-mixing random variables with identical distribution and the conditions of Lemma 2.1 hold, then we have

$$\begin{aligned} E\left(\max_{1\leq i\leq n}|S_k(i)|^q\right) &\leq K\left[\left(nEX_1^2\right)^{q/2} + nE|X_1|^q\right], \\ E\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} X_j\right|^q\right) &\leq K\left[\left(nEX_1^2\right)^{q/2} + nE|X_1|^q\right] \\ &= K\left[\left(\sum_{j=1}^{n} EX_j^2\right)^{q/2} + \sum_{j=1}^{n} E\left|X_j\right|^q\right]. \end{aligned} \tag{2.3}$$

The inequality above is the Rosenthal type inequality of identical distributed ρ-mixing random variables.

Proof of Theorem 1.2. It is easily seen that $f(x) = (a + x)^{-\alpha}$ is a convex function of x on $[0, \infty)$, therefore, we have by Jensen's inequality that

$$E(a + X_n)^{-\alpha} \geq (a + EX_n)^{-\alpha}, \tag{2.4}$$

which implies that

$$\liminf_{n\to\infty}(a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} \geq 1. \tag{2.5}$$

To prove (1.4), it is enough to prove that

$$\limsup_{n\to\infty}(a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} \leq 1. \tag{2.6}$$

In order to prove (2.6), we need only to show that for all $\delta \in (0, 1)$,

$$\limsup_{n\to\infty}(a + EX_n)^{\alpha} E(a + X_n)^{-\alpha} \leq (1 - \delta)^{-\alpha}. \tag{2.7}$$

By (iii), we can see that for all $\delta \in (0, 1)$,

$$EZ_1 I(Z_1 > bB_n) \leq \frac{\delta}{2} EZ_1, \quad n \geq n_0. \tag{2.8}$$

Let

$$U_n = B_n^{-1} \sum_{k=1}^{n} Z_k I(Z_k \le bB_n), \tag{2.9}$$

$$\begin{aligned} E(a+X_n)^{-\alpha} &= E(a+X_n)^{-\alpha} I\left(U_n \ge \mu_n - \delta\mu_n\right) + E(a+X_n)^{-\alpha} I\left(U_n < \mu_n - \delta\mu_n\right) \\ &\doteq Q_1 + Q_2. \end{aligned} \tag{2.10}$$

For Q_1, since $X_n \ge U_n$, we have

$$Q_1 \le E(a+X_n)^{-\alpha} I\left(X_n \ge \mu_n - \delta\mu_n\right) \le \left(a + \mu_n - \delta\mu_n\right)^{-\alpha}. \tag{2.11}$$

By (2.8), we have for $n \ge n_0$ that

$$\mu_n - EU_n = B_n^{-1} \sum_{k=1}^{n} EZ_k I(Z_k > bB_n) \le \frac{\delta\mu_n}{2}. \tag{2.12}$$

Therefore, by (2.12), Markov's inequality, Remark 2.2 and C_r's inequality, for any $p > 2$ and all n sufficiently large,

$$\begin{aligned} Q_2 &\le a^{-\alpha} P\left(U_n < \mu_n - \delta\mu_n\right) \\ &= a^{-\alpha} P\left(EU_n - U_n > \delta\mu_n - \left(\mu_n - EU_n\right)\right) \\ &\le a^{-\alpha} P\left(EU_n - U_n > \frac{\delta\mu_n}{2}\right) \\ &\le a^{-\alpha} P\left(|U_n - EU_n| > \frac{\delta\mu_n}{2}\right) \le C\mu_n^{-p} E|U_n - EU_n|^p \\ &\le C\mu_n^{-p} \left[B_n^{-2} nEZ_1^2 I(Z_1 \le bB_n)\right]^{p/2} + C\mu_n^{-p} \left[B_n^{-p} nEZ_1^p I(Z_1 \le bB_n)\right] \\ &\le C\mu_n^{-p} \left[B_n^{-1} nEZ_1 I(Z_1 \le bB_n)\right]^{p/2} + C\mu_n^{-p} B_n^{-1} nEZ_1 I(Z_1 \le bB_n) \\ &\le C\mu_n^{-p} \left(\mu_n^{p/2} + \mu_n\right) = C\left(\mu_n^{-p/2} + \mu_n^{-(p-1)}\right). \end{aligned} \tag{2.13}$$

Taking $p > \max\{2, 2\alpha, \alpha+1\}$, we have by (2.10), (2.11), and (2.13) that

$$\begin{aligned} &\limsup_{n\to\infty} \left(a+\mu_n\right)^{\alpha} E(a+X_n)^{-\alpha} \\ &\quad \le \limsup_{n\to\infty} \left(a+\mu_n\right)^{\alpha} \left(a+\mu_n-\delta\mu_n\right)^{-\alpha} + \limsup_{n\to\infty} \left(a+\mu_n\right)^{\alpha} \left[C\mu_n^{-p/2} + C\mu_n^{-(p-1)}\right] \\ &\quad = (1-\delta)^{-\alpha}, \end{aligned} \tag{2.14}$$

which implies (2.7). This completes the proof of the theorem. □

Proof of Corollary 1.3. The condition $B_n = O(n^\delta)$ for some $0 < \delta < 1$ implies that

$$\mu_n \doteq EX_n = B_n^{-1}\sum_{k=1}^{n} EZ_k = nB_n^{-1}EZ_1, \tag{2.15}$$

thus, $\mu_n \ge Cn^{1-\delta} \to \infty$ as $n \to \infty$.

The fact $0 < EZ_1 < \infty$ and $B_n \to \infty$ yield that $EZ_1 I(Z_1 > bB_n) \to 0$ as $n \to \infty$, which implies that for all $0 < \varepsilon < 1$, there exists $n_0 > 0$ such that

$$EZ_1 I(Z_1 > bB_n) \le \varepsilon EZ_1, \quad n \ge n_0. \tag{2.16}$$

That is to say condition (iii) of Theorem 1.2 holds. Therefore, the desired result follows from Theorem 1.2 immediately. □

Proof of Theorem 1.4. Firstly, we will examine $\operatorname{Var} X_n$. By Remark 2.2, $0 < EZ_1^2 < \infty$ and the condition $B_n \ge Cn^{1/2}$ for all n large enough, we can get that

$$\operatorname{Var} X_n = B_n^{-2}\operatorname{Var}\left(\sum_{i=1}^{n} Z_i\right) \le B_n^{-2}E\left(\sum_{i=1}^{n} Z_i\right)^2 \le CnB_n^{-2}EZ_1^2 \le C_1 \tag{2.17}$$

for all n large enough.

Denote $\phi(x) = (a+x)^{-\alpha}$ for $x \ge 0$. By Taylor's expansion, we can see that

$$\phi(X_n) = \phi(EX_n) + \phi'(\xi_n)(X_n - EX_n), \tag{2.18}$$

where ξ_n is between X_n and EX_n. It is easily seen that $\{\phi'(x)\}^2$ is decreasing in $x \ge 0$. Therefore, by (2.18), Cauchy-Schwartz inequality, (2.17) and (1.4), we have

$$\begin{aligned}
\left[E\phi(X_n) - \phi(EX_n)\right]^2 &= E\left[\phi'(\xi_n)(X_n - EX_n)\right]^2 \\
&\le E\left[\phi'(\xi_n)\right]^2 \operatorname{Var} X_n \le C_1 E\left[\phi'(\xi_n)\right]^2 \\
&= C_1 E\left[\phi'(\xi_n)\right]^2 I(X_n \le EX_n) + C_1 E\left[\phi'(\xi_n)\right]^2 I(X_n > EX_n) \\
&\le C_1 E\left[\phi'(X_n)\right]^2 + C_1 E\left[\phi'(EX_n)\right]^2 \\
&\sim 2C_1 E\left[\phi'(EX_n)\right]^2 = 2C_1\alpha^2(a + EX_n)^{-2(\alpha+1)}.
\end{aligned} \tag{2.19}$$

This leads to (1.8). The proof is complete. □

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.4. In place of $\operatorname{Var} X_n \le C_1$, we make the use of $\operatorname{Var} X_n \le CnB_n^{-2}EZ_1^2 \doteq C_2 nB_n^{-2}$. The proof is complete. □

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