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Research Article

Dynamics of a Nonautonomous Leslie-Gower Type Food Chain Model with Delays

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A nonautonomous Leslie-Gower type food chain model with time delays is investigated. It is proved the general nonautonomous system is permanent and globally asymptotically stable under some appropriate conditions. Furthermore, if the system is periodic one, some sufficient conditions are established, which guarantee the existence, uniqueness, and global asymptotic stability of a positive periodic solution of the system. The conditions for the permanence, global stability of system, and the existence, uniqueness of positive periodic solution depend on delays; so, time delays are profitless.

1. Introduction

Among the relationships between the species living in the same outer environment, the predator-prey theory plays an important and fundamental role. The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. Food-chain predator-prey system, as one of the most important predator-prey system, has been extensively studied by many scholars, many excellent results concerned with the permanent property and positive periodic solution of the system, see [2–4] and the references cited therein. Recently, Nindjin and Aziz-Alaoui [5] proposed the following autonomous delayed predator-prey model with modified Leslie-Gower functional response

$$\dot{x}_1(t) = x_1(t)\left(a_1 - a_{11}x_1(t-\tau_{11}) - \frac{a_{12}x_2(t)}{x_1(t)+d_1}\right),$$

$$\dot{x}_2(t) = x_2(t)\left(-a_2 + \frac{a_{21}x_1(t-\tau_{21})}{x_1(t-\tau_{21})+d_1} - a_{22}x_2(t-\tau_{22}) - \frac{a_{23}x_3(t)}{x_2(t)+d_2}\right),$$
$$\dot{x}_3(t) = x_3(t)\left(a_3 - \frac{a_{33}x_3(t-\tau_{33})}{x_2(t-\tau_{32})+d_3}\right), \tag{1.1}$$

system (1) represents an ecological situation where a prey population x_1 is the only food for a predator x_2. This specialist predator x_2, in turns, serves as the prey of a top-predator x_3. The interaction between species x_2 and its prey x_1 has been modeled by the Volterra scheme. But, the interaction between species x_3 and its prey x_2 has been modeled by a modified version of Leslie-Gower scheme. About Leslie-Gower scheme one could refer to [6–13] and the references cited therein.

In [5], the authors showed that the system is uniformly persistent under some appropriate conditions and obtained sufficient conditions for global stability of the positive equilibrium of system (1).

We note that any biological or environment parameters are naturally subject to fluctuation in time, and if a model is to take into account such fluctuation then the model must be nonautonomous. On the other hand, time delays occur so often in nature, a number of models can be formulated as systems of differential equations with time delays (see, e.g., [2, 14, 15] and the references cited therein). Motivated by above considerations, in this paper, we consider the following general nonautonomous Leslie-Gower type food chain model with time delays of the form

$$\dot{x}_1(t) = x_1(t)\left(a_1(t) - a_{11}(t)x_1(t-\tau_{11}(t)) - \frac{a_{12}(t)x_2(t)}{x_1(t)+d_1(t)}\right),$$
$$\dot{x}_2(t) = x_2(t)\left(-a_2(t) + \frac{a_{21}(t)x_1(t-\tau_{21}(t))}{x_1(t-\tau_{21}(t))+d_1(t)} - a_{22}(t)x_2(t-\tau_{22}(t)) - \frac{a_{23}(t)x_3(t)}{x_2(t)+d_2(t)}\right), \tag{1.2}$$
$$\dot{x}_3(t) = x_3(t)\left(a_3(t) - \frac{a_{33}(t)x_3(t-\tau_{33}(t))}{x_2(t-\tau_{32}(t))+d_3(t)}\right),$$

where $\tau_{ii}(t)$, $i = 1, 2, 3$ denote the time delays due to negative feedbacks of the prey, specialist predator and top-predator, respectively. $\tau_{21}(t)$ is a time delay due to gestation, that is, mature adult predators can only contribute to the reproduction of predator biomass. $\tau_{32}(t)$ can be regarded as a gestation period.

In this paper, for system (1.2) we always assume that for all $i, j = 1, 2, 3$

(H_1) $a_i(t), a_{ij}(t), d_i(t)$ are continuous and bounded above and below by positive constants on $[0, +\infty)$, and

(H_2) $\tau_{ij}(t)$ are continuous and differentiable bounded functions on $[0, +\infty)$, and $\dot{\tau}_{ij}(t)$ is uniformly continuous with respect to t on $[0, +\infty)$ and $\inf_{t\in[0,+\infty)}\{1 - \dot{\tau}_{ij}(t)\} > 0$.

Let $\tau = \sup\{\tau_{ij}(t) : t \in [0, +\infty),\ i, j = 1, 2, 3\}$, then we have $0 \le \tau < +\infty$. Let $\sigma_{ij}(t) = t - \tau_{ij}(t)$, then the function $\sigma_{ij}^{-1}(t)$ is the inverse function of the function $\sigma_{ij}(t)$,

$i, j = 1, 2, 3$. Motivated by the application of system (1.2) to population dynamics, we assume that solutions of system (1.2) satisfies the following initial conditions

$$x_i(\theta) = \phi_i(\theta) \geq 0, \quad \theta \in [-\tau, 0],\ \phi_i(0) > 0,$$
$$\phi_i \in C([-\tau, 0], R_+), \quad i = 1, 2, 3. \tag{1.3}$$

It is well known that by the fundamental theory of functional differential equation [16], one can prove the solution of system (1.2) with initial conditions (1.3) exists and remains positive for $t \geq 0$, we call such a solution the positive solution of system (1.2).

The organization of this paper is as follows. In Section 2, by using comparison theorem and further developing the analytical technique of [2, 14], we obtain a set of sufficient conditions, which ensure the permanence of the system (1.2). In Section 3, by constructing a suitable Lyapunov function, we establish a set of sufficient conditions, which ensure the global stability of the system (1.2). In Section 4, we will explore the existence and stability of the solutions of the periodic system (1.2). At last, the conclusion ends with brief remarks.

2. Permanence

In this section, we establish a permanent result for system (1.2).

Here, for any bounded function $\{f(t)\}$

$$f^u = \lim_{t\to+\infty} \sup\{f(t)\}, \qquad f^l = \lim_{t\to+\infty} \inf\{f(t)\}. \tag{2.1}$$

Definition 2.1. System (1.2) is said to be permanent, if there are positive constants m_i and M_i, such that each positive solution $(x_1(t), x_2(t), x_3(t))$ of system (1.2) satisfies

$$m_i \leq \lim_{t\to+\infty} \inf x_i(t) \leq \lim_{t\to+\infty} \sup x_i(t) \leq M_i, \quad i = 1, 2, 3. \tag{2.2}$$

Theorem 2.2. *Assume that* (H_1) *and* (H_2) *hold assume further that*

(H_3) $a_1^l > a_{12}^u M_2/d_1^l$, $a_{21}^l m_1/(m_1 + d_1^u) > a_2^u + a_{23}^u M_3/d_2^l$

hold. Then system (1.2) *is permanent.*

Proof. From the first equation of the system (1.2) it follows that

$$\begin{aligned}\dot{x}_1(t) &\leq x_1(t)(a_1(t) - a_{11}(t)x_1(t - \tau_{11}(t))) \\ &\leq x_1(t)\left(a_1^u - a_{11}^l x_1(t - \tau_{11}(t))\right).\end{aligned} \tag{2.3}$$

Let $\tau = \sup\{\tau_{ij}(t) : t \in [0, +\infty),\ i, j = 1, 2, 3\}$, by (H_2), then we have $0 \leq \tau < +\infty$. Taking $\widetilde{M}_1 = (a_1^u/a_{11}^l)(1 + h_1)$, where $0 < h_1 < \exp\{a_1^u \tau\} - 1$. Firstly, suppose $x_1(t)$ is not oscillatory about $\widetilde{M}_1$. That is, there exists a $T_1 > 0$, for $t > T_1$ such that

$$x_1(t) < \widetilde{M}_1, \tag{2.4}$$

or

$$x_1(t) > \widetilde{M}_1. \tag{2.5}$$

If (2.4) holds, then our aim is obtained. Suppose (2.5) holds, then for $t \geq T_1 + \tau$, we obtain

$$\dot{x}_1(t) < -h_1 a_1^u x_1(t), \tag{2.6}$$

thus $x_1(t) < x_1(0)\exp[-h_1 a_1^u t] \to 0$, as $t \to +\infty$, which is contradiction with (2.5). Hence there must exist $T_1^* > T_1 + \tau$ such that $x_1(t) < \widetilde{M}_1$ for $t > T_1^*$. Secondly now assume that $x_1(t)$ is oscillatory about $\widetilde{M}_1$ for $t \geq T_1$, that is, there exists a time sequence $\{t_n\}$ such that $\tau < t_1 < t_2 < \cdots < t_n < \cdots$ is a sequence of zeros of $x_1(t_n) - \widetilde{M}_1$ with $\lim_{n\to\infty} t_n = +\infty$ and $x_1(t_n) = \widetilde{M}_1$. Set $\widetilde{t}_n$ a point where $x_1(t)$ attends its maximum in (t_n, t_{n+1}). Thus we get $x_1(\widetilde{t}_n) \geq x_1(t_n) = \widetilde{M}_1$. Then it follows from (2.3) that

$$0 = \dot{x}_1(t)|_{t=\widetilde{t}_n} \leq x_1\left(\widetilde{t}_n\right)\left(a_1^u - a_{11}^l x_1\left(\widetilde{t}_n - \tau_{11}\left(\widetilde{t}_n\right)\right)\right), \tag{2.7}$$

which leads to

$$x_1\left(\widetilde{t}_n - \tau_{11}\left(\widetilde{t}_n\right)\right) \leq \frac{a_1^u}{a_{11}^l}. \tag{2.8}$$

Integrating the both sides of (2.3) from $\widetilde{t}_n - \tau_{11}(\widetilde{t}_n)$ to $\widetilde{t}_n$, it follows that

$$\ln \frac{x_1\left(\widetilde{t}_n\right)}{x_1\left(\widetilde{t}_n - \tau_{11}\left(\widetilde{t}_n\right)\right)} \leq \int_{\widetilde{t}_n - \tau_{11}(\widetilde{t}_n)}^{\widetilde{t}_n} \left(a_1^u - a_{11}^l x_1(t - \tau_{11}(t))\right) dt \leq a_1^u \tau_{11}\left(\widetilde{t}_n\right). \tag{2.9}$$

From (2.8) and (2.9) we have

$$x_1\left(\widetilde{t}_n\right) \leq \frac{a_1^u}{a_{11}^l} \exp\{a_1^u \tau\}. \tag{2.10}$$

Since $x_1(\widetilde{t}_n)$ is an arbitrary local maximum of $x_1(t)$, we can see that there exists a $T_2 > T_1$ such that for all $t \geq T_2$

$$x_1(t) \leq \frac{a_1^u}{a_{11}^l} \exp\{a_1^u \tau\} := M_1. \tag{2.11}$$

Thus

$$\lim_{t\to+\infty} \sup\, x_1(t) \leq M_1. \tag{2.12}$$

For $t \geq T_2 + \tau$, from (2.11) and the second equation of the system (1.2) it follows that

$$\dot{x}_2(t) \leq x_2(t)\left(-a_2^l + \frac{a_{21}^u M_1}{M_1 + d_1^l} - a_{22}^l x_2(t - \tau_{22}(t))\right). \tag{2.13}$$

Similar to the argument above, from (2.13) we obtain

$$\lim_{t\to+\infty} \sup x_2(t) \leq \frac{-a_2^l + a_{21}^u M_1/\left(M_1 + d_1^l\right)}{a_{22}^l} \exp\left\{\left(-a_2^l + \frac{a_{21}^u M_1}{M_1 + d_1^l}\right)\tau\right\} := M_2. \tag{2.14}$$

Similarly, from the third equation of the system (1.2), we have

$$\dot{x}_3(t) \leq x_3(t)\left(a_3^u - \frac{a_{33}^l x_3(t - \tau_{33}(t))}{M_2 + d_3^u}\right), \tag{2.15}$$

and so

$$\lim_{t\to+\infty} \sup x_3(t) \leq \frac{a_3^u\left(M_2 + d_3^u\right)}{a_{33}^l} \exp\left\{a_3^u \tau\right\} := M_3. \tag{2.16}$$

Condition (H_3) of Theorem 2.2 also implies that we could choose $\varepsilon > 0$ small enough such that

$$a_1^l > \frac{a_{12}^u(M_2 + \varepsilon)}{d_1^l}, \qquad \frac{a_{21}^l(m_1 - \varepsilon)}{(m_1 - \varepsilon) + d_1^u} > a_2^u + \frac{a_{23}^u(M_3 + \varepsilon)}{d_2^l}, \tag{2.17}$$

hence, for $\varepsilon > 0$ satisfies (2.17), from (2.12), (2.14), and (2.16), we know that there exists $T_3 > T_2 + \tau$ such that for $i = 1, 2, 3$ and $t \geq T_3$

$$x_i(t) \leq M_i + \varepsilon. \tag{2.18}$$

From the first equation and (2.18) it follows that for $t \geq T_3 + \tau$,

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a_1(t) - a_{11}(t)x_1(t - \tau_{11}(t)) - \frac{a_{12}(t)x_2(t)}{x_1(t) + d_1(t)}\right) \\ &\geq x_1(t)\left(a_1^l - a_{11}^u(M_1 + \varepsilon) - \frac{a_{12}^u(M_2 + \varepsilon)}{d_1^l}\right). \end{aligned} \tag{2.19}$$

Note that $a_1^u / a_{11}^l \leq M_1$ implies that

$$a_1^l - a_{11}^u(M_1 + \varepsilon) - \frac{a_{12}^u(M_2 + \varepsilon)}{d_1^l} \leq a_1^l - a_{11}^u(M_1 + \varepsilon) \leq a_1^u - a_{11}^l(M_1 + \varepsilon) \leq 0. \tag{2.20}$$

Now we consider the following two cases.

Case 1. Suppose $a_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)/d_1^l = 0$, then for $t \geq T_3+\tau$, from the positivity of the solution and (2.19) it follows that

$$\dot{x}_1(t) = 0. \tag{2.21}$$

Then from (2.12) and (2.21) it follows that $\lim_{t\to+\infty} x_1(t) =: p_1 < a_1^l/a_{11}^u$, then there exists $T_4 > T_3+\tau$ such that for $t \geq T_4$

$$x_1(t) < p_1 + \frac{a_1^l/a_{11}^u - p_1}{2} < \frac{a_1^l}{a_{11}^u} < M_1. \tag{2.22}$$

From the positivity of the solution, (2.19) and (2.22) it follows that

$$\begin{aligned}
\dot{x}_1(t) &\geq x_1(t)\left(a_1^l - a_{11}^u\frac{a_1^l/a_{11}^u + p_1}{2} - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l}\right) \\
&> x_1(t)\left(a_1^l - a_{11}^u(M_1+\varepsilon) - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l}\right) \\
&= 0, \quad t \geq T_4+\tau, \\
x_1(t) &\geq x_1(T_4+\tau)\exp\left\{\left(a_1^l - a_{11}^u\frac{a_1^l/a_{11}^u + p_1}{2} - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l}\right)(t-(T_4+\tau))\right\},
\end{aligned} \tag{2.23}$$

then we can see that $x_1(t) \to +\infty$ as $t \to +\infty$, which is contradiction with (2.12). Hence we have $\lim_{t\to+\infty} x_1(t) \geq a_1^l/a_{11}^u$, which implies that there exists $T_4^* > T_4+\tau$ such that $x_1(t) \geq a_1^l/2a_{11}^u$ for $t \geq T_4^*$.

Case 2. Suppose $a_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)/d_1^l < 0$, from (2.19), for $t \geq T_3+\tau$, it follows that

$$\dot{x}_1(t) \geq x_1(t)\left(a_1^l - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l} - a_{11}^u x_1(t-\tau_{11}(t))\right). \tag{2.24}$$

Let

$$\widetilde{m}_1 = \frac{a_1^l - a_{12}^u(M_2+\varepsilon)/d_1^l}{a_{11}^u}(1-\sigma_1), \tag{2.25}$$

where $0 < \sigma_1 < 1 - \exp\{(a_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)/d_1^l)\tau\}$.

Firstly, suppose $x_1(t)$ is not oscillatory about $\widetilde{m}_1$. That is, there exists a $T_5 > 0$, for $t > T_5$ such that

$$x_1(t) > \widetilde{m}_1, \tag{2.26}$$

or

$$x_1(t) < \widetilde{m}_1. \tag{2.27}$$

If (2.26) holds, then our aim is obtained. Suppose (2.27) holds, then for $t \geq T_5 + \tau$, we obtain

$$\dot{x}_1(t) \geq \sigma_1\left(a_1^l - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l}\right)x_1(t). \tag{2.28}$$

thus there must exist $T_5^* > T_5 + \tau$ such that $x_1(t) > \widetilde{m}_1$ for $t > T_5^*$, which is a contradiction. Hence, (2.27) could not hold. Secondly now assume that $x_1(t)$ is oscillatory about $\widetilde{m}_1$ for $t \geq T_3 + \tau$, that is, there exists a time sequence $\{t_n\}$ such that $\tau < t_1 < t_2 < \cdots < t_n < \cdots$ is a sequence of zeros of $x_1(t_n) - \widetilde{m}_1$ with $\lim_{n\to\infty} t_n = +\infty$ and $x_1(t_n) = \widetilde{m}_1$. Set $\widehat{t}_n$ be a point where $x_1(t)$ attends its minimum in (t_n, t_{n+1}). Thus, we get $x_1(\widehat{t}_n) \leq x_1(t_n) = \widetilde{m}_1$. Then it follows from (2.24) that

$$0 = \dot{x}_1(t)|_{t=\widehat{t}_n} \geq x_1\left(\widehat{t}_n\right)\left(a_1^l - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l} - a_{11}^u x_1\left(\widehat{t}_n - \tau_{11}\left(\widehat{t}_n\right)\right)\right), \tag{2.29}$$

which implies that

$$x_1\left(\widehat{t}_n - \tau_{11}\left(\widehat{t}_n\right)\right) \geq \frac{a_1^l - a_{12}^u(M_2+\varepsilon)/d_1^l}{a_{11}^u}. \tag{2.30}$$

Integrating (2.24) on the interval $[\widehat{t}_n - \tau_{11}(\widehat{t}_n), \widehat{t}_n]$, we have

$$\begin{aligned}
\ln\frac{x_1\left(\widehat{t}_n\right)}{x_1\left(\widehat{t}_n - \tau_{11}\left(\widehat{t}_n\right)\right)} &\geq \int_{\widehat{t}_n-\tau_{11}(\widehat{t}_n)}^{\widehat{t}_n}\left(a_1^l - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l} - a_{11}^u x_1(t-\tau_{11}(t))\right)dt \\
&\geq \int_{\widehat{t}_n-\tau_{11}(\widehat{t}_n)}^{\widehat{t}_n}\left(a_1^l - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l} - a_{11}^u(M_1+\varepsilon)\right)dt \\
&= \left(a_1^l - a_{11}^u(M_1+\varepsilon) - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l}\right)\tau_{11}\left(\widehat{t}_n\right).
\end{aligned} \tag{2.31}$$

From (2.30) and (2.31), we get that

$$x_1\left(\hat{t}_n\right) \geq \frac{a_1^l - a_{12}^u(M_2+\varepsilon)/d_1^l}{a_{11}^u} \exp\left\{\left(a_1^l - a_{11}^u(M_1+\varepsilon) - \frac{a_{12}^u(M_2+\varepsilon)}{d_1^l}\right)\tau\right\}. \tag{2.32}$$

Since $x_1(\hat{t}_n)$ is an arbitrary local minimum of $x_1(t)$, we can have that there exists a $q_1 \leq \tilde{m}_1$ such that for all $t \geq T_6^*$

$$x_1(t) \geq q_1, \tag{2.33}$$

where

$$0 < q_1 \leq \frac{a_1^l - a_{12}^u M_2/d_1^l}{a_{11}^u} \exp\left\{\left(a_1^l - a_{11}^u M_1 - \frac{a_{12}^u M_2}{d_1^l}\right)\tau\right\}. \tag{2.34}$$

Taken $m_1 = \min\{a_1^l/2a_{11}^u, q_1\}$, thus, we have

$$\lim_{t\to+\infty} \inf x_1(t) \geq m_1. \tag{2.35}$$

Thus, for $\varepsilon > 0$ satisfies (2.17), it follows from (2.35) that there exists large enough T_6 such that for $t \geq T_6$

$$x_1(t) \geq m_1 - \varepsilon. \tag{2.36}$$

For $t \geq T_6 + \tau$, by using (2.36), from the second equation of system (1.2) it follows that

$$\begin{aligned}
\dot{x}_2(t) &= x_2(t)\left(-a_2(t) + \frac{a_{21}(t)x_1(t-\tau_{21}(t))}{x_1(t-\tau_{21}(t)) + d_1(t)} - a_{22}(t)x_2(t-\tau_{22}(t)) - \frac{a_{23}(t)x_3(t)}{x_2(t)+d_2(t)}\right)\\
&\geq x_2(t)\left(-a_2^u + \frac{a_{21}^l(m_1-\varepsilon)}{(m_1-\varepsilon)+d_1^u} - a_{22}^u(M_2+\varepsilon) - \frac{a_{23}^u(M_3+\varepsilon)}{d_2^l}\right),\\
\dot{x}_2(t) &\geq x_2(t)\left(-a_2^u + \frac{a_{21}^l(m_1-\varepsilon)}{(m_1-\varepsilon)+d_1^u} - \frac{a_{23}^u(M_3+\varepsilon)}{d_2^l} - a_{22}(t)x_2(t-\tau_{22}(t))\right);
\end{aligned} \tag{2.37}$$

from (2.37), by a procedure similar to the discussion above, we can verify that

$$\lim_{t\to+\infty} \inf x_2(t) \geq m_2, \tag{2.38}$$

where

$$m_2 = \min\left\{ \frac{-a_2^u + a_{21}^l m_1/(m_1 + d_1^u)}{2a_{22}^u}, q_2 \right\},$$
$$0 < q_2 \le \frac{-a_2^u + a_{21}^l m_1/(m_1 + d_1^u) - a_{23}^u M_3/d_2^l}{a_{22}^u} \exp\left\{ \left(-a_2^u + \frac{a_{21}^l m_1}{m_1 + d_1^u} - a_{22}^u M_2 - \frac{a_{23}^u M_3}{d_2^l} \right) \tau \right\}. \tag{2.39}$$

From (2.38), we see that there exists large enough T_7 such that for $t \ge T_7$

$$x_2(t) \ge m_2 - \varepsilon. \tag{2.40}$$

Substituting (2.18) to the last equation of system (1.2), it follows that

$$\dot{x}_3(t) \ge x_3(t)\left(a_3^l - \frac{a_{33}^u (M_3 + \varepsilon)}{d_3^l} \right),$$
$$\dot{x}_3(t) \ge x_3(t)\left(a_3^l - \frac{a_{33}^u x_3(t - \tau_{33}(t))}{d_3^l} \right); \tag{2.41}$$

from (2.41), similar to the argument of (2.35), we also have

$$\lim_{t \to +\infty} \inf x_3(t) \ge m_3, \tag{2.42}$$

where

$$m_3 = \min\left\{ \frac{a_3^l d_3^l}{2a_{33}^u}, q_3 \right\},$$
$$0 < q_3 \le \frac{a_3^l d_3^l}{a_{33}^u} \exp\left\{ \left(a_3^l - \frac{a_{33}^u M_3}{d_3^l} \right) \tau \right\}. \tag{2.43}$$

Consequently, (2.12), (2.14), (2.16), (2.35), (2.38), and (2.42) show that under the assumption (H_1)–(H_3), for any positive solution $(x_1(t), x_2(t), x_3(t))$ of system (1.2), one has

$$m_i \le \lim_{t \to +\infty} \inf x_i(t) \le \lim_{t \to +\infty} \sup x_i(t) \le M_i, \quad i = 1, 2, 3, \tag{2.44}$$

where m_i and M_i, $i = 1, 2, 3$ are independent of the solution of system (1.2), thus system (1.2) is permanent. This completes the proof of Theorem 2.2. □

3. Global Stability

Now we study the global stability of the positive solution of system (1.2). We say a positive solution of system (1.2) is globally asymptotically stable if it attracts all other positive solution of the system.

The following lemma is from [17], and will be employed in establishing the global stability of positive solution of system (1.2).

Lemma 3.1. *Let h be a real number and f be a nonnegative function defined on $[h;+\infty)$ such that f is integrable on $[h;+\infty)$ and is uniformly continuous on $[h;+\infty)$, then $\lim_{t\to+\infty} f(t) = 0$.*

Theorem 3.2. *In addition to (H_1)–(H_3), assume further that*

(H_4) *there exist constants $\lambda_i > 0$ such that*

$$\lim_{t\to+\infty}\inf\{A_i(t)\} > 0, \quad i = 1,2,3, \tag{3.1}$$

where

$$\begin{aligned}
A_1(t) = \lambda_1\Bigg(& a_{11}(t) - \frac{a_{12}(t)M_2}{d_1^2(t)} - \frac{a_{11}(\sigma_{11}^{-1}(t))M_1}{1-\dot{\tau}_{11}(\sigma_{11}^{-1}(t))}\int_{\sigma_{11}^{-1}(t)}^{\sigma_{11}^{-1}(\sigma_{11}^{-1}(t))} a_{11}(s)ds \\
& -\left[a_1(t) + a_{11}(t)M_1 + \frac{a_{12}(t)M_2}{d_1(t)}\right]\int_t^{\sigma_{11}^{-1}(t)} a_{11}(s)ds\Bigg) \\
& - \lambda_2\left(\frac{a_{21}(\sigma_{21}^{-1}(t))}{d_1(\sigma_{21}^{-1}(t))(1-\dot{\tau}_{21}(\sigma_{21}^{-1}(t)))} + \frac{a_{21}(\sigma_{21}^{-1}(t))M_2}{d_1(\sigma_{21}^{-1}(t))(1-\dot{\tau}_{21}(\sigma_{21}^{-1}(t)))}\int_{\sigma_{21}^{-1}(t)}^{\sigma_{22}^{-1}(\sigma_{21}^{-1}(t))} a_{22}(s)ds\right),
\end{aligned}$$

$$\begin{aligned}
A_2(t) = \lambda_2\Bigg(& a_{22}(t) - \frac{a_{23}(t)M_3}{d_2^2(t)} - \frac{a_{22}(\sigma_{22}^{-1}(t))M_2}{1-\dot{\tau}_{22}(\sigma_{22}^{-1}(t))}\int_{\sigma_{22}^{-1}(t)}^{\sigma_{22}^{-1}(\sigma_{22}^{-1}(t))} a_{22}(s)ds \\
& -\left[a_2(t) + a_{22}(t)M_2 + a_{21}(t) + \frac{a_{23}(t)}{d_2(t)}M_3\right]\int_t^{\sigma_{22}^{-1}(t)} a_{22}(s)ds\Bigg) \\
& - \lambda_1 a_{12}(t)\left(\frac{1}{d_1(t)} + \int_t^{\sigma_{11}^{-1}(t)} a_{11}(s)ds\right) \\
& - \lambda_3\left(\frac{a_{33}(\sigma_{32}^{-1}(t))M_3}{d_3^2(\sigma_{32}^{-1}(t))(1-\dot{\tau}_{32}(\sigma_{32}^{-1}(t)))} + \frac{a_{33}(\sigma_{32}^{-1}(t))M_3^2}{d_3^2(\sigma_{32}^{-1}(t))(1-\dot{\tau}_{32}(\sigma_{32}^{-1}(t)))}\int_{\sigma_{32}^{-1}(t)}^{\sigma_{33}^{-1}(\sigma_{32}^{-1}(t))} \frac{a_{33}(s)}{d_3(s)}ds\right),
\end{aligned}$$

$$A_3(t) = \lambda_3\left(\frac{a_{33}(t)}{M_2+d_3(t)} - \left[a_3(t)+\frac{a_{33}(t)M_3}{d_3(t)}\right]\int_t^{\sigma_{33}^{-1}(t)}\frac{a_{33}(s)}{d_3(s)}ds\right.$$
$$\left. - \frac{a_{33}(\sigma_{33}^{-1}(t))M_3}{d_3(\sigma_{33}^{-1}(t))(1-\dot{\tau}_{33}(\sigma_{33}^{-1}(t)))}\int_{\sigma_{33}^{-1}(t)}^{\sigma_{33}^{-1}(\sigma_{33}^{-1}(t))}\frac{a_{33}(s)}{d_3(s)}ds\right)$$
$$- \lambda_2 a_{23}(t)\left(\frac{1}{d_2(t)}+\int_t^{\sigma_{22}^{-1}(t)}a_{22}(s)ds\right). \tag{3.2}$$

Then for any positive solutions $(x_1(t), x_2(t), x_3(t))$ *and* $(x_1^*(t), x_2^*(t), x_3^*(t))$ *of system* (1.2), *one has*

$$\lim_{t\to+\infty}\left[\left|x_i(t)-x_i^*(t)\right|\right] = 0, \quad i = 1,2,3. \tag{3.3}$$

Proof. For two arbitrary nontrivial solutions $(x_1(t), x_2(t), x_3(t))$ and $(x_1^*(t), x_2^*(t), x_3^*(t))$ of system (1.2), we have from Theorem 2.2 that there exist positive constants $T > T_7$ and M_i, m_i $(i = 1,2,3)$ such that for all $t \geq T$ and $i = 1,2,3$

$$m_i \leq x_i(t) \leq M_i. \tag{3.4}$$

We define

$$V_{11}(t) = \left|\ln x_1(t) - \ln x_1^*(t)\right|. \tag{3.5}$$

Calculating the upper right derivative of $V_{11}(t)$ along the solution of system (1.2), for $t \geq T+\tau$, it follow that

$$D^+V_{11}(t) = \left(\frac{\dot{x}_1(t)}{x_1(t)} - \frac{\dot{x}_1^*(t)}{x_1^*(t)}\right)\operatorname{sgn}(x_1(t)-x_1^*(t))$$
$$= \operatorname{sgn}(x_1(t)-x_1^*(t))$$
$$\times\left\{-a_{11}(t)(x_1(t-\tau_{11}(t)) - x_1^*(t-\tau_{11}(t))) - \frac{a_{12}(t)x_2(t)}{x_1(t)+d_1(t)} + \frac{a_{12}(t)x_2^*(t)}{x_1^*(t)+d_1(t)}\right\}$$
$$= \operatorname{sgn}(x_1(t)-x_1^*(t))$$
$$\times\left\{-a_{11}(t)(x_1(t)-x_1^*(t)) - \frac{a_{12}(t)}{x_1(t)+d_1(t)}(x_2(t)-x_2^*(t))\right.$$
$$+ \frac{a_{12}(t)x_2^*(t)}{(x_1(t)+d_1(t))(x_1^*(t)+d_1(t))}(x_1(t)-x_1^*(t))$$
$$\left. + a_{11}(t)\int_{t-\tau_{11}(t)}^{t}(\dot{x}_1(u)-\dot{x}_1^*(u))du\right\}$$

$$
\begin{aligned}
&\leq -a_{11}(t)\left|x_1(t)-x_1^*(t)\right| + \frac{a_{12}(t)}{d_1(t)}\left|x_2(t)-x_2^*(t)\right| \\
&\quad + \frac{a_{12}(t)M_2}{d_1^2(t)}\left|x_1(t)-x_1^*(t)\right| \\
&\quad + a_{11}(t)\left|\int_{t-\tau_{11}(t)}^{t}\left(\dot{x}_1(u)-\dot{x}_1^*(u)\right)du\right|.
\end{aligned}
\tag{3.6}
$$

On substituting (1.2) into (3.6), we derive that

$$
\begin{aligned}
D^+V_{11}(t) \leq{}& -a_{11}(t)\left|x_1(t)-x_1^*(t)\right| + \frac{a_{12}(t)}{d_1(t)}\left|x_2(t)-x_2^*(t)\right| + \frac{a_{12}(t)M_2}{d_1^2(t)}\left|x_1(t)-x_1^*(t)\right| \\
&+ a_{11}(t)\left|\int_{t-\tau_{11}(t)}^{t}\left\{a_1(u) - a_{11}(u)x_1^*(u-\tau_{11}(u))\right.\right. \\
&\qquad - \frac{a_{12}(u)d_1(u)x_2^*(u)}{\left(x_1(u)+d_1(u)\right)\left(x_1^*(u)+d_1(u)\right)}\left(x_1(u)-x_1^*(u)\right) \\
&\qquad - a_{11}(u)x_1(u)\left(x_1(u-\tau_{11}(u)) - x_1^*(u-\tau_{11}(u))\right) \\
&\qquad \left.\left. - \frac{a_{12}(u)x_1(u)}{x_1(u)+d_1(u)}\left(x_2(u)-x_2^*(u)\right)\right\}du\right|.
\end{aligned}
\tag{3.7}
$$

It follows (3.4) and (3.7) that for $t > T + \tau$

$$
\begin{aligned}
D^+V_{11}(t) \leq{}& -a_{11}(t)\left|x_1(t)-x_1^*(t)\right| + \frac{a_{12}(t)}{d_1(t)}\left|x_2(t)-x_2^*(t)\right| \\
&+ \frac{a_{12}(t)M_2}{d_1^2(t)}\left|x_1(t)-x_1^*(t)\right| \\
&+ a_{11}(t)\int_{t-\tau_{11}(t)}^{t}\left\{\left[a_1(u) + a_{11}(u)M_1 + \frac{a_{12}(u)M_2}{d_1(u)}\right]\left|x_1(u)-x_1^*(u)\right|\right. \\
&\qquad + a_{11}(u)M_1\left|x_1(u-\tau_{11}(u)) - x_1^*(u-\tau_{11}(u))\right| \\
&\qquad \left. + a_{12}(u)\left|x_2(u)-x_2^*(u)\right|\right\}du.
\end{aligned}
\tag{3.8}
$$

Let $\sigma_{11}(t) = t - \tau_{11}(t)$, by (H_2), we can obtain the inverse function of the function $\sigma_{11}(t)$ denoted by $\sigma_{11}^{-1}(t)$.

Define

$$V_{12}(t)=\int_t^{\sigma_{11}^{-1}(t)}\int_{\sigma_{11}(s)}^{t}a_{11}(s)\times\left\{\left[a_1(u)+a_{11}(u)M_1+\frac{a_{12}(u)M_2}{d_1(u)}\right]\left|x_1(u)-x_1^*(u)\right|+a_{11}(u)M_1\left|x_1(u-\tau_{11}(u))-x_1^*(u-\tau_{11}(u))\right|+a_{12}(u)\left|x_2(u)-x_2^*(u)\right|\right\}du\,ds. \tag{3.9}$$

We obtain from (3.8) and (3.9) for $t>T+\tau$

$$\begin{aligned}D^+V_{11}(t)+\dot{V}_{12}(t)\le{}&-\left(a_{11}(t)-\frac{a_{12}(t)M_2}{d_1^2(t)}\right)\left|x_1(t)-x_1^*(t)\right|+\frac{a_{12}(t)}{d_1(t)}\left|x_2(t)-x_2^*(t)\right|\\&+\int_t^{\sigma_{11}^{-1}(t)}a_{11}(s)ds\\&\times\left\{\left[a_1(t)+a_{11}(t)M_1+\frac{a_{12}(t)M_2}{d_1(t)}\right]\left|x_1(t)-x_1^*(t)\right|\right.\\&+a_{11}(t)M_1\left|x_1(t-\tau_{11}(t))-x_1^*(t-\tau_{11}(t))\right|\\&\left.+a_{12}(t)\left|x_2(t)-x_2^*(t)\right|\right\}.\end{aligned}\tag{3.10}$$

We now define

$$V_1(t)=V_{11}(t)+V_{12}(t)+V_{13}(t),\tag{3.11}$$

where

$$V_{13}(t)=M_1\int_{t-\tau_{11}(t)}^{t}\int_{\sigma_{11}^{-1}(l)}^{\sigma_{11}^{-1}(\sigma_{11}^{-1}(l))}\frac{a_{11}(s)a_{11}\left(\sigma_{11}^{-1}(l)\right)}{1-\dot{\tau}_{11}\left(\sigma_{11}^{-1}(l)\right)}\left|x_1(l)-x_1^*(l)\right|ds\,dl.\tag{3.12}$$

It then follows from (3.10) that for $t>T+\tau$

$$\begin{aligned}D^+V_1(t)\le{}&-\left(a_{11}(t)-\frac{a_{12}(t)M_2}{d_1^2(t)}\right)\left|x_1(t)-x_1^*(t)\right|+a_{12}(t)\\&\times\left[\frac{1}{d_1(t)}+\int_t^{\sigma_{11}^{-1}(t)}a_{11}(s)ds\right]\left|x_2(t)-x_2^*(t)\right|\end{aligned}$$

$$+\left[a_1(t)+a_{11}(t)M_1+\frac{a_{12}(t)M_2}{d_1(t)}\right]\int_t^{\sigma_{11}^{-1}(t)}a_{11}(s)ds\left|x_1(t)-x_1^*(t)\right|$$
$$+\frac{a_{11}\left(\sigma_{11}^{-1}(t)\right)M_1}{1-\dot{\tau}_{11}\left(\sigma_{11}^{-1}(t)\right)}\int_{\sigma_{11}^{-1}(t)}^{\sigma_{11}^{-1}(\sigma_{11}^{-1}(t))}a_{11}(s)ds\left|x_1(t)-x_1^*(t)\right|. \tag{3.13}$$

Similarly, we define

$$V_2(t)=V_{21}(t)+V_{22}(t)+V_{23}(t), \tag{3.14}$$

where

$$V_{21}(t)=\left|\ln x_2(t)-\ln x_2^*(t)\right|,$$
$$V_{22}(t)=\int_{t-\tau_{21}(t)}^{t}\frac{a_{21}\left(\sigma_{21}^{-1}(s)\right)}{d_1\left(\sigma_{21}^{-1}(s)\right)\left(1-\dot{\tau}_{21}\left(\sigma_{21}^{-1}(s)\right)\right)}\left|x_1(s)-x_1^*(s)\right|ds$$
$$+\int_t^{\sigma_{22}^{-1}(t)}\int_{\sigma_{22}(s)}^{t}a_{22}(s)$$
$$\times\left\{\left[a_2(u)+a_{22}(u)M_2+a_{21}(u)+\frac{a_{23}(u)M_3}{d_2(u)}\right]\left|x_2(u)-x_2^*(u)\right|\right.$$
$$+a_{23}(u)\left|x_3(u)-x_3^*(u)\right|+a_{22}(u)M_2\left|x_2(u-\tau_{22}(u))-x_2^*(u-\tau_{22}(u))\right|$$
$$\left.+\frac{a_{21}(u)M_2}{d_1(u)}\left|x_1(u-\tau_{21}(u))-x_1^*(u-\tau_{21}(u))\right|\right\}du\,ds,$$
$$V_{23}(t)=M_2\int_{t-\tau_{22}(t)}^{t}\int_{\sigma_{22}^{-1}(l)}^{\sigma_{22}^{-1}(\sigma_{22}^{-1}(l))}\frac{a_{22}(s)a_{22}\left(\sigma_{22}^{-1}(l)\right)}{1-\dot{\tau}_{22}\left(\sigma_{22}^{-1}(l)\right)}\left|x_2(l)-x_2^*(l)\right|ds\,dl$$
$$+M_2\int_{t-\tau_{21}(t)}^{t}\int_{\sigma_{21}^{-1}(l)}^{\sigma_{22}^{-1}(\sigma_{21}^{-1}(l))}\frac{a_{22}(s)a_{21}\left(\sigma_{21}^{-1}(l)\right)}{d_1\left(\sigma_{21}^{-1}(l)\right)\left(1-\dot{\tau}_{21}\left(\sigma_{21}^{-1}(l)\right)\right)}\left|x_1(l)-x_1^*(l)\right|ds\,dl. \tag{3.15}$$

Calculating the upper right derivative of $V_2(t)$ along solutions of (1.2), we derive that for $t>T+\tau$

$$D^+V_2(t)\le-\left(a_{22}(t)-\frac{a_{23}(t)M_3}{d_2^2(t)}\right)\left|x_2(t)-x_2^*(t)\right|$$
$$+a_{23}(t)\left[\frac{1}{d_2(t)}+\int_t^{\sigma_{22}^{-1}(t)}a_{22}(s)ds\right]\left|x_3(t)-x_3^*(t)\right|$$
$$+\frac{a_{21}\left(\sigma_{21}^{-1}(t)\right)}{d_1\left(\sigma_{21}^{-1}(t)\right)\left(1-\dot{\tau}_{21}\left(\sigma_{21}^{-1}(t)\right)\right)}\left|x_1(t)-x_1^*(t)\right|$$

$$
\begin{aligned}
&+\left[a_2(t)+a_{22}(t)M_2+a_{21}(t)+\frac{a_{23}(t)M_3}{d_2(t)}\right]\\
&\times\int_t^{\sigma_{22}^{-1}(t)} a_{22}(s)ds\left|x_2(t)-x_2^*(t)\right|\\
&+\frac{a_{22}\left(\sigma_{22}^{-1}(t)\right)M_2}{1-\dot{\tau}_{22}\left(\sigma_{22}^{-1}(t)\right)}\int_{\sigma_{22}^{-1}(t)}^{\sigma_{22}^{-1}(\sigma_{22}^{-1}(t))} a_{22}(s)ds\left|x_2(t)-x_2^*(t)\right|\\
&+\frac{a_{21}\left(\sigma_{21}^{-1}(t)\right)M_2}{d_1\left(\sigma_{21}^{-1}(t)\right)\left(1-\dot{\tau}_{21}\left(\sigma_{21}^{-1}(t)\right)\right)}\int_{\sigma_{21}^{-1}(t)}^{\sigma_{22}^{-1}(\sigma_{21}^{-1}(t))} a_{22}(s)ds\left|x_1(t)-x_1^*(t)\right|.
\end{aligned}
\tag{3.16}
$$

Similarly, we define

$$
V_3(t)=V_{31}(t)+V_{32}(t)+V_{33}(t),
\tag{3.17}
$$

where

$$
\begin{aligned}
V_{31}(t)&=\left|\ln x_3(t)-\ln x_3^*(t)\right|,\\
V_{32}(t)&=M_3\int_{t-\tau_{32}(t)}^{t}\frac{a_{33}\left(\sigma_{32}^{-1}(s)\right)}{d_3^2\left(\sigma_{32}^{-1}(s)\right)\left(1-\dot{\tau}_{32}\left(\sigma_{32}^{-1}(s)\right)\right)}\left|x_2(s)-x_2^*(s)\right|ds\\
&\quad+\int_t^{\sigma_{33}^{-1}(t)}\int_{\sigma_{33}(s)}^{t}\frac{a_{33}(s)}{d_3(s)}\\
&\qquad\times\left\{\left[a_3(u)+\frac{a_{33}(u)M_3}{d_3(u)}\right]\left|x_3(u)-x_3^*(u)\right|\right.\\
&\qquad\quad+\frac{a_{33}(u)M_3}{d_3(u)}\left|x_3(u-\tau_{33}(u))-x_3^*(u-\tau_{33}(u))\right|\\
&\qquad\quad\left.+\frac{a_{33}(u)M_3^2}{d_3^2(u)}\left|x_2(u-\tau_{32}(u))-x_2^*(u-\tau_{32}(u))\right|\right\}du\,ds,\\
V_{33}(t)&=M_3\int_{t-\tau_{33}(t)}^{t}\int_{\sigma_{33}^{-1}(l)}^{\sigma_{33}^{-1}(\sigma_{33}^{-1}(l))}\frac{a_{33}(s)a_{33}\left(\sigma_{33}^{-1}(l)\right)}{d_3(s)d_3\left(\sigma_{33}^{-1}(l)\right)\left(1-\dot{\tau}_{33}\left(\sigma_{33}^{-1}(l)\right)\right)}\left|x_3(l)-x_3^*(l)\right|ds\,dl\\
&\quad+M_3^2\int_{t-\tau_{32}(t)}^{t}\int_{\sigma_{32}^{-1}(l)}^{\sigma_{33}^{-1}(\sigma_{32}^{-1}(l))}\frac{a_{33}(s)a_{33}\left(\sigma_{32}^{-1}(l)\right)}{d_3(s)d_3^2\left(\sigma_{32}^{-1}(l)\right)\left(1-\dot{\tau}_{32}\left(\sigma_{32}^{-1}(l)\right)\right)}\left|x_2(l)-x_2^*(l)\right|ds\,dl.
\end{aligned}
\tag{3.18}
$$

Calculating the upper right derivative of $V_3(t)$ along solutions of (1.2), we derive that for $t > T + \tau$

$$\begin{aligned}
D^+V_3(t) \le &-\frac{a_{33}(t)}{M_2 + d_3(t)}\left|x_3(t) - x_3^*(t)\right| \\
&+ M_3\frac{a_{33}(\sigma_{32}^{-1}(t))}{d_3^2(\sigma_{32}^{-1}(t))(1-\dot{\tau}_{32}(\sigma_{32}^{-1}(t)))}\left|x_2(t) - x_2^*(t)\right| \\
&+ \left[a_3(t) + \frac{a_{33}(t)M_3}{d_3(t)}\right]\int_t^{\sigma_{33}^{-1}(t)}\frac{a_{33}(s)}{d_3(s)}ds\left|x_3(t) - x_3^*(t)\right| \\
&+ \frac{a_{33}(\sigma_{33}^{-1}(t))M_3}{d_3(\sigma_{33}^{-1}(t))(1-\dot{\tau}_{33}(\sigma_{33}^{-1}(t)))}\int_{\sigma_{33}^{-1}(t)}^{\sigma_{33}^{-1}(\sigma_{33}^{-1}(t))}\frac{a_{33}(s)}{d_3(s)}ds\left|x_3(t) - x_3^*(t)\right| \\
&+ \frac{a_{33}(\sigma_{32}^{-1}(t))M_3^2}{d_3^2(\sigma_{32}^{-1}(t))(1-\dot{\tau}_{32}(\sigma_{32}^{-1}(t)))}\int_{\sigma_{32}^{-1}(t)}^{\sigma_{33}^{-1}(\sigma_{32}^{-1}(t))}\frac{a_{33}(s)}{d_3(s)}ds\left|x_2(t) - x_2^*(t)\right|.
\end{aligned} \tag{3.19}$$

We now define a Lyapunov functional as

$$V(t) = \sum_{i=1}^{3}\lambda_i V_i(t). \tag{3.20}$$

It then follows from (3.13), (3.16), (3.19), and (3.20) that for $t > T + \tau$

$$D^+V(t) \le -\sum_{i=1}^{3}A_i(t)\left|x_i(t) - x_i^*(t)\right|. \tag{3.21}$$

By the hypothes is (H_4), there exist enough small positive constants α_i, $i = 1, 2, 3$ and a large enough constant $T^* \ge T + \tau$, such that for all $i = 1, 2, 3$ and $t \ge T^*$

$$A_i(t) \ge \alpha_i > 0. \tag{3.22}$$

Integrating both sides of (3.21) on interval $[T^*, t]$

$$V(t) + \sum_{i=1}^{3}\int_{T^*}^{t}A_i(s)\left|x_i(s) - x_i^*(s)\right|ds \le V(T^*). \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$V(t) + \sum_{i=1}^{3}\alpha_i\int_{T^*}^{t}\left|x_i(s) - x_i^*(s)\right|ds \le V(T^*). \tag{3.24}$$

Therefore, $V(t)$ is bounded on $[T^*, +\infty)$ and also

$$\int_{T^*}^{t} \left|x_i(s) - x_i^*(s)\right| ds < \infty, \quad i = 1, 2, 3. \tag{3.25}$$

By Theorem 2.2, we know that $|x_i(t) - x_i^*(t)|$, $i = 1, 2, 3$ are bounded on $[T^*, +\infty)$. On the other hand, it is easy to see that $\dot{x}_i(t)$, $i = 1, 2, 3$ are bounded for $t \geq T^*$. Therefore, $|x_i(t) - x_i^*(t)|$, $i = 1, 2, 3$ are uniformly continuous on $[T^*, +\infty)$. By Lemma 3.1, one can conclude that

$$\lim_{t \to +\infty} \left|x_i(t) - x_i^*(t)\right| = 0, \quad i = 1, 2, 3. \tag{3.26}$$

This completes the proof of Theorem 3.2. □

Remark 3.3. In the proof of the stability of system (1.2), we construct the Lyapunov functional, which need the condition (H_2). Similar method and conditon can be found in the [2, 14].

4. Existence and Stability of the Positive Periodic Solutions

In this section, we suppose that all the coefficients in system (1.2) are continuous and positive ω-periodic functions, then the system (1.2) is an ω-periodic system for this case.

We let the following denote the unique solution of periodic system (1.2) for initial value $Z^0 = \{x_1^0, x_2^0, x_3^0\}$:

$$\begin{gathered} Z\left(t, Z^0\right) = \left\{x_1\left(t, Z^0\right), x_2\left(t, Z^0\right), x_3\left(t, Z^0\right)\right\}, \quad \text{for } t > 0, \\ Z\left(0, Z^0\right) = Z^0. \end{gathered} \tag{4.1}$$

Now define Poincâre transformation $A : R_+^3 \to R_+^3$ is

$$A\left(Z^0\right) = Z\left(\omega, Z^0\right). \tag{4.2}$$

In this way, the existence of periodic solution of system (1.2) will be equal to the existence of the fixed point A.

Theorem 4.1. *Assume that the conditions of* (H_1)–(H_3) *hold, then system* (1.2) *with initial condition* (1.3) *has at least one positive* ω*-periodic solution.*

Proof. If assumption (H_1)–(H_3) are satisfied, then from Theorem 2.2 we have that there exist positive constants $T > T_7$ and M_i, m_i $(i = 1, 2, 3)$ such that for all $t \geq T$ and $i = 1, 2, 3$

$$m_i \leq x_i(t) \leq M_i. \tag{4.3}$$

Let

$$K = \{x_1(t), x_2(t), x_3(t) \mid m_i \leq x_i(t) \leq M_i,\ i = 1, 2, 3\}, \tag{4.4}$$

then the compact region $K \subset R_+^3$ is a positive invariant set of system (1.2), and K is also a close bounded convex set. So we have

$$Z^0 \in K \Longrightarrow Z\left(t, Z^0\right) \in K, \tag{4.5}$$

also $Z(\omega, Z^0) \in K$, thus $AK \subset K$. The operator A is continuous because the solution is continuous about the initial value. Using the fixed point theorem of Brower, we can obtain that A has at least one fixed point in K, then there exists at least one strictly positive ω-periodic solution of system (1.2). This ends the proof of Theorem 4.1. □

By constructing similar Lyapunov functional to those of Theorem 3.2, and using Theorem 4.1, we have the following theorem.

Theorem 4.2. *Assume that the conditions of* (H_1)–(H_4) *hold, then system* (1.2) *has a unique positive* ω*-periodic solution which is globally asymptotically stable.*

5. Concluding Remarks

In this paper, a nonautonomous Leslie-Gower type food chain model with time delays is investigated, which is based on the Holling type II and a Leslie-Gower modified functional response. By using comparison theorem, we prove the system is permanent under some appropriate conditions. Further, by constructing the suitable Lyapunov functional, we show that the system is globally asymptotically stable under some appropriate conditions. If the system is periodic one, some sufficient conditions are established, which guarantee the existence, uniqueness and global asymptotic stability of a positive periodic solution of the system. Our results have showed that the permanence, global stability of system and the existence, uniqueness of positive periodic solution depend on delays, and so time delays are profitless.

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