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Research Article

On the Dynamics of a Higher-Order Rational Difference Equation

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The aim of this paper is to investigate the global asymptotic stability and the periodic character for the rational difference equation $x_{n+1} = \alpha x_{n-1}/(\beta + \gamma \Pi_{i=l}^{k} x_{n-2i}^{p_i})$, $n = 0, 1, 2, \ldots$, where the parameters $\alpha, \beta, \gamma, p_l, p_{l+1}, \ldots, p_k$ are nonnegative real numbers, and l, k are nonnegative integers such that $l \leq k$.

1. Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic process, and so forth.

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

Amleh et al. [1] investigated the third-order rational difference equation

$$x_{n+1} = \frac{a + bx_{n-1}}{A + Bx_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where a, b, A, B are nonnegative real numbers and the initial conditions are nonnegative real numbers.

Ahmed [2] studied the global asymptotic behavior and the periodic character of solutions of the third-order rational difference equation

$$x_{n+1} = \frac{bx_{n-1}}{A + Bx_n^p x_{n-2}^q}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where the parameters b, A, B, p, q are nonnegative real numbers, and the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary nonnegative real numbers.

For other related results, see [3] and also [4–15].

In this paper, the global asymptotic behavior and the periodic character of solutions of the rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \prod_{i=l}^{k} x_{n-2i}^{p_i}}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

where the parameters $\alpha, \beta, \gamma, p_l, p_{l+1}, \ldots, p_k$ are nonnegative real numbers, l, k are nonnegative integers such that $l \leq k$, and the initial conditions $x_{-2k}, x_{-2k+1}, \ldots, x_0$ are arbitrary nonnegative real numbers such that

$$\beta + \gamma \prod_{i=l}^{k} x_{n-2i}^{p_i} > 0, \quad \forall n \geq 0, \tag{1.4}$$

will be investigated.

Let I be an interval of real numbers, and let $f : I^{2k+1} \to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-2k}), \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

with $x_{-2k}, x_{-2k+1}, \ldots, x_0 \in I$. Let $\overline{x}$ be the equilibrium point of (1.5). The linearized equation of (1.5) about $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + \cdots + c_{2k+1} y_{n-2k}, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

where

$$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \ldots, \overline{x}), \qquad c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \ldots, \overline{x}), \ldots, \qquad c_{2k+1} = \frac{\partial f}{\partial x_{n-2k}}(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.7}$$

The characteristic equation of (1.5) is

$$\lambda^{2k+1} - c_1 \lambda^{2k} - c_2 \lambda^{2k-1} - \cdots - c_{2k+1} = 0. \tag{1.8}$$

Definition 1.1. Let $\overline{x}$ be an equilibrium point of (1.5).

(i) The equilibrium point $\overline{x}$ of (1.5) is called *locally stable* if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x_{-2k}, x_{-2k+1}, \ldots, x_0 \in I$ with $|x_{-2k} - \overline{x}| + |x_{-2k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta$, we have $|x_n - \overline{x}| < \epsilon$ for all $n \geq -2k$.

(ii) The equilibrium point $\overline{x}$ of (1.5) is called *locally asymptotically stable* if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_{-2k}, x_{-2k+1}, \ldots, x_0 \in I$ with $|x_{-2k} - \overline{x}| + |x_{-2k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma$, we have $\lim_{n \to \infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of (1.5) is called *global attractor* if, for every x_{-2k}, $x_{-2k+1}, \ldots, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of (1.5) is called *globally asymptotically stable* if it is locally stable and global attractor.

(v) The equilibrium point $\overline{x}$ of (1.5) is called *unstable* if it is not stable.

(vi) The equilibrium point $\overline{x}$ of (1.5) is called *source or repeller* if there exists $r > 0$ such that, for all $x_{-2k}, x_{-2k+1}, \ldots, x_0 \in I$ with $0 < |x_{-2k}-\overline{x}|+|x_{-2k+1}-\overline{x}|+\cdots+|x_0-\overline{x}| < r$, there exists $N \geq 1$ such that $|x_N - \overline{x}| \geq r$. Clearly, a repeller is an unstable equilibrium.

Theorem A (linearized stability theorem). *The following statements are true.*

(1) *If all roots of* (1.8) *have modulus less than one, then the equilibrium point $\overline{x}$ of* (1.5) *is locally asymptotically stable.*

(2) *If at least one of the roots of* (1.8) *has modulus greater than one, then the equilibrium point $\overline{x}$ of* (1.5) *is unstable.*

The equilibrium point $\overline{x}$ of (1.5) *is called a "saddle point" if* (1.8) *has roots both inside and outside the unit disk.*

2. The Special Cases $\alpha\beta\gamma\Sigma_{i=l}^{k} p_i = 0$

In this section, we examine the character of solutions of (1.3) when one or more of the parameters in (1.3) are zero.

There are four such equations; namely,

$$x_{n+1} = 0, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

$$x_{n+1} = \frac{\alpha}{\beta} x_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

$$x_{n+1} = \frac{\alpha}{\beta+\gamma} x_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

$$x_{n+1} = \frac{\alpha x_{n-1}}{\gamma \Pi_{i=l}^{k} x_{n-2i}^{p_i}}, \quad n = 0, 1, 2, \ldots. \tag{2.4}$$

Equation (2.1) is trivial, (2.2) and (2.3) are linear, and (2.4) is a non-linear difference equation; the change of variables $x_n = e^{y_n}$ reduces it to a linear difference equation.

3. A General Oscillation Result

The change of variables $x_n = (\beta/\gamma)^{1/\sum_{i=l}^{k} p_i} y_n$ reduces (1.3) to the difference equation

$$y_{n+1} = \frac{r y_{n-1}}{1 + \prod_{i=l}^{k} y_{n-2i}^{p_i}}, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

where $r = \alpha/\beta > 0$.

Note that $\overline{y}_1 = 0$ is always an equilibrium point. When $r > 1$, (3.1) also possesses the unique positive equilibrium $\overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}$.

Theorem B (see [8]). *Assume that $F \in C([0,\infty)^{2k+1} \to [0,\infty))$ is nonincreasing in the odd arguments, and nondecreasing in the even arguments. Let $\overline{x}$ be an equilibrium point of the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-2k}), \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

and let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of (3.2) *such that either*

$$x_{-2k}, x_{-2k+2}, \ldots, x_0 \geq \overline{x}, \qquad x_{-2k+1}, x_{-2k+3}, \ldots, x_{-1} < \overline{x}, \tag{3.3}$$

or

$$x_{-2k}, x_{-2k+2}, \ldots, x_0 < \overline{x}, \qquad x_{-2k+1}, x_{-2k+3}, \ldots, x_{-1} \geq \overline{x}. \tag{3.4}$$

Then $\{x_n\}_{n=-2k}^{\infty}$ oscillates about $\overline{x}$ with semicycles of length one.

Corollary 3.1. *Assume that $r > 1$; let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of* (3.1) *such that either*

$$\begin{gathered} y_{-2k}, y_{-2k+2}, \ldots, y_0 \geq \overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}, \\ y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} < \overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}, \end{gathered} \tag{3.5}$$

or

$$\begin{gathered} y_{-2k}, y_{-2k+2}, \ldots, y_0 < \overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i} \\ y_{-2k+1}, y_{-2k+3}, \ldots, y_{-1} \geq \overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}. \end{gathered} \tag{3.6}$$

Then $\{y_n\}_{n=-2k}^{\infty}$ oscillates about the positive equilibrium point $\overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}$ with semicycles of length one.

Proof. The proof follows immediately from Theorem B. □

4. The Dynamics of (3.1)

In this section, we investigate the dynamics of (3.1) with nonnegative initial conditions.

Theorem 4.1. For (3.1), we have the following results.

(i) Assume that $r < 1$, then the zero equilibrium point is locally asymptotically stable.

(ii) Assume that $r > 1$, then the zero equilibrium point is saddle point.

(iii) The positive equilibrium point $\overline{y}_2$ is unstable.

Proof. The linearized equation associated with (3.1) about $\overline{y}_1 = 0$ has the form

$$z_{n+1} - rz_{n-1} = 0, \quad n = 0, 1, 2, \dots, \tag{4.1}$$

so, the characteristic equation of (3.1) about $\overline{y}_1 = 0$ is

$$\lambda^{2k+1} - r\lambda^{2k-1} = 0, \tag{4.2}$$

then the proof of (i) and (ii) follows immediately from Theorem A.

The linearized equation of (3.1) about $\overline{y}_2 = (r-1)^{1/\Sigma_{i=l}^{k} p_i}$ is

$$z_{n+1} - z_{n-1} + \sum_{i=l}^{k} p_i \left(1 - \frac{1}{r}\right) z_{n-2i} = 0, \quad n = 0, 1, 2, \dots, \tag{4.3}$$

so, the characteristic equation of (3.1) about $\overline{y}_2 = (r-1)^{1/\Sigma_{i=l}^{k} p_i}$ is

$$\lambda^{2k+1} - \lambda^{2k-1} + \sum_{i=l}^{k} p_i \left(1 - \frac{1}{r}\right) \lambda^{2k-2i} = 0. \tag{4.4}$$

Set

$$f(\lambda) = \lambda^{2k+1} - \lambda^{2k-1} + \sum_{i=l}^{k} p_i \left(1 - \frac{1}{r}\right) \lambda^{2k-2i}, \tag{4.5}$$

then $f(-1) = ((\Sigma_{i=l}^{k} p_i)(r-1))/r > 0$, and $\lim_{\lambda \to -\infty} f(\lambda) = -\infty$ so $f(\lambda)$ has at least a root in $(-\infty, -1)$. Then the proof of (iii) follows. □

Theorem 4.2. Assume $r < 1$. Then the zero equilibrium point of (3.1) is globally asymptotically stable.

Proof. We know by Theorem 4.1 that the equilibrium point $\overline{y}_1 = 0$ is locally asymptotically stable of (3.1), and so it suffices to show that $\overline{y}_1 = 0$ is a global atractor of (3.1) as follows:

$$0 \le y_{n+1} = \frac{ry_{n-1}}{1 + \prod_{i=l}^{k} y_{n-2i}^{p_i}} \le ry_{n-1}, \tag{4.6}$$

since $r < 1$, then

$$\lim_{n\to\infty} y_n = 0. \tag{4.7}$$

□

The next theorem shows that (3.1) has a prime-period two solutions when $r = 1$.

Theorem 4.3. For (3.1), we have the following results.

(a) Equation (3.1) possesses the prime-period two solutions

$$\ldots, \phi, 0, \phi, 0, \phi, \ldots \tag{4.8}$$

with $\phi > 0$, when $r = 1$.

(b) Assume that $r = 1$, then every solution of (3.1) converges to a period (not necessarily prime) two solutions (4.8) with $\phi \geq 0$.

Proof. (a) Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{4.9}$$

be period two solutions of (3.1). Then

$$\phi = \frac{r\phi}{1 + \psi^{\Sigma_{i=l}^{k} p_i}}, \qquad \psi = \frac{r\psi}{1 + \phi^{\Sigma_{i=l}^{k} p_i}}. \tag{4.10}$$

If $\phi \neq 0$ and $\psi \neq 0$, then $\phi = \psi = (r-1)^{1/\Sigma_{i=l}^{k} p_i}$, which is impossible. Hence, $\psi = 0$ which implies that $(r-1)\phi = 0$, so $r = 1$.

(b) Assume that $r = 1$, and let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of (3.1), then

$$y_{n+1} - y_{n-1} = \frac{-y_{n-1}\Pi_{i=l}^{k} y_{n-2i}^{p_i}}{1 + \Pi_{i=l}^{k} y_{n-2i}^{p_i}} \leq 0. \tag{4.11}$$

So the even terms of this solution decrease to a limit (say $\Phi \geq 0$), and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus,

$$\Phi = \frac{\Phi}{1 + \Psi^{\Sigma_{i=l}^{k} p_i}}, \qquad \Psi = \frac{\Psi}{1 + \Phi^{\Sigma_{i=l}^{k} p_i}}, \tag{4.12}$$

which implies that

$$\Phi\Psi^{\Sigma_{i=l}^{k} p_i} = 0, \qquad \Psi\Phi^{\Sigma_{i=l}^{k} p_i} = 0. \tag{4.13}$$

This completes the proof. □

The next theorem shows that when $r > 1$, (3.1) possesses unbounded solutions.

Theorem 4.4. Assume $r > 1$. Then (3.1) possesses unbounded solutions. In particular, every solution of (3.1) which oscillates about the equilibrium $\overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}$ with semicycles of length one is unbounded.

Proof. we will prove that every solution $\{y_n\}_{n=-2k}^{\infty}$ of (3.1) which oscillates with semicycles of length one is unbounded (see corollary 3.1).

Assume that $\{y_n\}_{n=-2k}^{\infty}$ is a solution of (3.1) such that

$$y_{2n+1} < \overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}, \qquad y_{2n} > \overline{y}_2 = (r-1)^{1/\sum_{i=l}^{k} p_i}, \ n \geq -k. \tag{4.14}$$

Then

$$\begin{aligned} y_{2n+2} &= \frac{r y_{2n}}{1 + \prod_{i=l}^{k} y_{2n+1-2i}^{p_i}} > \frac{r y_{2n}}{1 + \overline{y}_2^{\sum_{i=l}^{k} p_i}} = y_{2n}, \\ y_{2n+3} &= \frac{r y_{2n+1}}{1 + \prod_{i=l}^{k} y_{2n+2-2i}^{p_i}} < \frac{r y_{2n+1}}{1 + \overline{y}_2^{\sum_{i=l}^{k} p_i}} = y_{2n+1}. \end{aligned} \tag{4.15}$$

From which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = 0, \tag{4.16}$$

which completes the proof. □

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