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*Research Article*

# Chaotic Attractor Generation via Space Function Controls

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The analysis of chaotic attractor generation is given, and the generation of novel chaotic attractor is introduced in this paper. The underlying mechanism involves two simple linear systems with one-dimensional, two-dimensional, or three-dimensional space functions. Moreover, it is demonstrated by simulation that various attractor patterns are generated conveniently by adjusting suitable space functions' parameters and the statistic behavior is also discussed.

## 1. Introduction

Owing to theoretical development in mathematics and technological advances in engineering, complex phenomena are rapidly becoming possible to be studied systematically. As one of the essences of natural complexity, chaos has been found to be very useful in a variety of applications such as science, mathematics, and engineering communities [1–4] and various techniques such as identification and synchronization [3, 5, 6]. In recent years, people tend to introduce the chaos to many applications and for their purpose, (chaotic) attractors in different shapes may be needed for desired dynamical behaviors. As a result, effective generation of different chaotic attractors with simple techniques becomes an interesting problem in the past decade.

Many chaotic attractors have been found numerically, and experimentally and it is relatively easy to generate chaotic systems numerically, but it is usually very hard to analyze or verify the dynamical characteristics of nonsmooth systems, even for the switched systems with low dimensions [7, 8]. To deal with the stability of the equilibrium of switched (linear) systems, many efforts have been made and strict analysis has been carried out [9, 10]. However, in the studies of complex nonsmooth phenomena, there has not been any effective mathematical method, though differential inclusions provide a strict way to describe discontinuous dynamics [11].

Some new chaotic systems have been developed in [12–16], but there does not seem to be a general methodology for generating chaos. In [17], a chaotic attractor in a new funnel shape is introduced, simply by designing a switched system with hysteresis switching signal. It also could be regarded as a method of chaotic attractor generation with one-dimensional space function. In this paper, we propose a new method of chaotic attractor generation for two linear systems. It is shown that chaos can be generated by applying an appropriate rule of the space functions. This new rule can generate different types of chaos or chaos-like behaviors from different pairs of linear systems.

The rest of this paper is organized as follows. Section 2 presents the structure of two simple linear systems to generate a new chaotic attractor with one-dimensional space function. Section 3 introduces two-dimensional and three-dimensional space functions to generate new chaotic attractors. Then, Section 4 concentrates on the pattern changes of the generated attractors with parameters variation. A brief conclusion is given in Section 5.

## 2. Specification of the Chaotic Attractor Generation by One-Dimensional Space Function

In this section, we first introduce two simple linear systems for the generation of chaotic attractors. Consider the following system:

$$\dot{X}(t) = A_1 X(t), \tag{2.1}$$

$$\dot{X}(t) = A_2 X(t), \tag{2.2}$$

where the state $X = (x, y, z)^T \in R^3$ and

$$A_1 = \begin{pmatrix} a & b_1 & 0 \\ -b_1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & b_2 & 0 \\ -b_2 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}, \tag{2.3}$$

Furthermore, the parameters $a$, $b_1$, $b_2$, and $c$ are chosen to satisfy

$$a > 0, \quad c > 0, \quad a^2 - 4b_i^2 < 0, \quad i = 1, 2. \tag{2.4}$$

We introduce one-dimensional space function:

$$f(z) = z \in (z_1, z_2) \tag{2.5}$$

to make the system trajectory switch between $z_1$ and $z_2$, where $z_1$ and $z_2$ are positive constants satisfying $z_1 \leq z_2$. The switching rule is constructed as follows. When system (2.1) is

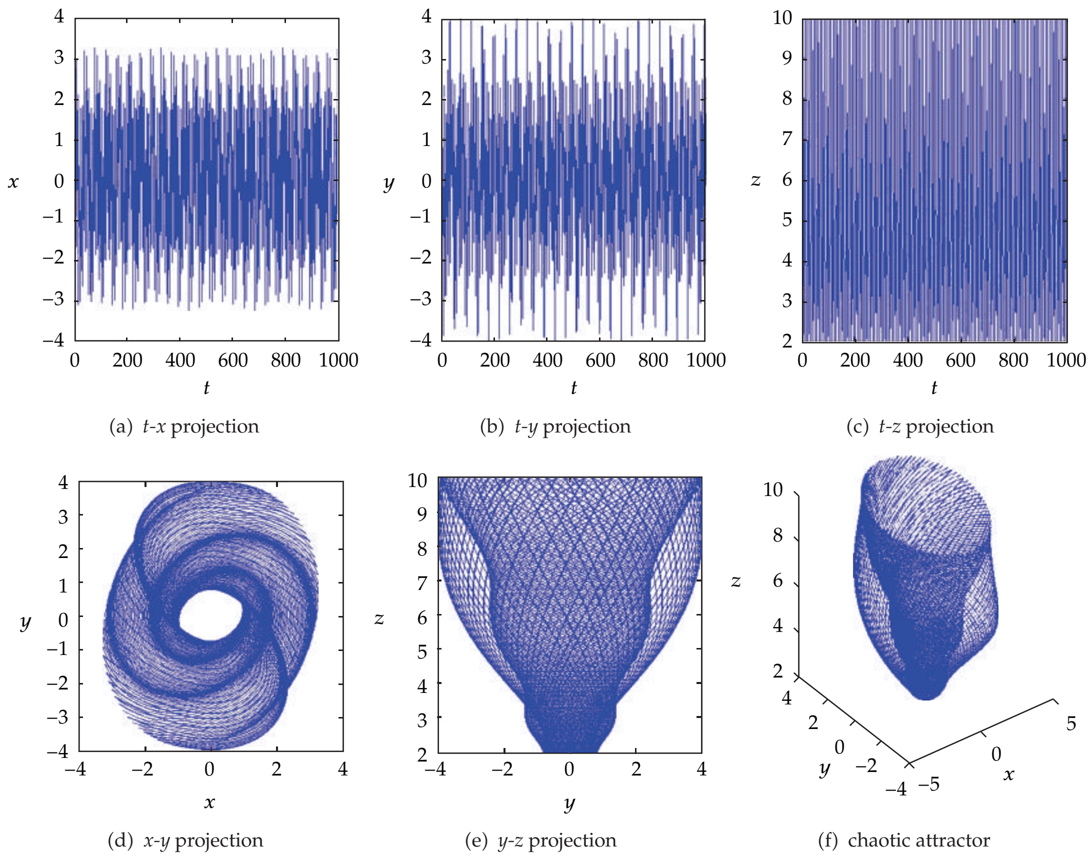



**Figure 1:** Chaotic attractor generated by one-dimensional space function (2.5).

active, it will switch to system (2.2) at time $t_1$ if $z(t_1) = z_2$. Similarly, when system (2.2) is active, it will switch to system (2.1) at time $t_2$ if $z(t_2) = z_1$.

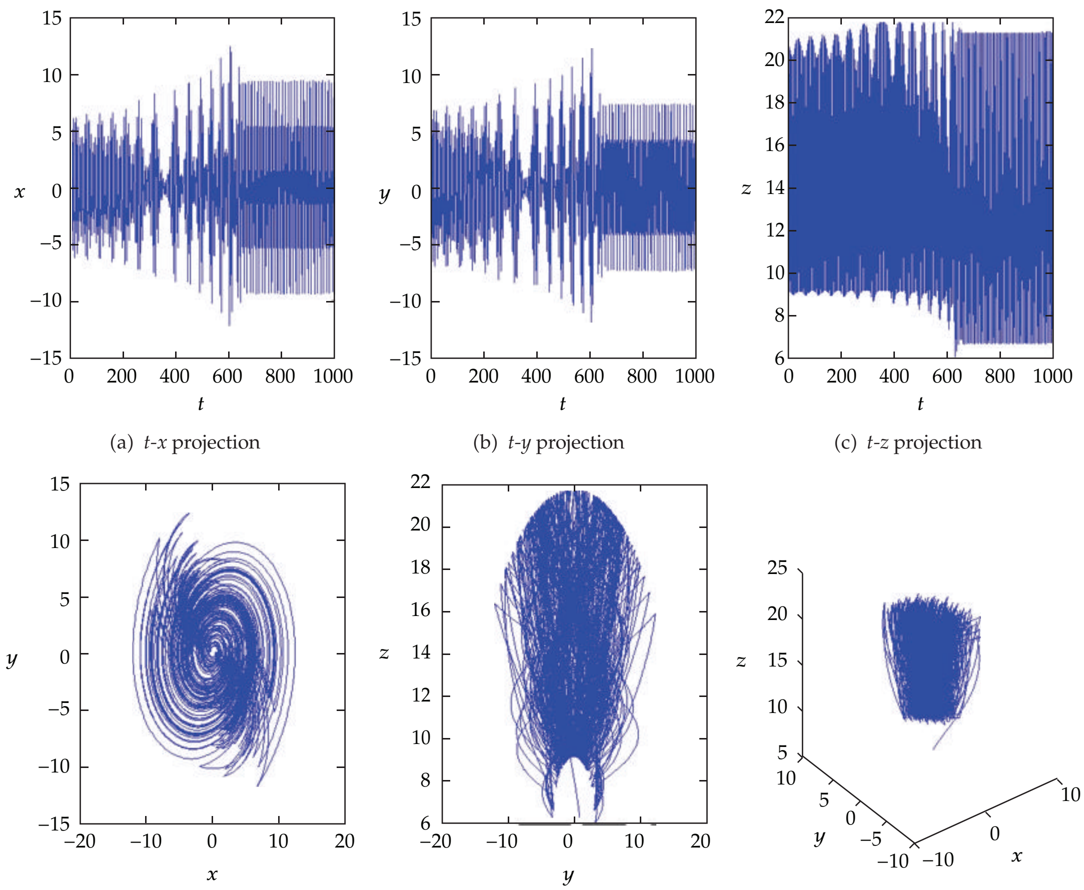
With this switching rule, the switched system will generate chaos or chaos-like behavior if the system parameters are properly chosen. As shown in Figure 1, the switched system has a chaotic attractor, where

$$a = 0.7, \qquad b_1 = 0.8, \qquad b_2 = -2, \qquad c = 0.4 \tag{2.6}$$

and we assume that $z_1 = 2$, $z_2 = 10$. The maximum Lyapunov exponent is $-6.7206e - 004$, which indicates the chaotic behavior of the switched system.

Solving $\dot{x} = \dot{y} = \dot{z} = 0$ yields the two linear systems equilibrium $(0\,0\,0)^T$. The equilibrium is unstable since all the real parts of the three eigenvalues of Jacobian $J$, at the origin for system (2.1),

$$J = \begin{pmatrix} a & b_1 & 0 \\ -b_1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{2.7}$$




(a) *t-x* projection (b) *t-y* projection (c) *t-z* projection

(d) *x-y* projection (e) *y-z* projection (f) chaotic attractor

**Figure 2:** Chaotic attractor generated by two-dimensional space function (3.1).

are positive due to $a > 0$ and $c > 0$. Obviously, the system (2.1) does not have a stable equilibrium.

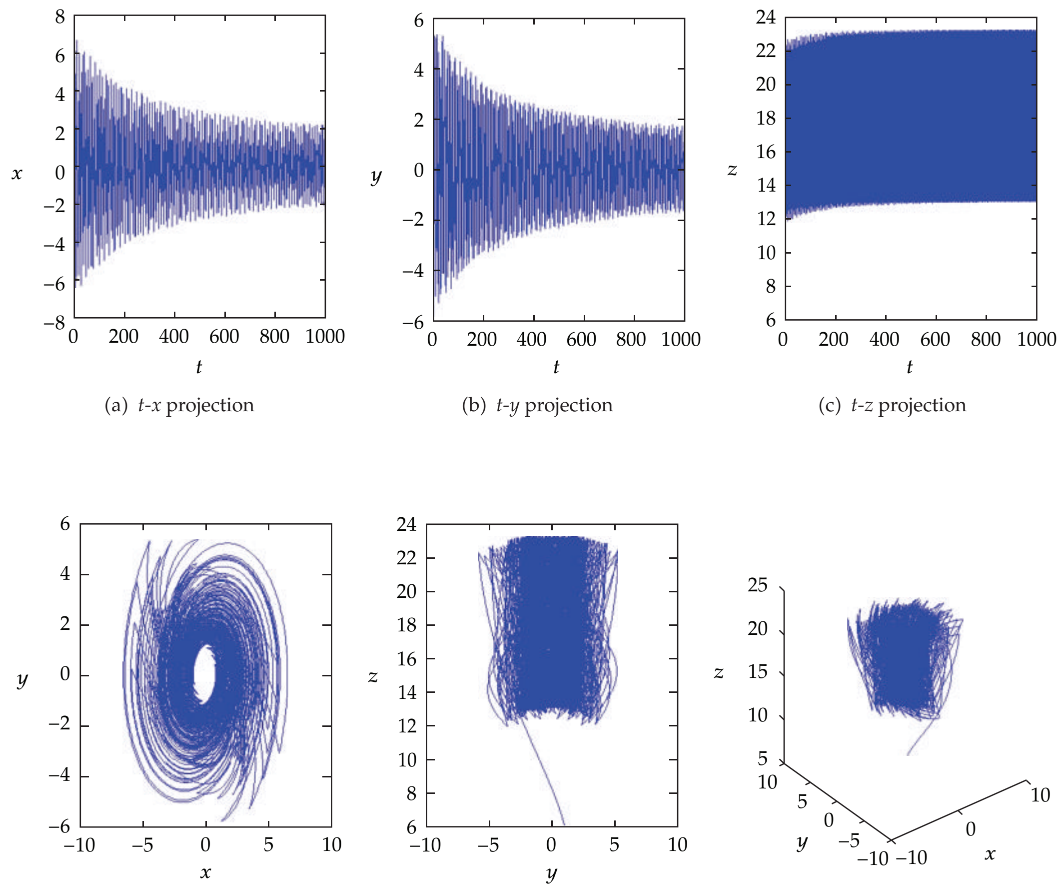
Similarly, the system (2.2) has a stable equilibrium since all the real parts of the three eigenvalues of Jacobian $J$, at the origin for system (2.2),

$$J = \begin{pmatrix} 0 & b_2 & 0 \\ -b_2 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}, \tag{2.8}$$

are negative due to $a > 0$ and $c > 0$.

In addition, it is easy to see that the state trajectory of the systems (2.1) and (2.2) stays in the region of

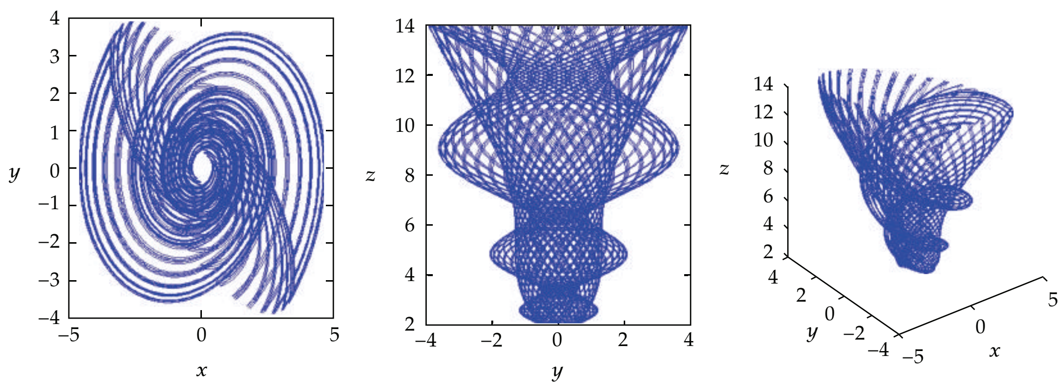
$$D = \left\{ (x, y, z)^T \in R^3 \mid 0 < z_1 \leq z \leq z_2 \right\}. \tag{2.9}$$




(a) $t$-$x$ projection (b) $t$-$y$ projection (c) $t$-$z$ projection

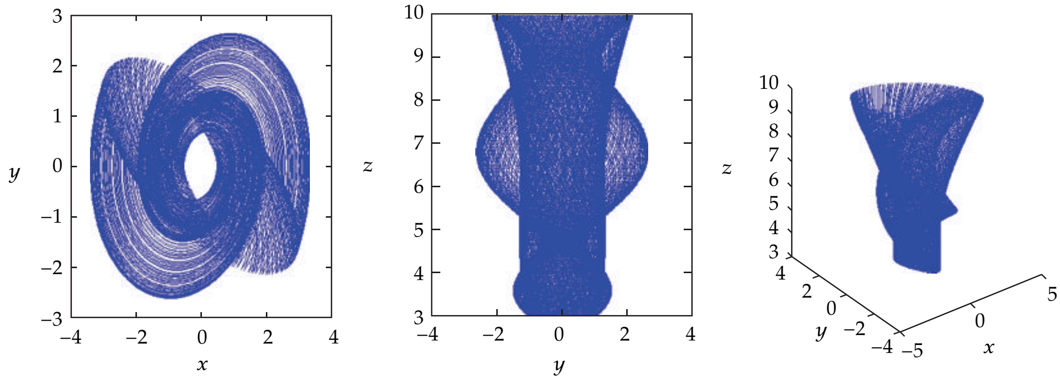
(d) $x$-$y$ projection (e) $y$-$z$ projection (f) chaotic attractor

**Figure 3:** Chaotic attractor generated by three-dimensional space function (3.2).



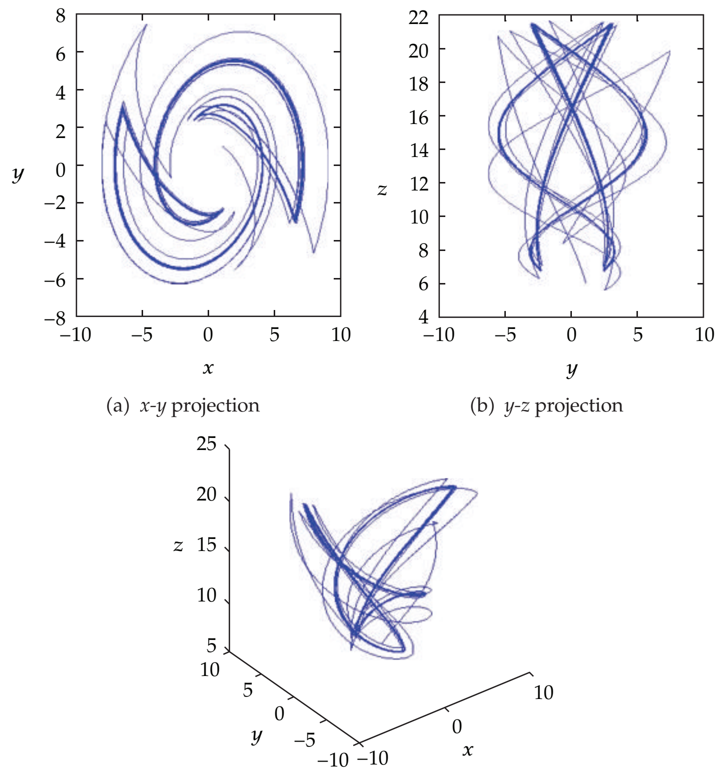

(a) $x$-$y$ projection (b) $y$-$z$ projection (c) chaotic attractor

**Figure 4:** Chaotic attractor generated by $z_1 = 2$, $z_2 = 14$.




(a) $x$-$y$ projection (b) $y$-$z$ projection (c) chaotic attractor

**Figure 5:** Chaotic attractor generated by $z_1 = 3$, $z_2 = 10$.

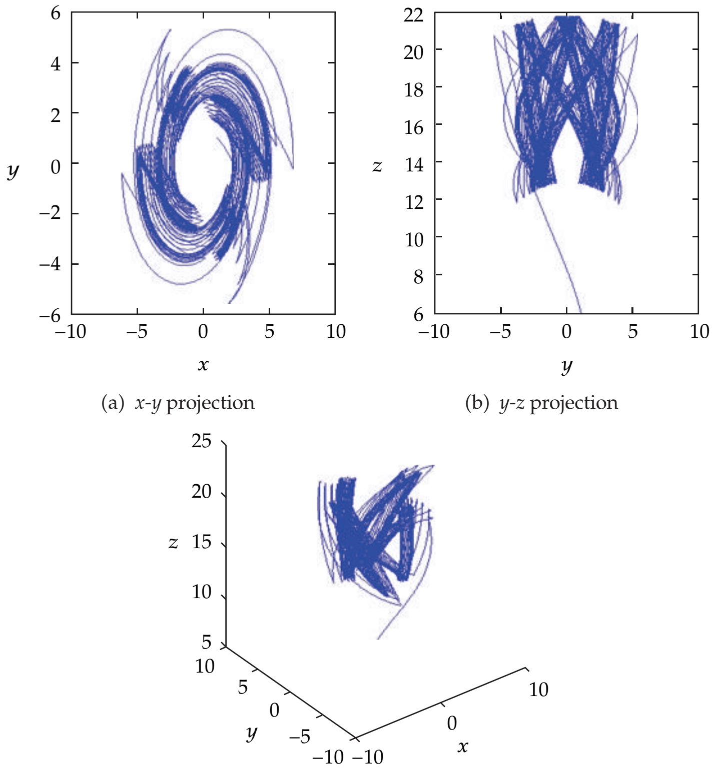



(a) $x$-$y$ projection (b) $y$-$z$ projection

(c) Chaotic attractor

**Figure 6:** Chaotic attractor generated by $F_1 = 6$, $F_2 = 250$.

In fact, once the state of the system (2.1) reaches the plane $\{z = z_2\}$, according to the switching rule, it will switch to system (2.2), then the velocity along the direction $z$ is $\dot{z} = -cz_2 < -cz_1 < 0$, which means that $z$ will not be greater than $z_2$. Similarly, when the state of the system (2.2) reaches the plane $\{z = z_1\}$, it will switch to system (2.1) and the velocity along the direction $z$ is $\dot{z} = cz_1 > 0$, which implies that $z$ will never be less than $z_1$. Hence, the system trajectory switches between $z_1$ and $z_2$.




(a) $x$-$y$ projection

(b) $y$-$z$ projection

(c) Chaotic attractor

**Figure 7:** Chaotic attractor generated by $F_1 = 48$, $F_2 = 250$.

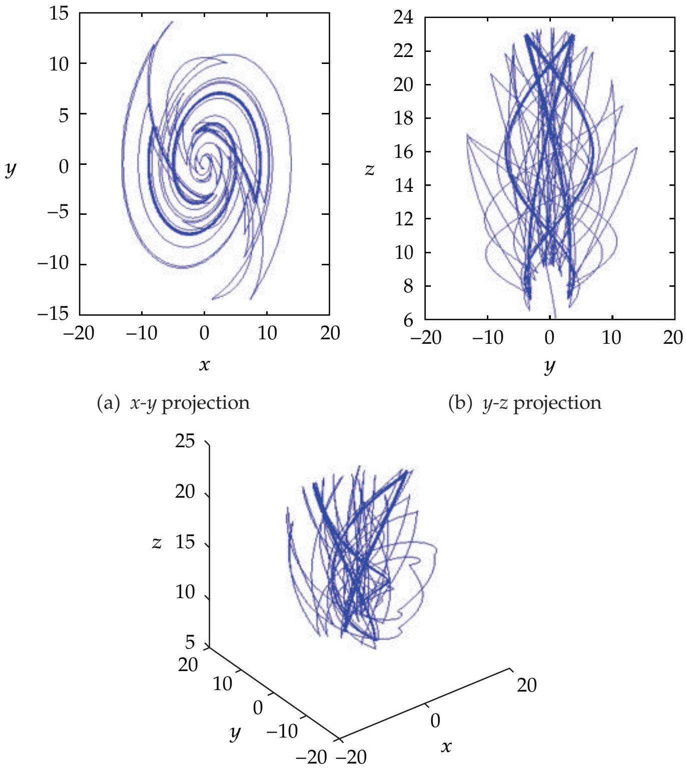
From the foregoing discussion, we observe that as $t \to +\infty$, the system switches between system (2.1) and system (2.2) and the state orbits never go out from the region $D$. Hence, the state orbits are folded and stretched repeatedly, leading to the generation of chaos or chaos-like behaviors.

## 3. Generating Chaotic Attractors by Two-Dimensional and Three-Dimensional Space Functions

From the analysis in Section 2, the region of the system orbits is restricted by one-dimensional space function, leading to the generation of chaos or chaos-like behaviors. Similarly, we can introduce two-dimensional and three-dimensional space functions to generate new chaotic attractors. Still considering the system (2.1) and the system (2.2), we introduce a simple two-dimensional space function:

$$f(y,z) = y^2 + (z-6)^2 \in (F_1, F_2) \tag{3.1}$$

to make the system trajectory switch between $F_1$ and $F_2$, where $F_1$ and $F_2$ are positive constants satisfying $F_1 \leq F_2$. The switching rule is constructed as follows. When system (2.1)




(a) $x$-$y$ projection (b) $y$-$z$ projection

(c) Chaotic attractor

**Figure 8:** Chaotic attractor generated by $F_1 = 10$, $F_2 = 300$.

is active, it will switch to system (2.2) at time $t_1^*$ if $f(y(t_1^*), z(t_1^*)) = F_2$. Similarly, when system (2.2) is active, it will switch to system (2.1) at time $t_2^*$ if $f(y(t_2^*), z(t_2^*)) = F_1$.

With this switching rule, we can generate chaos or chaos-like behavior by the system parameters chosen as (2.6) and we assume that $F_1 = 10$, $F_2 = 250$. As shown in Figure 2, the maximum Lyapunov exponent is 0.0017.

Similarly, we can introduce a three-dimensional space function as following:

$$g(x, y, z) = x^2 + y^2 + (z - 6)^2 \in (G_1, G_2) \tag{3.2}$$

to make the system trajectory switch between $G_1$ and $G_2$, where $G_1$ and $G_2$ are positive constants satisfying $G_1 \le G_2$. The switching rule is constructed as follows. When system (2.1) is active, it will switch to system (2.2) at time $t_1^{**}$ if $g(x(t_1^{**}), y(t_1^{**}), z(t_1^{**})) = G_2$. Similarly, when system (2.2) is active, it will switch to system (2.1) at time $t_2^{**}$ if $g(x(t_2^{**}), y(t_2^{**}), z(t_2^{**})) = G_1$.

With this switching rule, we can generate chaos or chaos-like behavior by the system parameters chosen as (2.6) and we assume that $G_1 = 50$, $G_2 = 300$. As shown in Figure 3, the maximum Lyapunov exponent is $5.1735e - 004$.

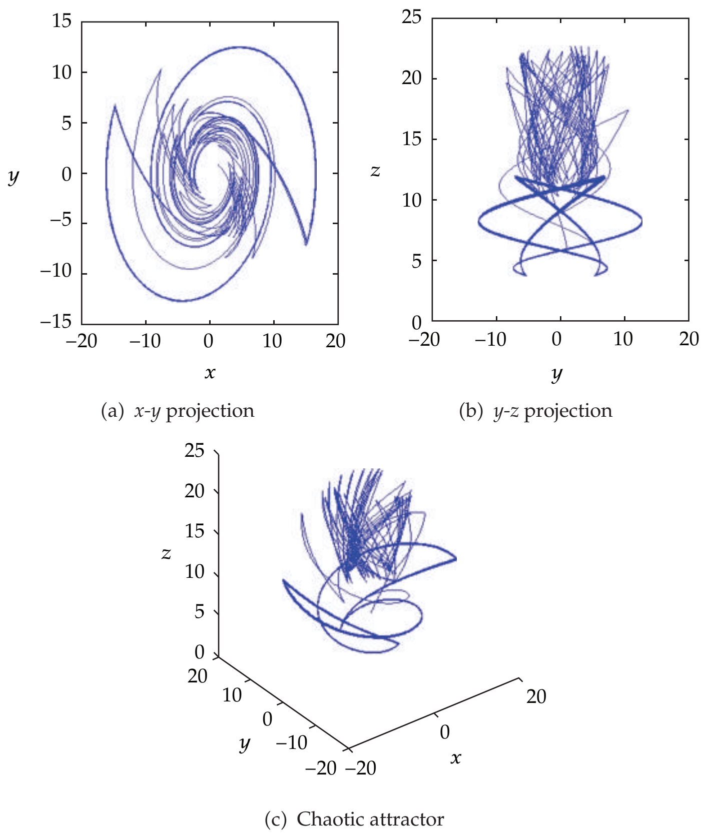



**Figure 9:** Chaotic attractor generated by $G_1 = 40$, $G_2 = 300$.

## 4. Various Patterns with Parameter Changing

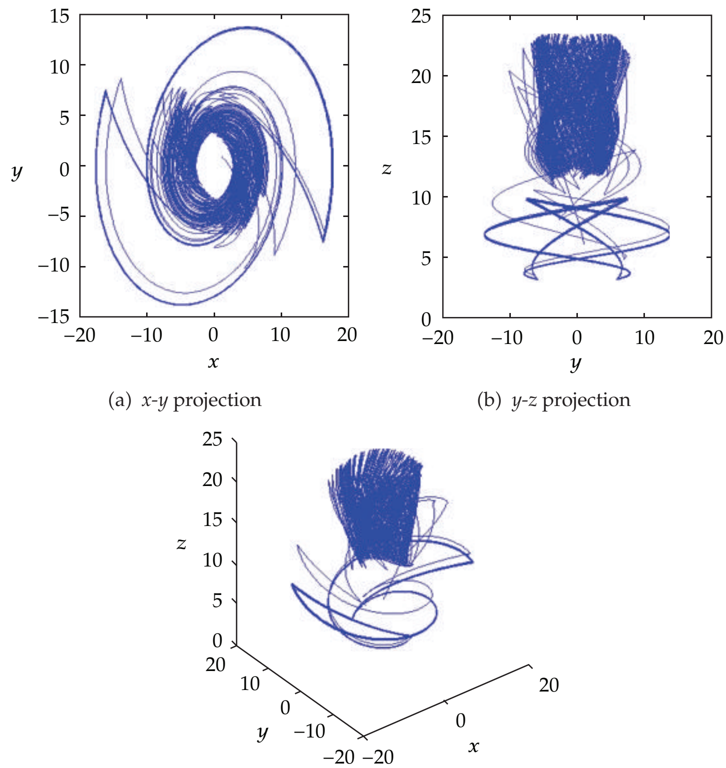
In this section, we pay attention to the dynamical behaviors of the system (2.1) and the system (2.2) with parameter of space functions selected in the "chaotic" regions in order to show the effective generation of various patterns of attractors based on the parameter selection.

At first, we consider one-dimensional space function: let $z_1$ and $z_2$ change in the stability intervals. Then, the system displays different patterns for different values of $z_1$ and $z_2$, as shown in Figures 4 and 5.

In the two cases, the largest Lyapunov exponents are

$$\begin{aligned} \text{LE} &= 1.4720e - 004 \quad (z_1 = 2, z_2 = 14), \\ \text{LE} &= -7.9221e - 004 \quad (z_1 = 3, z_2 = 10). \end{aligned} \tag{4.1}$$

Then, we consider two-dimensional space function: let $F_1$ and $F_2$ change in the stability intervals. Then, the system displays different patterns for different values of $F_1$ and $F_2$, as shown in Figures 6, 7, and 8.




(a) $x$-$y$ projection

(b) $y$-$z$ projection

(c) Chaotic attractor

**Figure 10:** Chaotic attractor generated by $G_1 = 50$, $G_2 = 330$.

The largest Lyapunov exponents are given as following:

$$\begin{gathered}
\mathrm{LE} = 7.5329e\text{–}004 \quad (F_1 = 6, F_2 = 250), \\
\mathrm{LE} = 0.0012 \quad (F_1 = 48, F_2 = 250), \\
\mathrm{LE} = 0.0016 \quad (F_1 = 10, F_2 = 300).
\end{gathered} \tag{4.2}$$

Finally, we consider three-dimensional space function; the system displays different patterns for different values of $G_1$ and $G_2$, as shown in Figures 9 and 10.

The largest Lyapunov exponents are given as following:

$$\mathrm{LE} = 0.0021 \quad (G_1 = 40, G_2 = 300), \qquad \mathrm{LE} = 0.0016 \quad (G_1 = 50, G_2 = 330). \tag{4.3}$$

These numerical simulations verify that space functions dominate rich complex patterns when adjusting parameters. From this, we can see that the proposed space functions are quite effective in the generation of attractor with obviously quasiperiodic or chaotic behaviors based on the change of parameters. Moreover, the statistic behavior is also researched by giving the largest Lyapunov exponents.

## 5. Conclusion

This paper has presented a new control method for generating chaos or chaos-like dynamics. The generation of novel chaotic attractors via two simple three-dimensional linear systems with various space functions is introduced. The results once again support the long-accepted belief that properly designed simple systems can perform complex dynamical behaviors. Moreover, this system can produce various attractor patterns within a wide range of parameter values and the statistic behavior which reveals the regularities in the complex dynamics is also discussed; other space functions also can be chosen and can generate chaotic attractors with various system parameters too. In addition, the method which has been developed in this paper can also be applied to nonlinear dynamical systems and other fields. It is desirable that one could design more chaos generators by means of the method proposed in this paper.

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