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Research Article

New Construction Weighted (h, q)-Genocchi Numbers and Polynomials Related to Zeta Type Functions

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The fundamental aim of this paper is to construct (h, q)-Genocchi numbers and polynomials with weight α. We shall obtain some interesting relations by using p-adic q-integral on $\mathbb{Z}_p$ in the sense of fermionic. Also, we shall derive the (h, q)-extensions of zeta type functions with weight α from the Mellin transformation of this generating function which interpolates the (h, q)-Genocchi numbers and polynomials with weight α at negative integers.

1. Introduction, Definitions, and Notations

Let p be a fixed odd prime number. Throughout this paper we use the following notations. $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The p-adic absolute value is defined by $|p|_p = 1/p$. In this paper, we assume $|q-1|_p < 1$ as an indeterminate. In [1–3], Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.1}$$

$[x]_q$ is a q-extension of x which is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.2}$$

see [1–15].

Note that $\lim_{q\to 1}[x]_q = x$.

Let $f_n(x) = f(x+n)$. By the definition (1.1) we easily get

$$\begin{aligned} -qI_{-q}(f_1) &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x+1)(-q)^{x+1} \\ &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^{x} - (1+q)\lim_{N\to\infty}\frac{f(p^N)q^{p^N}+f(0)}{1+q^{p^N}} \\ &= I_{-q}(f) - [2]_q f(0). \end{aligned} \tag{1.3}$$

Continuing this process, we obtain easily the relation

$$q^n I_{-q}(f_n) + (-1)^{n-1}I_{-q}(f) = [2]_q\sum_{l=0}^{n-1}(-1)^{n-l-1}q^l f(l), \tag{1.4}$$

(h,q)-Genocchi numbers are defined as follows:

$$G_{0,q}^{(h)} = 0, \qquad q^{h-2}\left(qG_q^{(h)}+1\right)^n + G_{n,q}^{(h)} = \begin{cases} [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.5}$$

with the usual convention about replacing $(G_q^{(h)})^n$ by $G_{n,q}^{(h)}$ (see [6]).

In this paper, we constructed (h,q)-Genocchi numbers and polynomials with weight α. By using fermionic p-adic q-integral equations on $\mathbb{Z}_p$, we investigated some interesting identities and relations on the (h,q)-Genocchi numbers and polynomials with weight α. Furthermore, we derive the q-extensions of zeta type functions with weight α from the Mellin transformation of this generating function which interpolates the (h,q)-Genocchi polynomials with weight α.

2. On the Weighted (h,q)-Genocchi Numbers and Polynomials

In this section, by using fermionic p-adic q-integral equations on $\mathbb{Z}_p$, some interesting identities and relation on the (h,q)-Genocchi numbers and polynomials with weight α are shown.

Definition 2.1. Let $\alpha, n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Then the (h,q)-Genocchi numbers with weight α defined by as follows:

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} = [2]_q\sum_{m=0}^{\infty}(-1)^m q^{mh}[m]_{q^\alpha}^n. \tag{2.1}$$

If we take $h = 1$ to (2.1), then we have, $\widetilde{G}_{n+1,q}^{(\alpha,1)} = \widetilde{G}_{n+1,q}^{(\alpha)}$ (see [5]).
From (2.1), we obtain

$$\begin{aligned}\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} &= \frac{[2]_q}{(1-q^\alpha)^n}\sum_{m=0}^{\infty}(-1)^m q^{mh}(1-q^{m\alpha})^n \\ &= \frac{[2]_q}{(1-q^\alpha)^n}\left[\sum_{m=0}^{\infty}(-1)^m q^{mh}\sum_{l=0}^{n}\binom{n}{l}(-1)^l (q^{m\alpha})^l\right] \\ &= \frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\sum_{m=0}^{\infty}(-1)^m q^{m\alpha l+mh} \\ &= \frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{\alpha l+h}}\end{aligned} \tag{2.2}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $\alpha, n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Then*

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} = \frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{\alpha l+h}}. \tag{2.3}$$

In (1.1), one takes $f(x) = q^{(h-1)x}[x]_{q^\alpha}^n$,

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{(h-1)x}[x]_{q^\alpha}^n d\mu_{-q}(x) &= \frac{1}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\int_{\mathbb{Z}_p} q^{x(\alpha l+h-1)}d\mu_{-q}(x) \\ &= \frac{1}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1}\left(-q^{\alpha l+h}\right)^x \\ &= \frac{1}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{(1+q)}{1+q^{\alpha l+h}}\lim_{N\to\infty}\frac{1+(q^{\alpha l+h})^{p^N}}{1+q^{p^N}} \\ &= \frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{\alpha l+h}} \\ &= \frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1}.\end{aligned} \tag{2.4}$$

From [12], we obtain (h,q)-Genocchi numbers with weight α witt's type formula as follows.

Theorem 2.3. *For $\alpha, n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Then*

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} = \int_{\mathbb{Z}_p} q^{(h-1)x}[x]_{q^\alpha}^n d\mu_{-q}(x). \tag{2.5}$$

From (2.1), one easily gets

$$\int_{\mathbb{Z}_p} q^{(h-1)x} e^{t[x]_{q^\alpha}} d\mu_{-q}(x) = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{t[m]_{q^\alpha}}. \tag{2.6}$$

By (2.6), one has

$$\sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha,h)} \frac{t^n}{n!} = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{t[m]_{q^\alpha}}. \tag{2.7}$$

Therefore, we obtain the following corollary.

Corollary 2.4. *If $\widetilde{G}_{0,q}^{(\alpha,h)} = 0$. Let $D_q^{(\alpha,h)}(t) = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha,h)}(t^n/n!)$. Then*

$$D_q^{(\alpha,h)}(t) = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{t[m]_{q^\alpha}}. \tag{2.8}$$

Now, one considers the (h,q)-Genocchi polynomials with weight α as follows:

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{(h-1)y} [x+y]_{q^\alpha}^n d\mu_{-q}(y), \quad n \in \mathbb{N},\ \alpha \in \mathbb{N}^*. \tag{2.9}$$

From (2.9), one sees that

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1} = \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{1+q^{\alpha l+h}} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh} [m+x]_{q^\alpha}^n. \tag{2.10}$$

Let $D_q^{(\alpha,h)}(t,x) = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha,h)}(x)(t^n/n!)$. Then, one has

$$D_q^{(\alpha,h)}(t,x) = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{t[m+x]_{q^\alpha}} = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha,h)}(x) \frac{t^n}{n!}. \tag{2.11}$$

By (1.4), one sees that

$$q^{hn} \frac{\widetilde{G}_{m+1,q}^{(\alpha,h)}(n)}{m+1} + (-1)^{n-1} \frac{\widetilde{G}_{m+1,q}^{(\alpha,h)}}{m+1} = [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} q^{hl} [l]_{q^\alpha}^m. \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *For $m, h \in \mathbb{N}$, and $\alpha, n \in \mathbb{N}^*$, one has*

$$q^{hn}\frac{\widetilde{G}^{(\alpha,h)}_{m+1,q}(n)}{m+1}+(-1)^{n-1}\frac{\widetilde{G}^{(\alpha,h)}_{m+1,q}}{m+1}=[2]_q\sum_{l=0}^{n-1}(-1)^{n-l-1}q^{hl}[l]^m_{q^\alpha}. \tag{2.13}$$

In (1.3), it is known that

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0). \tag{2.14}$$

If we take $f(x)=q^{(h-1)x}e^{t[x]_{q^\alpha}}$, then one has

$$\begin{aligned}[2]_q&=q\int_{\mathbb{Z}_p}q^{(h-1)(x+1)}e^{t[x+1]_{q^\alpha}}d\mu_{-q}(x)+\int_{\mathbb{Z}_p}q^{(h-1)x}e^{t[x]_{q^\alpha}}d\mu_{-q}(x)\\&=\sum_{m=0}^{\infty}\left(q^h\frac{\widetilde{G}^{(\alpha,h)}_{m+1,q}(1)}{m+1}+\frac{\widetilde{G}^{(\alpha,h)}_{m+1,q}}{m+1}\right)\frac{t^m}{m!}.\end{aligned} \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.6. *For $\alpha \in \mathbb{N}^*$ and $m, h \in \mathbb{N}$, one has*

$$\widetilde{G}^{(\alpha,h)}_{0,q}=0,\qquad q^h\frac{\widetilde{G}^{(\alpha,h)}_{m+1,q}(1)}{m+1}+\frac{\widetilde{G}^{(\alpha,h)}_{m+1,q}}{m+1}=\begin{cases}[2]_q, & \text{if } m=0,\\ 0, & \text{if } m\neq 0.\end{cases} \tag{2.16}$$

From (2.9), one can easily derive the following:

$$\begin{aligned}\int_{\mathbb{Z}_p}q^{(h-1)y}[x+y]^n_{q^\alpha}d\mu_{-q}(y)&=\frac{[d]^n_{q^\alpha}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^{ha}\int_{\mathbb{Z}_p}q^{dy(h-1)}\left[\frac{x+a}{d}+y\right]^n_{q^{d\alpha}}d\mu_{(-q)^d}(y)\\&=\frac{[d]^n_{q^\alpha}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^{ha}\frac{\widetilde{G}^{(\alpha,h)}_{n+1,q^d}((x+a)/d)}{n+1}.\end{aligned} \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.7. *For $d \equiv 1 (\mathrm{mod}\ 2)$, $n \in \mathbb{N}^*$ and $\alpha, h \in \mathbb{N}$*

$$\widetilde{G}^{(\alpha,h)}_{n+1,q}(x)=\frac{[d]^n_{q^\alpha}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^a q^{ha}\widetilde{G}^{(\alpha,h)}_{n+1,q^d}\left(\frac{x+a}{d}\right). \tag{2.18}$$

3. Interpolation Function of the Polynomials $\widetilde{G}_{n,q}^{(\alpha,h)}(x)$

In this section, we give interpolation function of the generating functions of (h,q)-Genocchi polynomials with weight α. For $s \in \mathbb{C}$ and $h \in \mathbb{N}$, by applying the Mellin transformation to (2.11), we obtain

$$\Im_q^{(\alpha,h)}(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\left(-D_q^{(\alpha,h)}(-t,x)\right)dt = [2]_q\sum_{m=0}^{\infty}(-1)^m q^{mh}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t[m+x]_{q^\alpha}}dt, \quad (3.1)$$

so we have

$$\Im_q^{(\alpha,h)}(s,x) = [2]_q\sum_{m=0}^{\infty}\frac{(-1)^m q^{mh}}{[m+x]_{q^\alpha}^s}. \quad (3.2)$$

We define q-extension zeta type function as follows.

Theorem 3.1. *For $s \in \mathbb{C}$, $h \in \mathbb{N}$, and $\alpha \in \mathbb{N}^*$. One has*

$$\Im_q^{(\alpha,h)}(s,x) = [2]_q\sum_{m=0}^{\infty}\frac{(-1)^m q^{mh}}{[m+x]_{q^\alpha}^s}. \quad (3.3)$$

$\Im_q^{(\alpha,h)}(s,x)$ *can be continued analytically to an entire function.*

By subsituting $s=-n$ into (3.3) one easily gets

$$\Im_q^{(\alpha,h)}(-n,x) = \frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1}. \quad (3.4)$$

We obtain the following theorem.

Theorem 3.2. *For $h \in \mathbb{N}$ and $q, s \in \mathbb{C}$, $|q| < 1$. Then one defines*

$$\Im_q^{(\alpha,h)}(-n,x) = \frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1}. \quad (3.5)$$

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