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Research Article

Mean Convergence Rate of Derivatives by Lagrange Interpolation on Chebyshev Grids

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We consider the rate of mean convergence of derivatives by Lagrange interpolation operators based on the Chebyshev nodes. Some estimates of error of the derivatives approximation in terms of the error of best approximation by polynomials are derived. Our results are sharp.

1. Introduction and Main Results

Mean convergence of Lagrange interpolation based on the zeros of orthogonal polynomials (and possibly some additional points) has been studied for at least 70 years. There is a vast literature on this topic. The authors of [1–3] considered the simultaneous approximation by the Hermite interpolation operators, and we will consider the simultaneous approximation by Lagrange interpolation operators based on the zeros of Chebyshev polynomials. The relevant results can be found in [4–6]. We introduce these results below.

Let

$$\omega(x) = \prod_{k=1}^{N} \left|x - y_k\right|^{\Gamma_k} \left(|x| \le 1; -1 = y_1 < y_2 < \cdots < y_N = 1; \Gamma_k > -1; k = 1, \ldots, N\right) \tag{1.1}$$

be a so-called generalized Jacobi weight ($\omega \in GJ$), and let

$$-1 \le x_1 < x_2 < \cdots < x_n \le 1 \tag{1.2}$$

be the zeros of the nth orthogonal polynomial $p_n(\omega)$ associated with the weight-function $\omega \in GJ$. Let $L_n(\omega, f)$ denote the Lagrange interpolating polynomial which interpolates f at the zeros of $p_n(\omega)$. By using Markov-Bernstein type inequalities in L_p metric, J. Szabados and A. K. Varma [5] reduced the weighted mean convergence of derivatives $L_n^{(r)}(\omega, f, x)$ to the weighted mean convergence of $L_n(\omega, f, x)$ and obtained the following. If L^p means functional space equipped with L_p norm and

$$\omega(x) \in GJ, \quad \frac{\omega(x)^{1/p-1/2}}{(1-x^2)^{1/4}} \in L^p, \tag{$*$}$$

then, for $f^{(r)}(x) \in C[-1,1]$, we have

$$\int_{-1}^{1} \left| f^{(r)}(x) - L_n^{(r)}(f,x) \right|^p \left(1-x^2\right)^{rp/2} \omega(x)dx \le C_r E_{n-r-1}^p\left(f^{(r)}\right) \quad (n \ge r+1). \tag{1.3}$$

Here and in the following, the constant C_r (may be different in the same expression) is independent of n and f but depends on r, and $E_n(\cdot)$ denotes the error of the best polynomial approximation of degree n of the corresponding function in the L_∞ metric.

Mastroianni and Nevai [4] get sharper estimates in terms of modulus of continuity instead of the best approximation. It improves some old results. But its proof also needs weighted Markov-Bernstein type inequality in L^p metric and the idea of additional points. For the weight functions not satisfying $(*)$, it is not possible to discuss by their method. To deal with these case, Du and Xu [7] consider the most important special case $\omega(x) = 1/\sqrt{1-x^2}$. Let

$$t_k = t_{kn} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, \ldots, n, \tag{1.4}$$

be the zeros of $T_n(x) = \cos n\theta,\ x = \cos\theta$, the nth degree Chebyshev polynomial of the first kind. If $f \in C[-1,1]$, then the well-known Lagrange interpolation polynomial of f based on $\{t_k\}_{k=1}^n$ is given by (see [8])

$$L_n(f,x) = \sum_{k=1}^{n} f(t_k)\ell_k(x), \tag{1.5}$$

where

$$\ell_k(x) = \frac{(-1)^{k+1}\sqrt{1-t_k^2}\,T_n(x)}{n(x-t_k)}, \quad k = 1, \ldots, n. \tag{1.6}$$

Du and Xu [7] obtained the following.

Theorem A. *Let $L_n(f,x)$ be as defined as above. Then, for $f \in C^1_{[-1,1]}$, we have*

$$\left(\int_{-1}^{1} \left|f'(x)-L_n'(f,x)\right|^p \left(1-x^2\right)^{\alpha} dx\right)^{1/p} \le \begin{cases} CE_{n-2}(f'), & \alpha > \dfrac{p}{2}-1, \\ C(\ln n)^{1/p} E_{n-2}(f'), & \alpha = \dfrac{p}{2}-1, \\ Cn^{1-((2\alpha+2)/p)} E_{n-2}(f'), & -1 < \alpha < \dfrac{p}{2}-1, \end{cases} \tag{1.7}$$

and the estimation for $-1 < \alpha \le (p/2)-1$ is sharp.

We notice that although the sharp estimate is obtained, the upper bound is not $E_{n-2}(f')$ for $-1 \le \alpha \le (p/2)-1$. Now we will give a Lagrange interpolation to improve their results.

Let

$$x_k = x_{kn} = \cos\frac{k\pi}{n+1}, \quad k = 1,\ldots,n, \tag{1.8}$$

be the zeros of $U_n(x) = \sin(n+1)\theta/\sin\theta$, $x=\cos\theta$, the nth degree Chebyshev polynomial of the second kind. If $f \in C[-1,1]$, then the well-known Lagrange interpolation polynomial of f based on $\{x_k\}_{k=1}^n \cup \{x_0 = 1, x_{n+1} = -1\}$ is given by (see [9])

$$Q_{n+2}(f,x) = \sum_{k=0}^{n+1} f(x_k)\varphi_k(x), \tag{1.9}$$

where

$$\begin{gathered} \varphi_0(x) = \frac{(1+x)U_n(x)}{2(n+1)}, \qquad \varphi_{n+1}(x) = \frac{(-1)^n(x-1)U_n(x)}{2(n+1)}, \\ \varphi_k(x) = \frac{(-1)^{k+1}(1-x^2)U_n(x)}{(n+1)(x-x_k)}, \quad k=1,\ldots,n. \end{gathered} \tag{1.10}$$

Firstly, we obtain the following.

Theorem 1.1. *Let $Q_n(f,x)$ be as defined as above, $0 < p < +\infty$, $\alpha > -1$. Then, for $f \in C^1_{[-1,1]}$, we have*

$$\left(\int_{-1}^{1} \left|f'(x)-Q_n'(f,x)\right|^p \left(1-x^2\right)^{\alpha} dx\right)^{1/p} \le CE_{n-2}(f'). \tag{1.11}$$

By Theorem A and Theorem 1.1, we know that $Q_{n+2}(f,x)$ have better convergence rate than $L_n(f,x)$ in the case $-1 \le \alpha \le (p/2)-1$. But for continuous function approximation, we noticed that Q_n have the same approximation order with L_n, that is, if $0 < p < +\infty, \alpha > -1$, then, for $f \in C_{[-1,1]}$, from Hölder inequality [8, 9], it follows that

$$\left(\int_{-1}^{1} |f(x) - L_n(f,x)|^p \left(1 - x^2\right)^{\alpha} dx\right)^{1/p} \leq CE_{n-1}(f),$$
$$\left(\int_{-1}^{1} |f(x) - Q_n(f,x)|^p \left(1 - x^2\right)^{\alpha} dx\right)^{1/p} \leq CE_{n-1}(f). \tag{1.12}$$

For high derivatives approximation, how the cases are? Secondly, we will consider second derivative approximation by L_n and Q_n and obtain the following.

Theorem 1.2. *Let $Q_n(f,x)$ and $L_n(f,x)$ be as defined as above. Then, for $f \in C^2_{[-1,1]}$, we have*

$$\left(\int_{-1}^{1} |f''(x) - Q''_n(f,x)|^p \left(1 - x^2\right)^{\alpha} dx\right)^{1/p} \leq \begin{cases} CE_{n-3}(f''), & \alpha > \dfrac{p}{2} - 1, \\ C(\ln n)^{1/p} E_{n-3}(f''), & \alpha = \dfrac{p}{2} - 1, \\ Cn^{1-(2\alpha+2)/p} E_{n-3}(f''), & -1 < \alpha < \dfrac{p}{2} - 1, \end{cases}$$
$$\left(\int_{-1}^{1} |f''(x) - L''_n(f,x)|^p \left(1 - x^2\right)^{\alpha} dx\right)^{1/p} \leq \begin{cases} CE_{n-3}(f''), & \alpha > p - 1, \\ C(\ln n)^{1/p} E_{n-3}(f''), & \alpha = p - 1, \\ Cn^{2-(2\alpha+2)/p} E_{n-3}(f''), & -1 < \alpha < p - 1, \end{cases} \tag{1.13}$$

and the estimation for $-1 < \alpha \leq (p/2) - 1$ or ($-1 < \alpha \leq p - 1$) is sharp.

From Theorem 1.2, we know that for the second derivative approximation, Q_n have better approximation orders than L_n in the case $-1 < \alpha \leq p - 1$.

Using the same way as in the proof of Theorem 1.2, we can consider the r order derivatives approximation for $r \geq 3$, but the computation is more complicated, and we omit the detail.

2. Some Lemmas

We introduce some lemmas which are the main tools in our proof.

Lemma 2.1 (see [10, p. 519]). *If $f \in C^r_{[-1,1]}$, then there exists an algebraic polynomial $p_n(x)$ of degree at most n such that*

$$\left|f^{(j)}(x) - p_n^{(j)}(x)\right| \leq C\left[\frac{\sqrt{1 - x^2}}{n}\right]^{r-j} E_{n-r}\left(f^{(j)}\right), \quad j = 0, 1, \ldots, r. \tag{2.1}$$

In the past, the error estimate depended on the Markov-Bernstein type inequalities in L_p metric. In this paper, we will use the inequality in L_∞ metric.

Lemma 2.2 (see [7, p. 50]). *Let $\varphi_k(x)$ be as defined by* (1.10), $\alpha > -1$. *Then, for any fixed* $p > 0$,

$$\left(\int_{-1}^{1}\left|\sum_{k=1}^{n}A_k\varphi_k(x)\right|^p\left(1-x^2\right)^\alpha dx\right)^{1/p} \le C\max_{1\le k\le n}|A_k|. \tag{2.2}$$

To prove our results, we need to build another polynomial integral inequality in L_∞ metric. For its proof, we introduce two lemmas.

Lemma 2.3 (see [8, p. 914]). *Let $v_1, v_2, \ldots, v_{2N}$ be distinct integers between 1 and n. Then, we have*

$$\int_{-1}^{1}\ell_{v_1}(x)\ell_{v_2}(x)\cdots\ell_{v_{2N}}(x)\frac{dx}{\sqrt{1-x^2}} = 0, \tag{2.3}$$

and it is well known that

$$\sum_{k=1}^{n}\ell_k^2(x) \le 2. \tag{2.4}$$

Let $x_1, \ldots, x_n$ be independent variables, s are positive integers, and

$$V_s = \left(\sum_{k=1}^{n}x_k^s\right)^{1/s}. \tag{2.5}$$

By the mathematical induction we can obtain the following.

Lemma 2.4. *If N is a positive integer, $n > 2N$, then, the homogeneous symmetrical polynomial of degree $2N$:*

$$B_{2N} = \left(\sum_{i=1}^{n}x_i\right)^{2N} - (2N)!\sum_{k_1<k_2<\cdots<k_{2N}}x_{k_1}\cdots x_{k_{2N}}, \tag{2.6}$$

can be represented as a homogeneous polynomial of degree $2N$ about $V_1, \ldots, V_{2N}$:

$$B_{2N} = \sum_{t_1\le 2N-2,\, t_i\ge 0}B_{t_1\cdots t_{2N}}V_1^{t_1}\cdots V_{2N}^{t_{2N}}. \tag{2.7}$$

Now we give the inequality in L_∞ metric which plays a key role in our paper.

Lemma 2.5. *Let $\ell_k(x)$ be as defined by* (2.1), $\alpha > -1$. *Then, for any fixed* $p > 0$,

$$\left(\int_{-1}^{1}\left|\sum_{k=1}^{n}A_k\ell_k(x)\right|^p\left(1-x^2\right)^\alpha dx\right)^{1/p} \le C\max_{1\le k\le n+1}|A_k|. \tag{2.8}$$

Proof. Firstly, we will consider the special case $p = 2N, \alpha = -1/2$ by induction on N. For $N = 1$, by (2.3) and (2.4), we obtain

$$\begin{aligned}\int_{-1}^{1}\left|\sum_{k=1}^{n}A_k\ell_k(x)\right|^2\frac{dx}{\sqrt{1-x^2}} &= \sum_{k=1}^{n}A_k^2\int_{-1}^{1}\ell_k^2(x)\frac{dx}{\sqrt{1-x^2}} + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1}A_kA_j\int_{-1}^{1}\ell_k(x)\ell_j(x)\frac{dx}{\sqrt{1-x^2}}\\ &\le \max_{1\le k\le n}|A_k|^2\int_{-1}^{1}\sum_{k=1}^{n}\ell_k^2(x)\frac{dx}{\sqrt{1-x^2}} \le 2\pi\max_{1\le k\le n}|A_k|^2.\end{aligned} \tag{2.9}$$

Suppose that for $0 < p \le 2(N-1)$, we have

$$\int_{-1}^{1}\left|\sum_{k=1}^{n}A_k\ell_k(x)\right|^p\frac{dx}{\sqrt{1-x^2}} \le C_p\max_{1\le k\le n}|A_k|^p. \tag{2.10}$$

For $p = 2N$, if $n \le 2N$, then, (2.4) gives

$$\int_{-1}^{1}\left|\sum_{k=1}^{n}A_k\ell_k(x)\right|^{2N}\frac{dx}{\sqrt{1-x^2}} \le \pi(4N)^{2N}\max_{1\le k\le n}|A_k|^{2N}. \tag{2.11}$$

If $n > 2N$, then by Lemma 2.4, we know

$$\begin{aligned}\left|\sum_{k=1}^{n}A_k\ell_k(x)\right|^{2N} &= (2N)!\sum_{k_1<k_2<\cdots<k_{2N}}A_{k_1}\cdots A_{k_{2N}}\ell_{k_1}(x)\cdots\ell_{k_{2N}}(x)\\ &\quad+\sum_{t_1\le 2N-2, t_i\ge 0}B_{t_1\cdots t_{2N}}V_1^{t_1}(x)\cdots V_{2N}^{t_{2N}}(x)\\ &= I_1(x)+I_2(x),\end{aligned} \tag{2.12}$$

where

$$V_s(x) = \left(\sum_{k=1}^{n}A_k^s\ell_k^s(x)\right)^{1/s}. \tag{2.13}$$

From (2.3), it follows that

$$\int_{-1}^{1}I_1(x)\frac{dx}{\sqrt{1-x^2}} = 0. \tag{2.14}$$

From (2.4), we know that, for $s \ge 2$,

$$|V_s(x)| \le \max_{1\le k\le n}|A_k|\left(\sum_{k=1}^{n}|\ell_k(x)|^s\right)^{1/s} \le \sqrt{2}\max_{1\le k\le n}|A_k|. \tag{2.15}$$

By virtue of (2.12) and (2.15), we have

$$\begin{aligned}\left|\int_{-1}^{1} I_2(x)\frac{dx}{\sqrt{1-x^2}}\right| &\leq \sum_{t_1\leq 2N-2, t_i\geq 0} |B_{t_1\cdots t_{2N}}| \int_{-1}^{1}\left|V_1^{t_1}(x)\cdots V_{2N}^{t_{2N}}(x)\right|\frac{dx}{\sqrt{1-x^2}} \\ &\leq \sum_{t_1\leq 2N-2, t_i\geq 0} 2^N |B_{t_1\cdots t_{2N}}| \max_{1\leq k\leq n}|A_k|^{2N-t_1}\int_{-1}^{1}\left|V_1^{t_1}(x)\right|\frac{dx}{\sqrt{1-x^2}} \\ &\leq \sum_{t_1\leq 2N-2, t_i\geq 0} 2^N |B_{t_1\cdots t_{2N}}|\left(\pi+\sum_{i=1}^{2N-2} C_i\right)\max_{1\leq k\leq n}|A_k|^{2N}.\end{aligned} \tag{2.16}$$

From (2.11), (2.12), (2.14), and (2.16), it follows that

$$\int_{-1}^{1}\left|\sum_{k=1}^{n} A_k\ell_k(x)\right|^{2N}\frac{dx}{\sqrt{1-x^2}} \leq C_{2N}\max_{1\leq k\leq n}|A_k|^{2N}. \tag{2.17}$$

Now we consider the general case. For arbitrary $p > 0$ and $\alpha > -1$, it is easy to see that we can choose a positive integer N satisfying $p/4N < 1$ and $(\alpha+(p/4N))/(1-(p/2N)) > -1$. By Hölder inequality and (2.17), we can obtain

$$\begin{aligned}&\int_{-1}^{1}\left|\sum_{k=1}^{n} A_k\ell_k(x)\right|^{p}\left(1-x^2\right)^{\alpha}dx \\ &\quad\leq \left(\int_{-1}^{1}\left|\sum_{k=1}^{n} A_k\ell_k(x)\right|^{2N}\frac{dx}{\sqrt{1-x^2}}\right)^{p/2N}\left(\int_{-1}^{1}\left(1-x^2\right)^{(\alpha+(p/4N))/(1-(p/2N))}dx\right)^{1-(p/2N)} \\ &\quad\leq C_p\max_{1\leq k\leq n}|A_k|^{p}.\end{aligned} \tag{2.18}$$

□

Remark 2.6. P. Erdös and E. Feldheim [8] give a proof for $p = 2, 4$ and $\alpha = -1/2$. We give a mathematical induction proof for completion.

3. Proof of Theorem 1.1

We will consider $Q_{n+2}(f, x)$ instead of $Q_n(f, x)$ for simplicity. For $f \in C^1_{[-1,1]}$, let $p_{n+1}(x)$ be the polynomial of degree at most $n + 1$ satisfying (2.1). It is easily checked that for $-1 \leq x \leq 1$,

$$f(x) - Q_{n+2}(f, x) = f(x) - p_{n+1}(x) + Q_{n+2}(p_{n+1} - f, x). \tag{3.1}$$

From (3.1), we can conclude that

$$f'(x) - Q'_{n+2}(f, x) = f'(x) - p'_{n+1}(x) + Q'_{n+2}(p_{n+1} - f, x) = I_1(x) + I_2(x). \tag{3.2}$$

From (2.1), we can derive

$$\int_{-1}^{1} |I_1(x)|^p \left(1-x^2\right)^{\alpha} dx \le CE_n^p(f') \int_{-1}^{1} \left(1-x^2\right)^{\alpha} dx \le CE_n^p(f'). \tag{3.3}$$

It is easy to see that $I_2(x)$ is a polynomial of degree at most n. Hence,

$$\begin{aligned} I_2(x) &= \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \varphi_k'(x) = L_{n+1}(I_2, x) \\ &= \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \varphi_k'(t_s) \right] \ell_s(x). \end{aligned} \tag{3.4}$$

By a direct computation, we know

$$\varphi_k'(t_s) = \frac{(-1)^{k+s+1}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^2} + \frac{(-1)^{k+s+1} t_s}{(n+1)\sqrt{1-t_s^2}(t_s-x_k)}. \tag{3.5}$$

Combining (3.4) and (3.5), we derive

$$\begin{aligned} I_2(x) &= \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^2} \right] \ell_s(x) \\ &\quad + \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1} t_s}{(n+1)\sqrt{1-t_s^2}(t_s-x_k)} \right] \ell_s(x) \\ &= J_1(x) + J_2(x). \end{aligned} \tag{3.6}$$

We consider $J_1(x)$ first. For an arbitrary $1 \le s \le n+1$,

$$\begin{aligned} \left| \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^2} \right| &\le \frac{CE_n(f')}{n^2} \sum_{k=1}^{n} \frac{\sqrt{1-x_k^2}\sqrt{1-t_s^2}}{(t_s-x_k)^2} \\ &\le \frac{CE_n(f')}{n^2} \left(\sum_{k=1}^{n} \frac{1-t_s^2}{(t_s-x_k)^2} + \sum_{k=1}^{n} \frac{1-x_k^2}{(t_s-x_k)^2} \right). \end{aligned} \tag{3.7}$$

Similar to [9, p. 71], we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{(1-x^2)U_n^2(x)}{(x-x_k)^2}&=\left(1-x^2\right)\left[\left(U_n'(x)\right)^2-U_n(x)U_n''(x)\right],\\ \sum_{k=1}^{n}\frac{(1-x_k^2)U_n^2(x)}{(x-x_k)^2}&=\sum_{k=1}^{n}\frac{(1-x^2)U_n^2(x)}{(x-x_k)^2}+2xU_n(x)U_n'(x)-nU_n^2(x).\end{aligned}\tag{3.8}$$

By [9, p. 71], we know

$$\left(1-x^2\right)U_n'(x)=xU_n(x)-(n+1)T_{n+1}(x),\tag{3.9}$$

$$\left(1-x^2\right)U_n''(x)=3xU_n'(x)-n(n+2)U_n(x).\tag{3.10}$$

Let $x=t_s$, then by (3.8), (3.9), and (3.10), we obtain

$$\sum_{k=1}^{n}\frac{1-t_s^2}{(t_s-x_k)^2}=n(n+2)-\frac{2t_s^2}{1-t_s^2}\le n(n+2),\tag{3.11}$$

$$\sum_{k=1}^{n}\frac{1-x_k^2}{(t_s-x_k)^2}=n^2+n.\tag{3.12}$$

From (3.7), (3.11), and (3.12), we obtain that for an arbitrary $1\le s\le n+1$,

$$\left|\sum_{k=1}^{n}\left(p_{n+1}(x_k)-f(x_k)\right)\frac{(-1)^{k+s+1}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^2}\right|\le CE_n(f').\tag{3.13}$$

From (2.8) and (3.13), we can obtain

$$\left(\int_{-1}^{1}|J_1(x)|^p\left(1-x^2\right)^{\alpha}dx\right)^{1/p}\le CE_n(f').\tag{3.14}$$

Now we consider $J_2(x)$. Exchanging the summation order, we have

$$\begin{aligned}J_2(x)&=\sum_{k=1}^{n}\frac{(-1)^k\left(p_{n+1}(x_k)-f(x_k)\right)}{n+1}\left[\sum_{s=1}^{n+1}\frac{(-1)^{s+1}t_s}{\sqrt{1-t_s^2}(t_s-x_k)}\ell_s(x)\right]\\ &=\sum_{k=1}^{n}\frac{(-1)^k\left(p_{n+1}(x_k)-f(x_k)\right)}{(n+1)^2}\left[\sum_{s=1}^{n+1}\frac{t_sT_{n+1}(x)}{(t_s-x_k)(x-t_s)}\right].\end{aligned}\tag{3.15}$$

It is easy to know

$$\sum_{s=1}^{n+1}\frac{T_{n+1}(x)}{x-t_s}=T'_{n+1}(x)=(n+1)U_n(x). \tag{3.16}$$

Let $x=x_k$, then, we have

$$\sum_{s=1}^{n+1}\frac{1}{x_k-t_s}=0. \tag{3.17}$$

By (3.16), (3.17), and the identity

$$\frac{t_sT_{n+1}(x)}{(t_s-x_k)(x-t_s)}=\frac{xT_{n+1}(x)}{(x-x_k)(x-t_s)}-\frac{x_kT_{n+1}(x)}{(x-x_k)(x_k-t_s)}, \tag{3.18}$$

we conclude that

$$\sum_{s=1}^{n+1}\frac{t_sT_{n+1}(x)}{(t_s-x_k)(x-t_s)}=\frac{(n+1)\ xU_n(x)}{x-x_k}. \tag{3.19}$$

From (3.15) and (3.19), it follows that

$$J_2(x)=\sum_{k=1}^{n}\frac{(-1)^k(p_{n+1}(x_k)-f(x_k))}{n+1}\cdot\frac{xU_n(x)}{x-x_k}. \tag{3.20}$$

For an arbitrary $1\le k\le n$, by (3.20), (2.1), $|U'_n(x_k)|=(n+1)/(1-x_k^2), k=1,2,\dots,n$, and a simple computation, we can obtain

$$|J_2(x_k)|=\left|\frac{(-1)^k(p_{n+1}(x_k)-f(x_k))}{n+1}\cdot x_kU'_n(x_k)\right|\le CE_n(f'). \tag{3.21}$$

For $k=0$, by (2.1), $U_n(1)=n+1$ and a simple computation we obtain

$$|J_2(1)|=\left|\sum_{k=1}^{n}\frac{(-1)^k(p_{n+1}(x_k)-f(x_k))}{1-x_k}\right|\le\frac{CE_n(f')}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1-x_k}}. \tag{3.22}$$

From $2x/\pi\le\sin x\le x$, for all $x\in[0,\pi/2]$, we derive

$$\sum_{k=1}^{n}\frac{1}{\sqrt{1-x_k}}=\sum_{k=1}^{n}\frac{1}{\sqrt{2}\sin k\pi/2(n+1)}\le\sum_{k=1}^{n}\frac{n+1}{k}\le Cn\ln n. \tag{3.23}$$

Hence,

$$|J_2(1)| \le C \ln nE_n(f'). \tag{3.24}$$

Similarly,

$$|J_2(-1)| \le C \ln nE_n(f'). \tag{3.25}$$

The fact that $J_2(x)$ is an algebraic polynomial of degree at most n implies

$$J_2(x) = Q_{n+2}(J_2, x) = J_2(1)\varphi_0(x) + J_2(-1)\varphi_{n+1}(x) + \sum_{k=1}^{n} J_2(x_k)\varphi_k(x). \tag{3.26}$$

Let $x = \cos\theta$. By (3.24) and a simple computation similar to [11, p. 204], we obtain that, for $p > 0$ and $\alpha > -1$,

$$\int_{-1}^{1} \left|J_2(1)\varphi_0(x)\right|^p \left(1-x^2\right)^{\alpha} dx \le \frac{C\ln^p nE_n^p(f')}{(n+1)^p} \int_0^{\pi} \frac{|\sin n\theta|^p}{\sin^{p-2\alpha-1}\theta} d\theta \le CE_n^p(f'). \tag{3.27}$$

Similarly,

$$\int_{-1}^{1} \left|J_2(-1)\varphi_{n+1}(x)\right|^p \left(1-x^2\right)^{\alpha} dx \le CE_n^p(f'). \tag{3.28}$$

By virtue of (2.2) and (3.21), we have

$$\int_{-1}^{1} \left|\sum_{k=1}^{n} J_2(x_k)\varphi_k(x)\right|^p \left(1-x^2\right)^{\alpha} dx \le CE_n^p(f'). \tag{3.29}$$

From (3.26), (3.27), (3.28), and (3.29), it follows that

$$\left(\int_{-1}^{1} |J_2(x)|^p \left(1-x^2\right)^{\alpha} dx\right)^{1/p} \le CE_n(f'). \tag{3.30}$$

By (3.2), (3.3), (3.6), (3.14), and (3.30), we obtain the upper estimate.

4. Proof of Theorem 1.2

We consider Q_n first. We will consider $Q_{n+2}(f,x)$ instead of $Q_n(f,x)$ for simplicity. For $f \in C^2_{[-1,1]}$, let $p_{n+1}(x)$ be the polynomial of degree at most $n+1$ satisfying (2.1). From (3.1), it follows that

$$f''(x) - Q''_{n+2}(f,x) = f''(x) - p''_{n+1}(x) + Q''_{n+2}(p_{n+1} - f, x) = M_1(x) + M_2(x). \tag{4.1}$$

From (2.1), we can derive

$$\int_{-1}^{1} |M_1(x)|^p \left(1-x^2\right)^{\alpha} dx \le C E_{n-1}^{p}(f''). \tag{4.2}$$

Similar to (3.4),

$$M_2(x) = \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \varphi_k''(t_s) \right] \ell_s(x). \tag{4.3}$$

By a direct computation, we get

$$\begin{aligned} \varphi_k''(t_s) = {} & \frac{2(-1)^{k+s}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^3} + \frac{2(-1)^{k+s} t_s}{(n+1)\sqrt{1-t_s^2}(t_s-x_k)^2} + \frac{(n+1)(-1)^{k+s+1}}{\sqrt{1-t_s^2}(t_s-x_k)} \\ & + \frac{(-1)^{k+s+1}}{(n+1)\left(1-t_s^2\right)^{3/2}(t_s-x_k)}. \end{aligned} \tag{4.4}$$

Equations (4.3) and (4.4) yield

$$\begin{aligned} M_2(x) = {} & \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{2(-1)^{k+s}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^3} \right] \ell_s(x) \\ & + \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{2(-1)^{k+s} t_s}{(n+1)\sqrt{1-t_s^2}(t_s-x_k)^2} \right] \ell_s(x) \\ & + \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(n+1)(-1)^{k+s+1}}{\sqrt{1-t_s^2}(t_s-x_k)} \right] \ell_s(x) \\ & + \sum_{s=1}^{n+1} \left[\sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1}}{(n+1)\left(1-t_s^2\right)^{3/2}(t_s-x_k)} \right] \ell_s(x) \\ = {} & N_1(x) + N_2(x) + N_3(x) + N_4(x). \end{aligned} \tag{4.5}$$

We consider $N_1(x)$ now. For an arbitrary $1 \le s \le n+1$, from (2.1), (3.12), and $\sum_{k=1}^{n} |\varphi_k(x)|^2 \le 2$ (see [9]), it follows that

$$\begin{aligned}\left|\sum_{k=1}^{n}(p_{n+1}(x_k)-f(x_k))\frac{2(-1)^{k+s}\sqrt{1-t_s^2}}{(n+1)(t_s-x_k)^3}\right| &\le \frac{CE_{n-1}(f'')}{(n+1)^3}\sum_{k=1}^{n}\frac{(1-x_k^2)\sqrt{1-t_s^2}}{|t_s-x_k|^3} \\ &= \frac{CE_{n-1}(f'')}{(n+1)^2}\sum_{k=1}^{n}\frac{(1-x_k^2)|\varphi_k(t_s)|}{|t_s-x_k|^2} \\ &\le \frac{CE_{n-1}(f'')}{(n+1)^2}\sum_{k=1}^{n}\frac{(1-x_k^2)}{|t_s-x_k|^2} \le CE_{n-1}(f'').\end{aligned} \tag{4.6}$$

From (2.8) and (4.6), we can obtain

$$\left(\int_{-1}^{1}|N_1(x)|^p\left(1-x^2\right)^{\alpha}dx\right)^{1/p} \le CE_{n-1}(f''). \tag{4.7}$$

Now we consider $N_2(x)$. From $2x/\pi \le \sin x \le x$, for all $x \in [0,\pi/2]$, it follows that $\sqrt{1-t_s^2} \ge \sin(\pi/2(n+1)) \ge 1/(n+1)$. By (2.1) and (3.12), we have that, for an arbitrary $1 \le s \le n+1$,

$$\left|\sum_{k=1}^{n}(p_{n+1}(x_k)-f(x_k))\frac{2(-1)^{k+s}t_s}{(n+1)\sqrt{1-t_s^2}(t_s-x_k)^2}\right| \le \frac{CE_{n-1}(f'')}{(n+1)^3\sqrt{1-t_s^2}}\sum_{k=1}^{n}\frac{1-x_k^2}{(t_s-x_k)^2} \le CE_{n-1}(f''). \tag{4.8}$$

From (2.8) and (4.8), we can obtain

$$\left(\int_{-1}^{1}|N_2(x)|^p\left(1-x^2\right)^{\alpha}dx\right)^{1/p} \le CE_{n-1}(f''). \tag{4.9}$$

For the $N_3(x)$, similar to (3.15), we have

$$\begin{aligned}N_3(x) &= \sum_{k=1}^{n}(-1)^k(p_{n+1}(x_k)-f(x_k))\left[\sum_{s=1}^{n+1}\frac{T_{n+1}(x)}{(t_s-x_k)(x-t_s)}\right] \\ &= (n+1)\sum_{k=1}^{n}(-1)^k(p_{n+1}(x_k)-f(x_k))\frac{U_n(x)}{x-x_k}.\end{aligned} \tag{4.10}$$

For an arbitrary $1 \le k \le n$, by (2.1) and a simple computation, we can obtain

$$|N_3(x_k)| = (n+1)\left|(p_{n+1}(x_k)-f(x_k))\cdot U_n'(x_k)\right| \le CE_{n-1}(f''). \tag{4.11}$$

For $k = 0$, (2.1) leads to

$$|N_3(1)| = (n+1)^2 \left| \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{1 - x_k} \right| \leq CnE_{n-1}(f''). \tag{4.12}$$

Similarly,

$$N_3(-1) \leq CnE_{n-1}(f''). \tag{4.13}$$

Similar to (3.30), from (4.10), (4.11), (4.12), and (4.13), it follows that

$$\left(\int_{-1}^{1} |N_3(x)|^p \left(1 - x^2\right)^{\alpha} dx \right)^{1/p} \leq \begin{cases} CE_{n-1}(f''), & \alpha > \frac{p}{2} - 1, \\ C(\ln n)^{1/p} E_{n-1}(f''), & \alpha = \frac{p}{2} - 1, \\ Cn^{1-(2\alpha+2)/p} E_{n-1}(f''), & -1 < \alpha < \frac{p}{2} - 1. \end{cases} \tag{4.14}$$

For the $N_4(x)$, similar to (3.15), we have

$$N_4(x) = \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{(n+1)^2} \left[\sum_{s=1}^{n+1} \frac{T_{n+1}(x)}{(1 - t_s^2)(t_s - x_k)(x - t_s)} \right]. \tag{4.15}$$

It is easy to verify

$$\frac{T_{n+1}(x)}{(1 - t_s^2)(t_s - x_k)(x - t_s)} = \frac{1}{x - x_k} \left[\frac{T_{n+1}(x)}{(1 - t_s^2)(x - t_s)} + \frac{T_{n+1}(x)}{(1 - t_s^2)(t_s - x_k)} \right]. \tag{4.16}$$

For $a \neq \pm 1$, it is easy to verify that

$$\frac{1}{(1 - x^2)(x - a)} = -\frac{1}{2(1+a)(1+x)} + \frac{1}{2(1-a)(1-x)} + \frac{1}{(1-a^2)(x-a)}. \tag{4.17}$$

From (4.17), (3.16) and

$$\sum_{s=1}^{n+1} \frac{1}{1 + t_s} = \sum_{s=1}^{n+1} \frac{1}{1 - t_s} = (n+1)^2, \tag{4.18}$$

we obtain

$$\begin{aligned}\sum_{s=1}^{n+1}\frac{T_{n+1}(x)}{(1-t_s^2)(x-t_s)} &= \frac{T_{n+1}(x)}{2(1+x)}\sum_{s=1}^{n+1}\frac{1}{1+t_s}-\frac{T_{n+1}(x)}{2(1-x)}\sum_{s=1}^{n+1}\frac{1}{1-t_s}+\frac{1}{1-x^2}\sum_{s=1}^{n+1}\frac{T_{n+1}(x)}{x-t_s}\\ &= -\frac{(n+1)^2xT_{n+1}(x)}{1-x^2}+\frac{(n+1)U_n(x)}{1-x^2},\\ \sum_{s=1}^{n+1}\frac{T_{n+1}(x)}{(1-t_s^2)(t_s-x_k)} &= \frac{(n+1)^2x_kT_{n+1}(x)}{1-x_k^2}.\end{aligned}\tag{4.19}$$

From (4.16), (4.19), (61), (3.9), and a direct computation, we get

$$\begin{aligned}\sum_{s=1}^{n+1}\frac{T_{n+1}(x)}{(1-t_s^2)(t_s-x_k)(x-t_s)} &= -\frac{(n+1)^2(1+xx_k)T_{n+1}(x)}{(1-x^2)(1-x_k^2)}+\frac{(n+1)U_n(x)}{(1-x^2)(x-x_k)}\\ &= \frac{(n+1)(1+xx_k)U_n'(x)}{1-x_k^2}+\frac{(n+1)x_kU_n(x)}{1-x_k^2}+\frac{(n+1)\,U_n(x)}{(1-x_k^2)(x-x_k)}.\end{aligned}\tag{4.20}$$

From (4.15) and (4.20), we obtain

$$\begin{aligned}N_4(x) &= \sum_{k=1}^{n}\frac{(-1)^k(p_{n+1}(x_k)-f(x_k))}{(n+1)}\left[\frac{(1+xx_k)U_n'(x)}{1-x_k^2}+\frac{x_kU_n(x)}{1-x_k^2}\right]\\ &\quad+\sum_{k=1}^{n}\frac{(-1)^k(p_{n+1}(x_k)-f(x_k))}{(n+1)}\frac{U_n(x)}{(1-x_k^2)(x-x_k)}=N_{41}(x)+N_{42}(x).\end{aligned}\tag{4.21}$$

For $N_{41}(x)$, from (2.1), we can obtain

$$|N_{41}(x)|\le\frac{|U_n'(x)|+|U_n(x)|}{(n+1)^2}E_{n-1}(f'').\tag{4.22}$$

By (3.9), Markov inequality, and $\|U_n(x)\|_\infty=n+1$, we obtain

$$\left|U_n'(x)\right|\le\frac{2(n+1)}{1-x^2},\quad \left|U_n'(x)\right|\le n^2(n+1).\tag{4.23}$$

So for an arbitrary $0\le A\le 1$,

$$\left|U_n'(x)\right|\le\frac{2^{1-A}(n+1)^{1+2A}}{(1-x^2)^{1-A}}\le\frac{8n^{1+2A}}{(1-x^2)^{1-A}}.\tag{4.24}$$

Let $x = \cos\theta$. From $2\alpha+1 > -1$, we can choose A such that $0 < A < 1$ and $2\alpha+1-2p+2pA > -1$. Then by (4.24) and $2x/\pi \le \sin x \le x$, for all $x \in [0, \pi/2]$, we can obtain

$$\begin{aligned}\int_{-1}^{1} |U_n'(x)|^p \left(1-x^2\right)^{\alpha} dx &= 2\int_{-1}^{0} |U_n'(x)|^p \left(1-x^2\right)^{\alpha} dx \\ &\le 2^{3p+1}\left(\int_{0}^{\pi/2(n+1)} n^{p(1+2A)} \sin^{2\alpha+1-2p+2pA}\theta d\theta \right. \\ &\qquad \left. + \int_{\pi/2(n+1)}^{\pi/2} n^p \sin^{1+2\alpha-2p}\theta d\theta \right) \\ &\le \begin{cases} Cn^p, & \alpha > p-1, \\ Cn^p \ln n, & \alpha = p-1, \\ Cn^{3p-2\alpha-2}, & -1 < \alpha < p-1. \end{cases}\end{aligned} \tag{4.25}$$

From $\|U_n(x)\|_\infty = n+1$, it follows that

$$\int_{-1}^{1} |U_n(x)|^p \left(1-x^2\right)^{\alpha} dx \le Cn^p. \tag{4.26}$$

From (4.22), (4.25), and (4.26), it follows that

$$\left(\int_{-1}^{1} |N_{41}(x)|^p \left(1-x^2\right)^{\alpha} dx\right)^{1/p} \le \begin{cases} CE_{n-1}(f''), & \alpha \ge \dfrac{p}{2}-1, \\ Cn^{1-(2\alpha+2)/p} E_{n-1}(f''), & -1 < \alpha < \dfrac{p}{2}-1. \end{cases} \tag{4.27}$$

For $N_{42}(x)$, from (2.1) and a simple computation, we can obtain that, for $1 \le k \le n$,

$$|N_{42}(x_k)| = \frac{1}{(n+1)(1-x_k^2)} \left|(p_{n+1}(x_k) - f(x_k)) \cdot U_n'(x_k)\right| \le CE_{n-1}(f''). \tag{4.28}$$

Let $x = 1$. Then, from (3.10) and

$$\sum_{s=1}^{n-1} \frac{U_n(x)}{x - x_s} = U_n'(x), \tag{4.29}$$

we obtain

$$\sum_{k=1}^{n} \frac{1}{1-x_k} = \frac{n(n+2)}{3}. \tag{4.30}$$

From (2.1), it follows that

$$|N_{42}(1)| = \left|\sum_{k=1}^{n}\frac{(-1)^k(p_{n+1}(x_k) - f(x_k))}{(1-x_k)(1-x_k^2)}\right| \le \frac{CE_{n-1}(f'')}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{1-x_k} \le CE_{n-1}(f''). \tag{4.31}$$

Similarly,

$$|N_{42}(-1)| \le CE_{n-1}(f''). \tag{4.32}$$

Similar to (3.30), from (4.28), (4.31), and (4.32), we can obtain

$$\left(\int_{-1}^{1}|N_{42}(x)|^p\left(1-x^2\right)^{\alpha}dx\right)^{1/p} \le CE_{n-1}(f''). \tag{4.33}$$

From (4.1), (4.2), (4.5), (4.7), (4.9), (4.14), (4.21), (4.27), and (4.33), we obtain the upper estimate.

On the other hand, for $p \ge 2\alpha + 2$, let $f(x) = (1-x^2)U_n(x)$. Then,

$$f''(x) = -2(n+2)(n+1)T_n(x) + q_{n-1}(x), \tag{4.34}$$

here, $q_{n-1}(x)$ is a polynomial of degree at most $n-1$. Hence,

$$E_{n-1}(f'') = 2(n+2)(n+1). \tag{4.35}$$

It is easy to verify that

$$\begin{gathered} Q''_{n+2}(f,x) = 0, \\ f''(x) = -(n+1)^2U_n(x) + \frac{-U_n(x) + (n+1)xT_{n+1}(x)}{1-x^2}. \end{gathered} \tag{4.36}$$

Let $x = \cos\theta$, then, $(2k\pi + \pi)/2(n+1) \le \theta \le (2k\pi + 2\pi)/2(n+1)$ implies that $T_{n+1}(x)U_n(x) \le 0$. Therefore,

$$\begin{aligned}
\int_{-1}^{1} \left|f''(x) - Q_n''(f,x)\right|^p \left(1-x^2\right)^{\alpha} dx &\ge \int_{0}^{1} \left|f''(x) - Q_n''(f,x)\right|^p \left(1-x^2\right)^{\alpha} dx \\
&\ge n^{2p} \sum_{k=0}^{[(n+1)/2]} \int_{2k\pi/2(n+1)}^{(2k\pi+\pi)/2(n+1)} \frac{|\sin(n+1)\theta|^p}{\sin^{p-2\alpha-1}\theta} d\theta \\
&\ge \begin{cases} Cn^{2p}\ln n, & \alpha = \dfrac{p}{2} - 1, \\ Cn^{3p-2\alpha-2}, & -1 < \alpha < \dfrac{p}{2} - 1, \end{cases} \\
&\ge \begin{cases} C\ln n E_{n-1}^{p}(f''), & \alpha = \dfrac{p}{2} - 1, \\ Cn^{p-2\alpha-2} E_{n-1}^{p}(f''), & -1 < \alpha < \dfrac{p}{2} - 1, \end{cases}
\end{aligned} \tag{4.37}$$

We consider L_n in the following. For $f \in C_{[-1,1]}^2$, let $p_{n-1}(x)$ be the polynomial of degree at most $n-1$ satisfying (2.1). Then,

$$f''(x) - L_n''(f,x) = f''(x) - p_{n-1}''(x) + L_n''(p_{n-1} - f, x) = K_1(x) + K_2(x). \tag{4.38}$$

From (2.1), we can derive

$$\int_{-1}^{1} |K_1(x)|^p \left(1-x^2\right)^{\alpha} dx \le CE_{n-3}^{p}(f''). \tag{4.39}$$

If $f \in C[-1,1]$, then the well-known Lagrange interpolation polynomial of f based on $\{x_k\}_{k=1}^{n}$ is given by

$$R_n(f,x) = \sum_{k=1}^{n} f(x_k)\phi_k(x), \tag{4.40}$$

where

$$\phi_k(x) = \frac{(-1)^{k+1}(1-x_k^2)U_n(x)}{(n+1)(x-x_k)}, \quad k = 1, \dots, n. \tag{4.41}$$

Similar to (3.4), we have

$$K_2(x) = R_{n-1}(K_2, x) = \sum_{s=1}^{n-1} \left[\sum_{k=1}^{n} (p_{n-1}(t_k) - f(t_k))\ell_k''(x_s)\right]\phi_s(x). \tag{4.42}$$

By a direct computation, we obtain

$$\ell_k''(x_s) = \frac{n(-1)^{k+s}\sqrt{1-t_k^2}}{(1-x_s^2)(x_s-t_k)} + \frac{2(-1)^{k+s+1}\sqrt{1-t_k^2}}{n(x_s-t_k)^3}, \quad s=1,\dots,n-1. \tag{4.43}$$

From (4.42) and (4.43), it follows that

$$\begin{aligned} K_2(x) &= \sum_{s=1}^{n-1}\left[\sum_{k=1}^{n}(p_{n-1}(t_k)-f(t_k))\frac{n(-1)^{k+s}\sqrt{1-t_k^2}}{(1-x_s^2)(x_s-t_k)}\right]\phi_s(x) \\ &\quad+\sum_{s=1}^{n-1}\left[\sum_{k=1}^{n}(p_{n-1}(t_k)-f(t_k))\frac{2(-1)^{k+s+1}\sqrt{1-t_k^2}}{n(x_s-t_k)^3}\right]\phi_s(x) \\ &= A_1(x)+A_2(x). \end{aligned} \tag{4.44}$$

Exchanging the summation order, we have

$$A_1(x) = \sum_{k=1}^{n}(-1)^{k+1}(p_{n-1}(t_k)-f(t_k))\sqrt{1-t_k^2}\left[\sum_{s=1}^{n-1}\frac{U_{n-1}(x)}{(x_s-t_k)(x-x_s)}\right]. \tag{4.45}$$

For an arbitrary $1 \le s \le n-1$,

$$\frac{U_{n-1}(x)}{(x_s-t_k)(x-x_s)} = \frac{1}{x-t_k}\left(\frac{U_{n-1}(x)}{x_s-t_k}+\frac{U_{n-1}(x)}{x-x_s}\right). \tag{4.46}$$

It is well known that

$$\sum_{s=1}^{n-1}\frac{U_{n-1}(x)}{x-x_s} = U_{n-1}'(x). \tag{4.47}$$

Let $x=t_k$. Then, from (4.47) and (3.9), it follows that

$$\sum_{s=1}^{n-1}\frac{1}{t_k-x_s} = \frac{U_{n-1}'(t_k)}{U_{n-1}(t_k)} = \frac{t_k}{1-t_k^2}. \tag{4.48}$$

Combining (4.47), (4.48), and (3.9), we obtain

$$\begin{aligned}
&\sum_{s=1}^{n-1}\frac{1}{x-t_k}\left(\frac{U_{n-1}(x)}{x_s-t_k}+\frac{U_{n-1}(x)}{x-x_s}\right)\\
&\quad=\frac{1}{x-t_k}\frac{-t_kU_{n-1}(x)+\left(1-t_k^2\right)U'_{n-1}(x)}{1-t_k^2}\\
&\quad=\frac{U_{n-1}(x)+(x+t_k)U'_{n-1}(x)}{1-t_k^2}+\frac{1}{x-t_k}\frac{-xU_{n-1}(x)+\left(1-x^2\right)U'_{n-1}(x)}{1-t_k^2}\\
&\quad=\frac{U_{n-1}(x)+(x+t_k)U'_{n-1}(x)}{1-t_k^2}-\frac{nT_n(x)}{(x-t_k)\left(1-t_k^2\right)}.
\end{aligned}\tag{4.49}$$

From (4.45) and (4.49), it follows that

$$\begin{aligned}
A_1(x)&=\sum_{k=1}^{n}(-1)^{k+1}\left(p_{n-1}(t_k)-f(t_k)\right)\frac{U_{n-1}(x)+(x+t_k)U'_{n-1}(x)}{\sqrt{1-t_k^2}}\\
&\quad+\sum_{k=1}^{n}(-1)^{k}\left(p_{n-1}(t_k)-f(t_k)\right)\frac{nT_n(x)}{(x-t_k)\sqrt{1-t_k^2}}=A_{11}(x)+A_{12}(x).
\end{aligned}\tag{4.50}$$

By (2.1), we have

$$|A_{11}(x)|\le\frac{CE_{n-3}(f'')\left(\left|U'_{n-1}(x)\right|+|U_n(x)|\right)}{n}.\tag{4.51}$$

From (4.51), (4.25), and (4.26), it follows that

$$\left(\int_{-1}^{1}|A_{11}(x)|^p\left(1-x^2\right)^{\alpha}dx\right)^{1/p}\le\begin{cases}CE_{n-3}(f''), & \alpha>p-1,\\ C(\ln n)^{1/p}E_{n-3}(f''), & \alpha=p-1,\\ Cn^{2-(2\alpha+2)/p}E_{n-3}(f''), & -1<\alpha<p-1.\end{cases}\tag{4.52}$$

From (2.1) and $|T'_n(t_k)|=n/\sqrt{1-t_k^2}$, it follows that, for $1\le k\le n$,

$$|A_{12}(t_k)|=\left|(-1)^{k+1}\left(p_{n-1}(t_k)-f(t_k)\right)\frac{nT'_n(t_k)}{\sqrt{1-t_k^2}}\right|\le CE_{n-3}(f'').\tag{4.53}$$

From (4.53), (2.8), and

$$A_{12}(x)=\sum_{k=1}^{n}A_{12}(t_k)\ell_k(x),\tag{4.54}$$

we know

$$\left(\int_{-1}^{1} |A_{12}(x)|^p \left(1-x^2\right)^{\alpha} dx\right)^{1/p} \le CE_{n-3}(f''). \tag{4.55}$$

We consider $A_2(x)$ now. For an arbitrary $1 \le s \le n-1$, by (2.1) and (2.4), we obtain

$$\begin{aligned}\left|\sum_{k=1}^{n}(p_{n-1}(t_k)-f(t_k))\frac{2(-1)^{k+s+1}\sqrt{1-t_k^2}}{n(x_s-t_k)^3}\right| &\le CE_{n-3}(f'')\sum_{k=1}^{n}\frac{\left(1-t_k^2\right)^{3/2}}{n^3|x_s-t_k|^3} \\ &= CE_{n-3}(f'')\sum_{k=1}^{n}\left|\ell_k^3(x_s)\right| \le CE_{n-3}(f'').\end{aligned} \tag{4.56}$$

From (4.56), (4.44), and (4.41), we can obtain

$$\begin{gathered}|A_2(x_s)| \le CE_{n-3}(f''), \quad 1 \le s \le n-1. \\ |A_2(1)| \le CnE_{n-3}(f''), \qquad |A_2(-1)| \le CnE_{n-3}(f'').\end{gathered} \tag{4.57}$$

Similar to the proof of (3.30), by (4.57), (95), and

$$A_2(x) = \sum_{s=0}^{n} A_2(x_s)\varphi_s(x), \tag{4.58}$$

we can obtain

$$\left(\int_{-1}^{1} |A_2(x)|^p \left(1-x^2\right)^{\alpha} dx\right)^{1/p} \le \begin{cases} CE_{n-1}(f''), & \alpha > \dfrac{p}{2}-1, \\ C(\ln n)^{1/p}E_{n-1}(f''), & \alpha = \dfrac{p}{2}-1, \\ Cn^{1-(2\alpha+2)/p}E_{n-1}(f''), & -1 < \alpha < \dfrac{p}{2}-1. \end{cases} \tag{4.59}$$

From (4.38), (4.39), (4.44), (4.50), (4.52), (4.55), and (4.59), we can obtain the upper estimate.

On the other hand, let $f(x) = T_n(x)$. Then, it is easy to see

$$L_n(f,x) = 0, \tag{4.60}$$

$$f''(x) = 4n(n-1)T_{n-2}(x) + q_{n-3}(x), \tag{4.61}$$

here, $q_{n-3}(x)$ is a polynomial of degree at most $n-3$. Consequently, due to (4.61), we get

$$E_{n-3}(f'') = 4n(n-1). \tag{4.62}$$

Let $x = \cos\theta$. From (4.60), (4.62), (3.9), $T_n''(x) = nU_{n-1}'(x)$, and the odevity of $U_{n-1}'(x)$, it follows that

$$\begin{aligned}\int_{-1}^{1}\left|f''(x)-L_n''(f,x)\right|^p\left(1-x^2\right)^{\alpha}dx &= \frac{2E_{n-3}^p(f'')}{4^p(n-1)^p}\int_0^1\left|U_{n-1}'(x)\right|^p\left(1-x^2\right)^{\alpha}dx\\ &\geq \frac{E_{n-3}^p(f'')}{8^p}\sum_{k=1}^{n/2}\int_{(k\pi-\pi)/4/n}^{k\pi/n}\frac{1}{|\sin\theta|^{2p-2\alpha-1}}d\theta\\ &\geq \begin{cases} CE_{n-3}^p(f''), & \alpha>p-1,\\ C\ln nE_{n-3}^p(f''), & \alpha=p-1,\\ Cn^{2p-2\alpha-2}E_{n-3}^p(f''), & -1<\alpha<p-1.\end{cases}\end{aligned} \tag{4.63}$$

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