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Research Article

On the Characterization of a Class of Difference Equations

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We focus on the behavior of solutions of the difference equation $x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_n x_0 + y_n$, $n = 1, 2, \ldots$, where (b_k) is a fixed sequence of complex numbers, and (y_k) is a fixed sequence in a complex Banach space. We give the general solution of this difference equation. To examine the asymptotic behavior of solutions, we compute the spectra of operators which correspond to such type of difference equations. These operators are represented by upper triangular or lower triangular infinite banded Toeplitz matrices.

1. Introduction

The theory of difference equations is one of the most important representations of real world problems. The situation of an event at a fixed time usually depends on the situations of the event in the history. One of the ways to mathematically model such an event is to find a difference equation that directly or asymptotically describes the dependence of the situation at a time to the situations of the event in the history.

Let X be a complex Banach space. In this work, we are interested in a difference equation of the form

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_n x_0 + y_n, \tag{1.1}$$

$n = 1, 2, \ldots$, where (b_k) and (y_k) are fixed sequences in $\mathbb{C}$ and X, respectively, and $x_0 = y_0$. The difference equation (1.1) is a generalization of the difference equations investigated by Copson [1], Popa [2], and Stević [3]. We will give the general solution of the system of equations (1.1) and examine the different types of stability conditions by determining the spectra of related matrix operators. We will also examine the system of equations which correspond to the transpose of these matrices.

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $\mathcal{R}(T)$, we denote the range of T, that is, $\mathcal{R}(T) = \{y \in Y : y = Tx;\ x \in X\}$. By $B(X)$, we denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ and $T : \mathfrak{D}(T) \to X$ be a linear operator with domain $\mathfrak{D}(T) \subset X$. With T, we associate the operator

$$T_\lambda = T - \lambda I, \tag{1.2}$$

where λ is a complex number, and I is the identity operator on $\mathfrak{D}(T)$. If T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is,

$$T_\lambda^{-1} = (T - \lambda I)^{-1} \tag{1.3}$$

and call it the *resolvent operator* of T. Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we will be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We will also ask for what λ's the domain of T_λ^{-1} is dense in X. For our investigation of T, T_λ, and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [4, page 370-371]).

Let $X \neq \{\theta\}$ be a complex normed space, and let $T : \mathfrak{D}(T) \to X$ be a linear operator with domain $\mathfrak{D}(T) \subset X$. A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* $\rho(T)$ of T is the set of all regular values λ of T. Its complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows. The *point spectrum* $\sigma_p(T)$ is the set such that T_λ^{-1} does not exist. A $\lambda \in \sigma_p(T)$ is called an *eigenvalue* of T. The *continuous spectrum* $\sigma_c(T)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2). The *residual spectrum* $\sigma_r(T)$ is the set such that T_λ^{-1} exists but does not satisfy (R3).

We will write ℓ_∞, c and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. By ℓ_p, we denote the space of all p-absolutely summable sequences, where $1 \leq p < \infty$. Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in γ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.4}$$

By (μ, γ), we denote the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu, \gamma)$ if and only if the series on the right side of (1.4) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \gamma$ for all $x \in \mu$.

Several authors have studied the spectrum and fine spectrum of linear operators defined by matrices over some sequence spaces. Rhoades [5] examined the fine spectra of the weighted mean operators. Reade [6] worked on the spectrum of the Cesàro operator over the sequence space c_0. Gonzáles [7] studied the fine spectrum of the Cesàro operator over the sequence space ℓ_p. Yıldırım [8] examined the fine spectra of the Rhally operators over the sequence spaces c_0 and c. Akhmedov and Başar [9] have determined the fine spectrum of the difference operator Δ over ℓ_p. Later, Bilgiç et al. [10] worked on the spectrum of the operator $B(r,s,t)$, defined by a triple-band lower triangular matrix, over the sequence spaces c_0 and c. Recently, Altun and Karakaya [11] determined the fine spectra of Lacunary operators.

Let

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.5}$$

which is the left shift operator, and let the transpose of L be

$$R = L^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.6}$$

which is the right shift operator. Let D be the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$.

Let $a = (a_0, a_1, a_2, \ldots)$. A lower triangular Toeplitz matrix corresponding to a is in the form

$$R_a = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.7}$$

And an upper triangular Toeplitz matrix corresponding to a is in the form $L_a = [R_a]^t$.

Lemma 1.1. *Let $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$. Then $R_a \in B(\mu)$ if and only if $a \in \ell_1$. Moreover $\|R_a\|_\mu = \|a\|_{\ell_1}$.*

Proof. Let us do the proof for $\mu = c$. The proof for $\mu = c_0$ or ℓ_∞ is similar. Let $\|\cdot\|$ denote the norm of c. Firstly, we have

$$\begin{aligned}\|R_a\|_c &= \sup_{\|x\|=1}\|R_a(x)\| = \sup_{\|x\|=1}\left(\sup_n\left|\sum_{k=0}^{n} a_k x_{n-k}\right|\right)\\ &\le \sup_{\|x\|=1}\left(\sup_n\sum_{k=0}^{n}|a_k||x_{n-k}|\right) \le \sup_n\sum_{k=0}^{n}|a_k| = \sum_{k=0}^{\infty}|a_k|.\end{aligned} \tag{1.8}$$

Now, fix $n \in \mathbb{N}$, and let $a' = (a'_k)$ be a sequence such that

$$a'_k = \begin{cases}\dfrac{|a_{n-k}|}{a_{n-k}} & \text{if } a_{n-k} \neq 0,\ k \le n,\\ 0 & \text{otherwise,}\end{cases} \tag{1.9}$$

then we have

$$\|R_a\|_c \ge \left\|R_a(a')\right\| \ge \left|\left(R_a(a')\right)_n\right| = \sum_{k=0}^{n}|a_k|. \tag{1.10}$$

Hence, $\|R_a\|_c = \|a\|_{\ell_1}$.

Now, let $\mu = \ell_1$, and let $\|\cdot\|$ denote the norm of ℓ_1. We have

$$\begin{aligned}\|R_a\|_{\ell_1} &= \sup_{\|x\|=1}\|R_a(x)\| = \sup_{\|x\|=1}\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} a_k x_{n-k}\right|\right)\\ &\le \sup_{\|x\|=1}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_k||x_{n-k}|\right)\\ &\le \sup_{\|x\|=1}\left(\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{j=0}^{n}|a_k|\left|x_j\right|\right)\\ &\le \left(\lim_{n\to\infty}\sum_{k=0}^{n}|a_k|\right) = \sum_{k=0}^{\infty}|a_k|.\end{aligned} \tag{1.11}$$

On the other hand,

$$\|R_a\|_{\ell_1} \ge \|R_a(1,0,0,\ldots)\| = \sum_{k=0}^{\infty}|a_k|. \tag{1.12}$$

So, we have the same norm $\|R_a\|_{\ell_1} = \|a\|_{\ell_1}$. □

Remark 1.2. We have $B(\mu) = (\mu, \mu)$ for the sequence spaces in Lemma 1.1 since these spaces are BK spaces.

We also have an L_a version of the last lemma, for which we leave the proof to the reader.

Lemma 1.3. *Let $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$. $L_a \in B(\mu)$ if and only if $a \in \ell_1$. Moreover $\|L_a\|_\mu = \|a\|_{\ell_1}$.*

For any sequence a, let us associate the function $f_a(z) = \sum_{k=0}^{\infty} a_k z^k$.

2. Spectra of the Operators

Theorem 2.1. *Let $a \in \ell_1$. Then $R_a = f_a(R)$ and $L_a = f_a(L)$.*

Proof. Let us do the proof for R_a. The proof for L_a can be done similarly. Let $a^{(n)} = a - (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$. Then

$$\sum_{k=0}^{n} a_k R^k = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & \cdots \\ 0 & a_n & a_{n-1} & a_{n-2} & \cdots & \cdots \\ 0 & 0 & a_n & a_{n-1} & a_{n-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad R_a - \sum_{k=0}^{n} a_k R^k = R_{a^{(n)}}. \tag{2.1}$$

So by Lemma 1.1,

$$\left\| R_a - \sum_{k=0}^{n} a_k R^k \right\|_{c_0} = \|R_{a^{(n)}}\|_{c_0} = \sum_{k=n+1}^{\infty} |a_k|. \tag{2.2}$$

Hence,

$$\lim_{n \to \infty} \left\| R_a - \sum_{k=0}^{n} a_k R^k \right\|_{c_0} = 0, \tag{2.3}$$

and so $R_a = f_a(R)$. □

Theorem 2.2. *Let μ be one of the sequence spaces c_0, c, ℓ_∞, or ℓ_p with $1 \le p < \infty$. Then L is a bounded linear operator over μ with $\|L\|_\mu = 1$ and $\sigma(L, \mu) = D$.*

Proof. Let us do the proof first for $\mu = \ell_p$ for $1 \leq p < \infty$. Let $x = (x_1, x_2, \ldots)$,

$$\|Lx\|_p = \|(x_2, x_3, \ldots)\|_p = \left(\sum_{i=2}^{\infty} |x_i|^p\right)^{1/p} \leq \|x\|_p \tag{2.4}$$

and $\|(0,1,0,0,\ldots)\|_p = 1 = \|L(0,1,0,0,\ldots)\|_p$, hence $\|L\|_p = 1$. In a similar way, we can show that $\|L\|_\mu = 1$ also for $\mu \in \{c_0, c, \ell_\infty\}$. This means the spectral radius is less or equal to 1 and so

$$\sigma(L, \mu) \subset D, \tag{2.5}$$

for $\mu \in \{c_0, c, \ell_p, \ell_\infty\}$.

Now, let us examine the eigenvalues for L. If $Lx = \lambda x$, then

$$\begin{aligned} x_2 &= \lambda x_1, \\ x_3 &= \lambda x_2, \\ &\vdots \end{aligned} \tag{2.6}$$

If $x_1 = 0$, then $x_k = 0$ for all k. So, let $x_1 \neq 0$, then

$$x_k = \lambda^{k-1} x_1. \tag{2.7}$$

Hence, for any λ with $|\lambda| < 1$, the sequence $x = (1, \lambda, \lambda^2, \ldots) \in \mu$ is an eigenvector for $\mu \in \{c_0, c, \ell_p, \ell_\infty\}$. Hence, $\sigma_p(L, \mu) \supset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Combining this with (2.5), we have

$$\sigma(L, \mu) = D, \tag{2.8}$$

for $\mu \in \{c_0, c, \ell_p, \ell_\infty\}$, since the spectrum is a closed set. □

Theorem 2.3. *Let μ be one of the sequence spaces c_0, c, ℓ_∞ or ℓ_p with $1 \leq p < \infty$. Then R is a bounded linear operator over μ with $\|R\|_\mu = 1$ and $\sigma(R, \mu) = D$.*

Proof. The boundedness of the operator can be proved as in the proof of Theorem 2.2. Now, we will use the fact that the spectrum of a bounded operator over a Banach space is equal to the spectrum of the adjoint operator. The adjoint operator is the transpose of the matrix for c_0 and ℓ_p with $1 \leq p < \infty$. Hence,

$$\begin{aligned} \sigma(R, \ell_1) &= \sigma(L, c_0) = D \\ \sigma(R, c_0) &= \sigma(L, \ell_1) = D \\ \sigma(R, \ell_p) &= \sigma(L, \ell_q) = D \qquad 1 < p < \infty. \\ \sigma(R, \ell_\infty) &= \sigma(L, \ell_1) = D \end{aligned} \tag{2.9}$$

It is known by Cartlidge [12] that if a matrix operator A is bounded on c, then $\sigma(A,c) = \sigma(A,\ell_\infty)$. Then we also have

$$\sigma(R,c) = \sigma(R,\ell_\infty) = D. \tag{2.10}$$

□

Theorem 2.4. *Let a be a sequence such that f_a is holomorphic in a region containing D. Then $\sigma(R_a,\mu) = \sigma(L_a,\mu) = f_a(D)$.*

Proof. By the spectral mapping theorem for holomorphic functions (see, e.g., [13, page 569]) we have

$$\begin{aligned}\sigma(R_a,\mu) &= \sigma(f_a(R),\mu) = f_a(\sigma(R,\mu)) = f_a(D),\\ \sigma(L_a,\mu) &= \sigma(f_a(L),\mu) = f_a(\sigma(L,\mu)) = f_a(D).\end{aligned} \tag{2.11}$$

□

Theorem 2.5. *Let μ be a sequence space. If R_a is not a multiple of the identity mapping, then*

$$\sigma_p(R_a,\mu) = \emptyset. \tag{2.12}$$

Proof. Suppose λ is an eigenvalue and $x = (x_0,x_1,\ldots) \in \mu$ is an eigenvector which corresponds to λ. Then $R_a x = \lambda x$ and so we have the following linear system of equations.

$$\begin{aligned} a_0x_0 &= \lambda x_0,\\ a_1x_0 + a_0x_1 &= \lambda x_1,\\ a_2x_0 + a_1x_1 + a_0x_2 &= \lambda x_2,\\ &\vdots \end{aligned} \tag{2.13}$$

Let x_k be the first nonzero entry of x. Then the system of equations reduces to

$$\begin{aligned} a_0x_k &= \lambda x_k,\\ a_1x_k + a_0x_{k+1} &= \lambda x_{k+1},\\ a_2x_k + a_1x_{k+1} + a_0x_{k+2} &= \lambda x_{k+2},\\ &\vdots \end{aligned} \tag{2.14}$$

From the first equation we get $\lambda = a_0$, and using the other equations in the given order we get $a_k = 0$ for $k > 0$. This means there is no solution if there exists any $k > 0$ with $a_k \neq 0$. □

3. Applications to the System of Equations

Let us consider the evolutionary difference equation

$$y_n = a_n x_0 + a_{n-1} x_1 + \cdots + a_0 x_n \quad (a_0 \neq 0), \tag{3.1}$$

$n = 0, 1, 2, \ldots$. Here (a_n) is a sequence of complex numbers. The condition $a_0 \neq 0$ is needed to make the system of equations (3.1) solvable. By solvability of a system of equations we mean that for any given sequence (y_n) of complex numbers there exists a unique sequence (x_n) of complex numbers satisfying the equations. Equation (3.1) may be written in the form

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_n x_0 + u_n, \tag{3.2}$$

for $n = 1, 2, \ldots$, and $x_0 = u_0$, by the change of variables $b_k = -a_k / a_0$ for $k \in \{1, 2, \ldots\}$ and where u_n is a function of n variables with $u_n = u_n(x_0, x_1, \ldots, x_{n-1}) = y_n / a_0$.

Let $0^0 = 1$ for the following theorem.

Theorem 3.1. *The solution of the difference equation* (3.2) *is*

$$x_n = u_n + c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_n u_0, \tag{3.3}$$

$n = 1, 2, \ldots$, *with*

$$c_k = \sum_{m_1, m_2, \ldots, m_k} \binom{m_1 + m_2 + \cdots + m_k}{m_1, m_2, \ldots, m_k} b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k}, \tag{3.4}$$

$k = 1, 2, \ldots$, *where the summation is over nonnegative integers satisfying* $m_1 + 2m_2 + \cdots + km_k = k$.

Proof. We have $b_1 = c_1$. For $k \geq 2$, let us show that

$$b_k + b_{k-1} c_1 + b_{k-2} c_2 + \cdots + b_1 c_{k-1} = c_k, \tag{3.5}$$

$k = 2, 3, \ldots$. We can see that the left side of (3.5) can be written in the form

$$\sum_{m_1, m_2, \ldots, m_k} a(m_1, m_2, \ldots, m_k) b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k}, \tag{3.6}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = k$. To show (3.5), let us fix $k \geq 2$ and the numbers $m_1, m_2, \ldots, m_k$ such that $m_1 + 2m_2 + \cdots + km_k = k$. Then for the coefficient $a(m_1, m_2, \ldots, m_k)$, we have two cases.

Case 1 ($m_k = 1$). $m_1 = m_2 = \cdots = m_{k-1} = 0$, and then

$$a(m_1, m_2, \ldots, m_k) = a(0, 0, \ldots, 0, 1) = 1 = \binom{1}{0, 0, \ldots, 0, 1} = \binom{m_1 + m_2 + \cdots + m_k}{m_1, m_2, \ldots, m_k}. \tag{3.7}$$

Case 2 ($m_k = 0$). By induction we have

$$a(m_1, m_2, \dots, m_k) = a(m_1, m_2, \dots, m_{k-1}, 0) = \sum_{m_j \geq 1, 1 \leq j \leq k-1} \binom{m_1 + m_2 + \cdots + m_{k-1} - 1}{m_1, m_2, \dots, m_{j-1}, m_j - 1, m_{j+1}, m_{j+2}, \dots, m_{k-1}} = \binom{m_1 + m_2 + \cdots + m_{k-1}}{m_1, m_2, \dots, m_{k-1}} = \binom{m_1 + m_2 + \cdots + m_k}{m_1, m_2, \dots, m_k}. \quad (3.8)$$

So, by Cases 1 and 2, we have

$$b_k + b_{k-1}c_1 + \cdots + b_1 c_{k-1} = \sum_{m_1, m_2, \dots, m_k} a(m_1, m_2, \dots, m_k) b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k} = \sum_{m_1, m_2, \dots, m_k} \binom{m_1 + m_2 + \cdots + m_k}{m_1, m_2, \dots, m_k} b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k} = c_k. \quad (3.9)$$

Now, we will prove the theorem by induction over n. $x_1 = u_1 + b_1 u_0 = u_1 + c_1 u_0$ and so (3.3) is true for $n = 1$. Suppose (3.3) is true for $n \leq s$ for a positive integer s. We have

$$\begin{aligned} x_{s+1} &= u_{s+1} + b_1 x_s + b_2 x_{s-1} + \cdots + b_{s+1} x_0 \\ &= u_{s+1} + b_1 (u_s + c_1 u_{s-1} + c_2 u_{s-2} + \cdots + c_s u_0) \\ &\quad + b_2 (u_{s-1} + c_1 u_{s-2} + c_2 u_{s-3} + \cdots + c_{s-1} u_0) + \cdots + b_{s+1} u_0 \\ &= u_{s+1} + b_1 u_s + (b_2 + b_1 c_1) u_{s-1} + (b_3 + b_2 c_1 + b_1 c_2) u_{s-2} \\ &\quad + \cdots + (b_{s+1} + b_s c_1 + b_{s-1} c_2 + \cdots + b_1 c_s) u_0 \\ &= u_{s+1} + c_1 u_s + c_2 u_{s-1} \cdots + c_{s+1} u_0. \end{aligned} \quad (3.10)$$

Hence, (3.3) is true for $n = s + 1$. □

A special case of (3.1) is the one where (a_n) consists of finitely many nonzero terms. So there exists a fixed $k \in \mathbb{N}$ such that the equations turn into the form

$$y_n = \begin{cases} a_k x_{n-k} + a_{k-1} x_{n-k+1} + \cdots + a_0 x_n & \text{if } n > k, \\ a_n x_0 + a_{n-1} x_1 + \cdots + a_0 x_n & \text{if } n \leq k, \end{cases} \quad (a_0 \neq 0), \quad (3.11)$$

$n = 0, 1, 2, \dots$.

We give the following theorem, which is a direct consequence of Lemma 1.1, to compare the results of it with the results of the next theorem.

Theorem 3.2. *Let (a_n) be a sequence of complex numbers such that the system of difference equations* (3.1) *hold for the complex number sequences $x = (x_n)$ and $y = (y_n)$. Then the following are equivalent:*

(i) *boundedness of (x_n) always implies boundedness of (y_n),*

(ii) *convergence of (x_n) always implies convergence of (y_n),*

(iii) $x_n \to 0$ *always implies* $y_n \to 0$,

(iv) $\sum |x_n| < \infty$ *always implies* $\sum |y_n| < \infty$,

(v) $\sum |a_n| < \infty$.

Theorem 3.3. *Suppose f_a is a nonconstant holomorphic function on a region containing D. Let (a_n) be a sequence of complex numbers such that the system of difference equations* (3.1) *hold for the complex number sequences $x = (x_n)$ and $y = (y_n)$. Then the following are equivalent:*

(i) *boundedness of (y_n) always implies boundedness of (x_n),*

(ii) *convergence of (y_n) always implies convergence of (x_n),*

(iii) $y_n \to 0$ *always implies* $x_n \to 0$,

(iv) $\sum |y_n|^p < \infty$ *always implies* $\sum |x_n|^p < \infty$,

(v) *$f_a(z)$ has no zero in the unit disc D.*

Proof. Let us prove only (i) $\Leftrightarrow$ (v). We will omit the proofs of (ii) $\Leftrightarrow$ (v), (iii) $\Leftrightarrow$ (v), (iv) $\Leftrightarrow$ (v) since they are similarly proved. Since f_a is a holomorphic function on a region containing D, we have $a \in \ell_1$ which means R_a is bounded by Lemma 1.1. Suppose boundedness of y_n implies boundedness of x_n. Then the operator $R_a \in (\ell_\infty, \ell_\infty)$ is onto. We have $R_a x = y$ and since f_a is not constant $R_a \in (\ell_\infty, \ell_\infty)$ is one to one by Theorem 2.5. Hence, R_a is bijective and by the open mapping theorem R_a^{-1} is continuous. This means that $\lambda = 0$ is not in the spectrum $\sigma(R_a, \ell_\infty)$, so $0 \notin f_a(D)$.

For the inverse implication, suppose $f_a(z)$ has no zero on the unit disc D. So, $\lambda = 0$ is in the resolvent set $\rho(R_a, \ell_\infty)$. Hence by Lemma 7.2-3 of [4] R_a^{-1} is defined on the whole space ℓ_∞, which means that the boundedness of (y_n) implies the boundedness of (x_n). □

Corollary 3.4. *Let a polynomial $P(z) = a_0 + a_1 z + \cdots + a_k z^k$ be given such that the system of difference equations* (3.11) *hold for the complex number sequences $x = (x_n)$ and $y = (y_n)$. Then the following are equivalent:*

(i) *boundedness of (y_n) always implies boundedness of (x_n),*

(ii) *convergence of (y_n) always implies convergence of (x_n),*

(iii) $y_n \to 0$ *always implies* $x_n \to 0$,

(iv) $\sum |y_n|^p < \infty$ *always implies* $\sum |x_n|^p < \infty$,

(v) *all zeros of $P(z)$ are outside the unit disc D.*

Now consider the system of equations

$$y_n = \sum_{j=0}^{\infty} a_j x_{n+j}, \tag{3.12}$$

$n = 0, 1, 2, \ldots$. Here (a_n) is a sequence of complex numbers.

A special case of (3.12) is the one where (a_n) consists of finitely many nonzero terms. So there exists a fixed $k \in \mathbb{N}$ such that the equations turn into the form

$$y_n = \sum_{j=0}^{k} a_j x_{n+j} \tag{3.13}$$

$n = 0, 1, 2, \ldots$.

Now, we again give a theorem, which is a direct consequence of Lemma 1.3, to compare the results of it with the results of the next theorem.

Theorem 3.5. *Let (a_n) be a sequence of complex numbers such that the system of difference equations* (3.12) *hold for the complex number sequences $x = (x_n)$ and $y = (y_n)$. Then the following are equivalent:*

(i) *boundedness of (x_n) always implies boundedness of (y_n),*

(ii) *convergence of (x_n) always implies convergence of (y_n),*

(iii) *$x_n \to 0$ always implies $y_n \to 0$,*

(iv) *$\sum |x_n| < \infty$ always implies $\sum |y_n| < \infty$,*

(v) *$\sum |a_n| < \infty$.*

Theorem 3.6. *Suppose that f_a is a holomorphic function on a region containing D. Let (a_n) be a sequence of complex numbers such that the system of difference equations* (3.12) *hold for the complex number sequences $x = (x_n)$ and $y = (y_n)$. Then the following are equivalent:*

(i) *boundedness of (y_n) always implies a unique bounded solution (x_n),*

(ii) *convergence of (y_n) always implies a unique convergent solution (x_n),*

(iii) *$y_n \to 0$ always implies a unique solution (x_n) with $x_n \to 0$,*

(iv) *$\sum |y_n|^p < \infty$ always implies a unique solution (x_n) with $\sum |x_n|^p < \infty$,*

(v) *$f_a(z)$ has no zero on the unit disc D.*

Proof. Let us prove only (i) $\Leftrightarrow$ (v). We will omit the proofs of (ii) $\Leftrightarrow$ (v), (iii) $\Leftrightarrow$ (v), (iv) $\Leftrightarrow$ (v) since they are similarly proved. Suppose boundedness of y_n implies a unique bounded solution x_n. Then the operator $L_a \in (\ell_\infty, \ell_\infty)$ is bijective. Since f_a is a holomorphic function on a region containing D, we have $a \in \ell_1$ which means L_a is bounded by Lemma 1.3. By the open mapping theorem L_a^{-1} is continuous. This means that $\lambda = 0$ is not in the spectrum $\sigma(L_a, \ell_\infty)$, so $0 \notin f_a(D)$.

For the inverse implication, suppose that $f_a(z)$ has no zero on the unit disc D. So, $\lambda = 0$ is in the resolvent set $\rho(L_a, \ell_\infty)$. Hence by Lemma 7.2-3 of [4] L_a^{-1} is defined on the whole space ℓ_∞, which means that the boundedness of (y_n) implies a bounded unique solution(x_n). □

Corollary 3.7. *Let a polynomial* $P(z) = a_0 + a_1 z + \cdots + a_k z^k$ *be given such that the system of difference equations* (3.13) *hold for the complex number sequences* $x = (x_n)$ *and* $y = (y_n)$. *Then the following are equivalent:*

(i) *boundedness of* (y_n) *always implies a bounded unique solution* (x_n),

(ii) *convergence of* (y_n) *always implies a convergent unique solution* (x_n),

(iii) $y_n \to 0$ *always implies a unique solution* (x_n) *with* $x_n \to 0$,

(iv) $\sum |y_n|^p < \infty$ *always implies a unique solution* (x_n) *with* $\sum |x_n|^p < \infty$,

(v) *all zeros of* $P(z)$ *are outside the unit disc* D.

Remark 3.8. We see that the unit disc also has an important role in examining the different types of stability conditions of difference equations or system of equations related with holomorphic functions.

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