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Research Article

On Generalized Bell Polynomials

Roberto B. Corcino[1] and Cristina B. Corcino[2]

[1] *Department of Mathematics, Mindanao State University, Marawi City 9700, Philippines*
[2] *Department of Mathematics, De La Salle University, Manila 1004, Philippines*

Correspondence should be addressed to Roberto B. Corcino, rcorcino@yahoo.com

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It is shown that the sequence of the generalized Bell polynomials $S_n(x)$ is convex under some restrictions of the parameters involved. A kind of recurrence relation for $S_n(x)$ is established, and some numbers related to the generalized Bell numbers and their properties are investigated.

1. Introduction

Hsu and Shiue [1] defined a kind of generalized Stirling number pair with three free parameters which is introduced via a pair of linear transformations between generalized factorials, viz,

$$
\begin{aligned}
(t \mid \alpha)_n &= \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t-\gamma \mid \beta)_k, \\
(t \mid \beta)_n &= \sum_{k=0}^{n} S(n,k;\beta,\alpha,-\gamma)(t+\gamma \mid \alpha)_k,
\end{aligned}
\tag{1.1}
$$

where $n \in N$ (set of nonnegative integers), α, β, and γ may be real or complex numbers with $(\alpha,\beta,\gamma) \neq (0,0,0)$, and $(t \mid \alpha)_n$ denotes the generalized factorial of the form

$$
(t \mid \alpha)_n = \prod_{j=0}^{n-1}(t-j\alpha), \quad n \geq 1, \ (t \mid \alpha)_0 = 1. \tag{1.2}
$$

In particular, $(t \mid 1)_n = (t)_n$ with $(t)_0 = 1$. Various well-known generalizations were obtained by special choices of the parameters α, β, and γ (cf. [1]), and the generalization of some properties of the classical Stirling numbers such as the recurrence relations

$$
S(n+1,k;\alpha,\beta,\gamma) = S(n,k-1;\alpha,\beta,\gamma) + (k\beta - n\alpha + \gamma)S(n,k;\alpha,\beta,\gamma), \tag{1.3}
$$

the exponential generating function

$$(1+\alpha t)^{\gamma/\alpha}\left[\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right]^{k} = k!\sum_{n\geq 0} S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!}, \tag{1.4}$$

the explicit formula

$$S(n,k;\alpha,\beta,\gamma) = \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+\gamma \mid \alpha)_n, \tag{1.5}$$

the congruence relation, and a kind of asymptotic expansion was established. As a follow-up study of these numbers, more properties were obtained in [2]. Furthermore, some combinatorial interpretations of $S(n,k;\alpha,\beta,\gamma)$ were given in [3] in terms of occupancy distribution and drawing of balls from an urn.

Hsu and Shiue [1] also defined a kind of generalized exponential polynomials $S_n(x) \equiv S_n(x;\alpha,\beta,\gamma)$ in terms of generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$ with $\alpha,\beta,$ and γ real or complex numbers as follows:

$$S_n(x) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)x^k. \tag{1.6}$$

We may call these polynomials *generalized Bell polynomials*. Note that when $x = 1$, we get

$$W_n = S_n(1) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma), \tag{1.7}$$

the *generalized Bell numbers*. A kind of generating function of the sequence $\{S_n(x)\}$ for the generalized exponential polynomials has been established by Hsu and Shiue, viz,

$$\sum_{n\geq 0} S_n(x)\frac{t^n}{n!} = (1+\alpha t)^{\gamma/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right], \tag{1.8}$$

where $\alpha,\beta\neq 0$. In particular, (1.8) gives the generating function for the generalized Bell numbers:

$$\sum_{n\geq 0} W_n\frac{t^n}{n!} = (1+\alpha t)^{\gamma/\alpha}\exp\left[\frac{\left((1+\alpha t)^{\beta/\alpha}-1\right)}{\beta}\right]. \tag{1.9}$$

Note that, when $\alpha \to 0$, $(1+\alpha t)^{\gamma/\alpha} \to \exp(\gamma t)$. Hence,

$$(1+\alpha t)^{\gamma/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right] \longrightarrow e^{\gamma t}\exp\left[\left(e^{\beta t}-1\right)\frac{x}{\beta}\right]. \tag{1.10}$$

If we define the polynomial $G_{n,\beta,r}(x)$ as

$$G_{n,\beta,r}(x) = \lim_{\alpha \to 0} S_n(x; \alpha, \beta, r), \tag{1.11}$$

then its exponential generating function is given by

$$\sum_{n \geq 0} G_{n,\beta,r}(x) \frac{t^n}{n!} = \exp\left[rt + \left(e^{\beta t} - 1\right) \frac{x}{\beta}\right]. \tag{1.12}$$

We may call $G_{n,\beta,r}(x)$ the *(r, β)-Bell polynomial*. Hence, with $x = 1$, this yields the exponential generating function for the (r, β)-Bell numbers. Now, if we use $S(n, k; \beta, \gamma)$ to denote the following limit:

$$S(n, k; \beta, \gamma) = \lim_{\alpha \to 0} S(n, k; \alpha, \beta, \gamma), \tag{1.13}$$

then, by (1.5),

$$S(n, k; \beta, \gamma) = \frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j + \gamma)^n, \tag{1.14}$$

$$G_{n,\beta,r}(x) = \sum_{k=0}^{n} S(n, k; \beta, \gamma) x^k. \tag{1.15}$$

Also obtained by Hsu and Shiue is an explicit formula for $S_n(x)$ of the form

$$S_n(x) = \left(\frac{1}{e}\right)^{x/\beta} \sum_{k=0}^{\infty} \frac{(x/\beta)^k}{k!} (k\beta + \gamma \mid \alpha)_n. \tag{1.16}$$

Consequently, with $x = 1$, we have

$$W_n = \left(\frac{1}{e}\right)^{1/\beta} \sum_{k=0}^{\infty} \frac{(k\beta + \gamma \mid \alpha)_n}{\beta^k k!}. \tag{1.17}$$

Note that, by taking $\alpha = 0$, (1.16) gives

$$G_{n,\beta,r}(x) = \left(\frac{1}{e}\right)^{x/\beta} \sum_{k=0}^{\infty} \frac{(x/\beta)^k}{k!} (k\beta + \gamma)^n, \tag{1.18}$$

the explicit formula for (r, β)-Bell polynomial. When $x = 1$, this gives

$$G_{n,\beta,r} = \left(\frac{1}{e}\right)^{1/\beta} \sum_{k=0}^{\infty} \frac{(1/\beta)^k}{k!} (k\beta + \gamma)^n, \tag{1.19}$$

a kind of the Dobinski formula for (r,β)-Bell numbers. This reduces further to the Dobinski formula for r-Bell numbers [4] when $\beta = 1$. Moreover, with $\gamma = 0$, we get

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k}{k!}, \tag{1.20}$$

which is the Dobinski formula for the ordinary Bell numbers [5].

In this paper, a recurrence relation and convexity of the generalized Bell numbers will be established and some numbers related to W_n will be investigated. Some theorems on (r,β)-Bell polynomials will be established including the asymptotic approximation of the (r,β)-Bell numbers.

2. More Properties of $S_n(x)$

Recurrence relation is one of the useful tools in constructing tables of values. The recurrence relation for the ordinary Bell numbers [6] is given by

$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}, \tag{2.1}$$

with initial condition $B_0 = 1$. Carlitz's Bell numbers [7] also satisfy the recurrence relation:

$$A_{n+1}(\lambda) = -\lambda n A_n(\lambda) + \sum_{k=0}^{n}k!\binom{n}{k}\binom{\mu}{k}\lambda^k A_{n-k}(\lambda), \quad \mu = \frac{1}{\lambda}, \tag{2.2}$$

with $A_0(\lambda) = 1$. Note that for $\lambda = 1$, $A_n(1) = B_n$ and (2.2) will reduce to (2.1). Moreover, Mező [4] obtained certain recurrence relations for the r-Bell polynomials, respectively, as

$$B_{n,r}(x) = rB_{n-1,r}(x) + x\sum_{k=0}^{n-1}\binom{n-1}{k}B_{k,r}(x). \tag{2.3}$$

The following theorem will generalize all of these recurrence relations.

Theorem 2.1. *The generalized exponential polynomials satisfy the following recurrence relation:*

$$S_{n+1}(x) = (\gamma - \alpha n)S_n(x) + \sum_{k=0}^{n}x\binom{n}{k}(\beta \mid \alpha)_k S_{n-k}(x) \tag{2.4}$$

with $S_0(x) = 1$. Moreover, the generalized Bell numbers $W_n = S_n(1)$ satisfy

$$W_{n+1} = (\gamma - \alpha n)W_n + \sum_{k=0}^{n}\binom{n}{k}(\beta \mid \alpha)_k W_{n-k}. \tag{2.5}$$

Proof. Differentiating both sides of (1.8) with respect to t will give

$$\sum_{n\geq 0} S_n(x)\frac{t^{n-1}}{(n-1)!} = (1+\alpha t)^{\gamma/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right]\left(\frac{(1+\alpha t)^{\beta/\alpha}x+\gamma}{1+\alpha t}\right). \tag{2.6}$$

Applying binomial theorem and Cauchy's rule for product of two power series will yield

$$\begin{aligned} (1+\alpha t)\sum_{n\geq 0} S_n(x)\frac{t^{n-1}}{(n-1)!} &= \left(\sum_{n\geq 0} S_n(x)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0}\binom{\frac{\beta}{\alpha}}{n}x\alpha^n t^n+\gamma\right), \\ \sum_{n\geq 0} S_n(x)\frac{t^{n-1}}{(n-1)!}+\sum_{n\geq 0} n\alpha S_n(x)\frac{t^n}{n!} &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} xk!\binom{n}{k}\binom{\frac{\beta}{\alpha}}{k}\alpha^k S_{n-k}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

Comparing the coefficients of $t^n/n!$, we obtain

$$S_{n+1}(x)+\alpha n S_n(x) = \gamma S_n(x)+\sum_{k=0}^{n} xk!\binom{n}{k}\binom{\frac{\beta}{\alpha}}{k}\alpha^k S_{n-k}(x), \tag{2.8}$$

which is precisely equivalent to (1.10). □

By taking $\alpha = 0$, Theorem 2.1 yields the recurrence relations for the (r,β)-Bell polynomials. More precisely,

$$G_{n+1,\beta,r}(x) = rG_{n,\beta,r}(x)+\sum_{k=0}^{n} x\binom{n}{k}\beta^k G_{n-k,\beta,r}(x). \tag{2.9}$$

These further give (2.3) when $\beta = 1$. Surely, (2.2) can be deduced from (2.5) by letting $(\alpha,\beta,\gamma) = (\lambda,1,0)$. Furthermore, for $(\alpha,\beta,\gamma) = (0,0,1)$, (2.4) gives

$$\overline{B}_{n+1}(x) = 2\overline{B}_n(x), \tag{2.10}$$

where $\overline{B}_n(x) = \sum_{k=0}^{n}\binom{n}{k}x^k$. If we let $\overline{B}_n = \overline{B}_n(1)$, we get

$$\overline{B}_{n+1} = 2\overline{B}_n, \tag{2.11}$$

which implies

$$\sum_{k=0}^{n}\binom{n+1}{k} = 2^{n+1}-1, \tag{2.12}$$

the number of distinct partitions of an $(n+2)$-set into 2 nonempty subsets, or simply $S(n+2,2)$, the classical Stirling number of the second kind.

Mathematicians have been aware for quite a while that the global behaviour of combinatorial sequences can be used in asymptotic estimates. One of these interesting behaviours is convexity [5]. A real sequence $v_k, k = 0,1,2,\ldots$ is called *convex* on an interval $[a,b]$ (containing at least 3 consecutive integers) when

$$v_k \leq \frac{1}{2}(v_{k-1} + v_{k+1}), \quad k \in [a+1, b-1]. \tag{2.13}$$

For instance, the sequence of binomial coefficients $\binom{n}{k}$ satisfies the convexity property since

$$\binom{n+2}{k} - 2\binom{n+1}{k} + \binom{n}{k} = \binom{n}{k-2} > 0, \quad \text{for } k \geq 2. \tag{2.14}$$

This implies that

$$\overline{B}_{n+1} \leq \frac{1}{2}\left(\overline{B}_n + \overline{B}_{n+2}\right), \tag{2.15}$$

that is, $\overline{B}_n$ is convex.

The next theorem asserts that the sequence of generalized exponential polynomials as well as the generalized Bell numbers is convex under some restrictions.

Theorem 2.2. *The sequence of generalized exponential polynomials $S_n(x)$ with $x > 0$, $\alpha \leq 0$, and $\beta, \gamma \geq 0$ possesses the convexity property, viz,*

$$S_{n+1}(x) \leq \frac{1}{2}(S_n(x) + S_{n+2}(x)), \quad n = 1,2,\ldots. \tag{2.16}$$

Proof. Since $\alpha \leq 0$ and $(k\beta + \gamma - n\alpha) \geq 0$, we have

$$\begin{gathered}
0 \leq \left[1 - (k\beta + \gamma - n\alpha)\right]^2 - \alpha(k\beta + \gamma - n\alpha), \\
0 \leq 1 - 2(k\beta + \gamma - n\alpha) + (k\beta + \gamma - n\alpha)^2 - \alpha(k\beta + \gamma - n\alpha), \\
2(k\beta + \gamma - n\alpha) \leq 1 + (k\beta + \gamma - n\alpha)(k\beta + \gamma - n\alpha - \alpha).
\end{gathered} \tag{2.17}$$

Multiplying both sides by $(k\beta + \gamma \mid \alpha)_n$, we get

$$2(k\beta + \gamma \mid \alpha)_{n+1} \leq (k\beta + \gamma \mid \alpha)_n + (k\beta + \gamma \mid \alpha)_{n+2}. \tag{2.18}$$

Thus, making use of (1.16), we obtain (2.16). □

Note that, for $(\alpha, \beta, \gamma, x) = (0, \beta, r, 1)$, (2.16) asserts the convexity of (r,β)-Bell polynomials which further imply the convexity of r-Bell polynomials when $\beta = 1$. Moreover, letting $(\alpha, \beta, \gamma, x) = (0,1,0,1)$, (2.16) yields (2.15) and implies the convexity of $\overline{B}_n$.

3. A Variation of Generalized Bell Numbers

Let us denote $\overline{A}(n,k;\alpha,\beta,\gamma) = k!\beta^k S(n,k;\alpha,\beta,\gamma)$ and define

$$B_n(\alpha,\beta,\gamma) = \sum_{k=1}^{n} \overline{A}(n,k;\alpha,\beta,\gamma). \tag{3.1}$$

The numbers $\overline{A}(n,k;\alpha,\beta,\gamma)$ were given combinatorial interpretation in [2], for nonnegative integers $\alpha,\beta,$ and γ, as the number of ways to distribute n distinct balls, one ball at a time, into $k+1$ distinct cells, first k of which has β distinct compartments and the last cell with γ distinct compartments such that

(i) the compartments in each cell are given cyclic ordered numbering,

(ii) the capacity of each compartment is limited to one ball,

(iii) each successive α available compartments in a cell can only have the leading compartment getting the ball,

(iv) the first k cells are nonempty.

Illustration of (iii)

Suppose the first ball lands in compartment 3 of cell 2. The compartment numbered 4, 5, 6,. . ., $\alpha, \alpha+1, \alpha+2$ will be closed. And suppose the second ball lands in compartment $\beta-2$ also of cell 2. Then compartments numbered $\beta-1, \beta, 1, 2, \alpha+3, \alpha+4, \alpha+5, \ldots, 2\alpha-3$ of cell 2 will be closed.

If $k+1$ cells will be changed to any number of cells with the last cell containing γ distinct compartments and the rest of the cells each has β distinct compartments such that only the last cell could be empty, then this gives the combinatorial interpretation of $B_n(\alpha,\beta,\gamma)$.

The following theorem contains a kind of exponential generating function for $B_n(\alpha,\beta,\gamma)$.

Theorem 3.1. *The numbers $B_n(\alpha,\beta,\gamma)$ have the following exponential generating function:*

$$\sum_{n\geq 0} B_n(\alpha,\beta,\gamma)\frac{t^n}{n!} = \frac{(1+\alpha t)^{\gamma/\alpha}}{2-(1+\alpha t)^{\beta/\alpha}}. \tag{3.2}$$

Proof. Using the exponential generating function in (1.4), we get

$$\begin{aligned}
\sum_{n\geq 0} B_n(\alpha,\beta,\gamma)\frac{t^n}{n!} &= \sum_{n\geq 0}\sum_{k\geq 0}\beta^k k! S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!} \\
&= (1+\alpha t)^{\gamma/\alpha}\sum_{k\geq 0}\left[(1+\alpha t)^{\beta/\alpha}-1\right]^k \\
&= (1+\alpha t)^{\gamma/\alpha}\frac{1}{1-\left[(1+\alpha t)^{\beta/\alpha}-1\right]}.
\end{aligned} \tag{3.3}$$

This is exactly the desired generating function. □

Differentiating both sides of (1.9) with respect to t, we yield

$$\overline{A}(n,k;\alpha,\beta,\gamma)=\frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\left((1+\alpha t)^{\beta/\alpha}-1\right)^k\right]_{t=0}. \tag{3.4}$$

Since $\overline{A}(n,k;\alpha,\beta,\gamma)$ vanishes when $k=0$ and $k>n$, we have

$$\begin{aligned}B_n(\alpha,\beta,\gamma)&=\sum_{k=0}^{\infty}\frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\left((1+\alpha t)^{\beta/\alpha}-1\right)^k\right]_{t=0}\\&=\frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\left(2-(1+\alpha t)^{\beta/\alpha}\right)^{-1}\right]_{t=0}\\&=\frac{1}{2}\frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\sum_{\nu=0}^{\infty}\left(\frac{1}{2}(1+\alpha t)^{\beta/\alpha}\right)^{\nu}\right]_{t=0}\\&=\frac{1}{2}\sum_{\nu=0}^{\infty}\frac{d^n}{dt^n}\left[(1+\alpha t)^{(\gamma+\beta\nu)/\alpha}\right]_{t=0}\frac{1}{2^{\nu}}.\end{aligned} \tag{3.5}$$

This result is embodied in the following theorem.

Theorem 3.2. *The number $B_n(\alpha,\beta,\gamma)$ is equal to*

$$B_n(\alpha,\beta,\gamma)=\frac{1}{2}\sum_{\nu=0}^{\infty}(\gamma+\beta\nu\mid\alpha)_n 2^{-\nu},\quad n\geq 1. \tag{3.6}$$

The next theorem provides a recurrence relation for the number $B_n(\alpha,\beta,\gamma)$ which can be used as a quick tool in computing its first values.

Theorem 3.3. *The following recurrence relation holds:*

$$B_n(\alpha,\beta,\gamma)=(\gamma\mid\alpha)_n+(\beta\mid\alpha)_n+\sum_{j=1}^{n-1}\binom{n}{j}(\beta\mid\alpha)_j B_{n-j}(\alpha,\beta,\gamma), \tag{3.7}$$

where $n\geq 1$.

Proof. Making use of (3.6), we have

$$\binom{\frac{\beta}{\alpha}}{j}\frac{B_{n-j}(\alpha,\beta,\gamma)}{\alpha^{n-j}(n-j)!}=\frac{1}{2}\sum_{\nu=0}^{\infty}\binom{\frac{\beta}{\alpha}}{j}\binom{\frac{\beta\nu+\gamma}{\alpha}}{n-j}2^{-\nu}. \tag{3.8}$$

Summing up both sides from $j = 0$ to $n - 1$ and using Vandermonde's formula, we get

$$\sum_{j=0}^{n-1}\binom{n}{j}(\beta \mid \alpha)_j \frac{B_{n-j}(\alpha,\beta,\gamma)}{\alpha^n n!} = \frac{1}{2}\sum_{\nu=0}^{\infty}\left(\sum_{j=0}^{n-1}\binom{\frac{\beta}{\alpha}}{j}\binom{\frac{\beta\nu+\gamma}{\alpha}}{n-j}\right)2^{-\nu}$$
$$= \frac{1}{2}\sum_{\nu=0}^{\infty}\binom{\frac{\beta+\nu\beta+\gamma}{\alpha}}{n}2^{-\nu} - \frac{1}{2}\sum_{\nu=0}^{\infty}\binom{\frac{\beta}{\alpha}}{n}2^{-\nu}. \tag{3.9}$$

Hence, we have

$$\sum_{j=0}^{n-1}\binom{n}{j}(\beta \mid \alpha)_j B_{n-j}(\alpha,\beta,\gamma) = \frac{1}{2}\sum_{\nu=0}^{\infty}((\nu+1)\beta+\gamma \mid \alpha)_n 2^{-\nu} - (\beta \mid \alpha)_n \frac{1}{2}\sum_{\nu=0}^{\infty}2^{-\nu}. \tag{3.10}$$

Now, by (3.6),

$$\frac{1}{2}\sum_{\nu=0}^{\infty}(\beta(\nu+1)+\gamma \mid \alpha)_n 2^{-\nu} = \sum_{\nu=0}^{\infty}(\beta(\nu+1)+\gamma \mid \alpha)_n 2^{-(\nu+1)}$$
$$= \sum_{\nu=0}^{\infty}(\beta\nu+\gamma \mid \alpha)_n 2^{-\nu} - (\gamma \mid \alpha)_n \tag{3.11}$$
$$= 2B_n(\alpha,\beta,\gamma) - (\gamma \mid \alpha)_n$$

and $(1/2)\sum_{\nu=0}^{\infty} 2^{-\nu} = 1$. Thus,

$$\sum_{j=1}^{n-1}\binom{n}{j}(\beta \mid \alpha)_j B_{n-j}(\alpha,\beta,\gamma) = B_n(\alpha,\beta,\gamma) - (\gamma \mid \alpha)_n - (\beta \mid \alpha)_n \tag{3.12}$$

which is precisely equivalent to (3.7). □

Note that when $n = 1$, (3.7) gives

$$B_1(\alpha,\beta,\gamma) = \gamma + \beta, \tag{3.13}$$

while (3.6) gives

$$B_1(\alpha,\beta,\gamma) = \gamma + \beta\left(\sum_{\nu=1}^{\infty}\frac{\nu}{2^{\nu+1}}\right). \tag{3.14}$$

This implies that

$$\sum_{\nu=1}^{\infty}\frac{\nu}{2^{\nu+1}} = 1. \tag{3.15}$$

The following theorem gives a kind of congruence relation for $B_n(\alpha,\beta,\gamma)$ with the restriction that $\alpha \to 0$. We use $\widehat{G}_{n,\beta,r}$ to denote the following limit:

$$\widehat{G}_{n,\beta,r} = \lim_{\alpha \to 0} B_n(\alpha,\beta,\gamma). \tag{3.16}$$

Theorem 3.4. *Let r and β be integers. Then for any odd prime p and $n \geq 1$, one has the following congruence relation:*

$$\widehat{G}_{n+p-1,\beta,r} - \widehat{G}_{n,\beta,r} \equiv 0 \pmod{2p}. \tag{3.17}$$

Proof. Note that the explicit formula in (1.14) can be expressed in terms of a kth difference operator. That is,

$$\left[\Delta^k(\beta t + r)^n\right]_{t=0} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + r)^n, \tag{3.18}$$

where Δ^k denotes the kth difference operator. Hence,

$$\widehat{G}_{n+p-1,\beta,r} = \sum_{k=0}^{\infty}\left[\Delta^k(\beta t + r)^{n+p-1}\right]_{t=0}. \tag{3.19}$$

Thus,

$$\widehat{G}_{n+p-1,\beta,r} - \widehat{G}_{n,\beta,r} = \sum_{k=0}^{\infty}\Delta^k\left\{(\beta t + r)^{n-1}\left[(\beta t + r)^p - (\beta t + r)\right]\right\}_{t=0}. \tag{3.20}$$

Since, by Fermat's little theorem, $(\beta t + r)^p - (\beta t + r)$ is divisible by p,

$$(\beta t + r)^{n-1}\left[(\beta t + r)^p - (\beta t + r)\right] = px, \tag{3.21}$$

for some integer x. Also, since $(\beta t + r)^n$ and $(\beta t + r)^{p-1} - 1$ are of different parity, $(\beta t + r)^n[(\beta t + r)^{p-1} - 1]$ is divisible by 2. Hence,

$$(\beta t + r)^n\left[(\beta t + r)^{p-1} - 1\right] = 2py, \tag{3.22}$$

for some integer y. Thus, we have

$$(\beta t + r)^n\left[(\beta t + r)^{p-1} - 1\right] \equiv 0 \pmod{2p}. \tag{3.23}$$

This completes the proof of the theorem. □

4. Some Theorems on (r,β)-Bell Polynomials

The (r,β)-Bell polynomials $G_{n,\beta,r}(x)$ have already possessed numerous properties. Some of them are obtained as special case of the properties of $S_n(x)$. However, there are properties of the ordinary Bell numbers or r-Bell numbers which are difficult to establish in $S_n(x)$ but can be done in $G_{n,\beta,r}(x)$. For instance, using the rational generating function for $S(n,k;\beta,r)$ in [2] which is given by

$$\sum_{n\geq k} S(n,k;\beta,r)t^n = \frac{t^k}{\prod_{j=0}^{k}[1-(\beta j+r)t]}, \tag{4.1}$$

we can have

$$\begin{aligned}\sum_{n\geq 0} S(n,k;\beta,r)t^n &= \frac{1}{\beta^{k+1}t}\frac{1}{\prod_{i=0}^{k}((1-rt)/(\beta t)-i)}\\ &= \frac{1}{\beta^{k+1}t}\frac{1}{((1-rt)/(\beta t))\prod_{i=1}^{k}((1-rt)/(\beta t)-i)}\\ &= \frac{-1}{\beta^{k}(rt-1)}\frac{(-1)^k}{\prod_{i=1}^{k}((rt-1)/(\beta t)+i)}.\end{aligned} \tag{4.2}$$

It can easily be shown that

$$\prod_{i=1}^{k}\left(\frac{rt-1}{\beta t}-i\right) = \left(\frac{(\beta+r)t-1}{\beta t}\right)_k. \tag{4.3}$$

Thus,

$$\begin{aligned}\sum_{k\geq 0}\left(\sum_{n\geq 0} S(n,k;\beta,r)t^n\right)x^k &= \sum_{k\geq 0}\left(\frac{-1}{\beta^k(rt-1)}\frac{(-1)^k}{(((\beta+r)t-1)/\beta t)_k}\right)x^k,\\ \sum_{n\geq 0}\left(\sum_{k=0}^{n} S(n,k;\beta,r)x^k\right)t^n &= \frac{-1}{rt-1}\sum_{k\geq 0}\frac{(1)_k}{(((\beta+r)t-1)/\beta t)_k}\frac{(-x/\beta)^k}{k!}.\end{aligned} \tag{4.4}$$

This can be expressed further as

$$\sum_{n\geq 0} G_{n,\beta,r}(x)t^n = \frac{-1}{rt-1}\cdot {}_1F_1\left(\left.\begin{matrix}1\\ \dfrac{(\beta+r)t-1}{\beta t}\end{matrix}\right| \frac{-x}{\beta}\right), \tag{4.5}$$

where ${}_1F_1$ is the hypergeometric function which is defined by

$${}_pF_q\left(\left.\begin{matrix}a_1, & a_2, & \dots, & a_p\\ b_1, & b_2, & \dots, & b_q\end{matrix}\right| t\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{t^k}{k!}, \tag{4.6}$$

where $(a_i)_j = a_i(a_i+1)(a_i+2)\cdots(a_i+j-1)$. Applying Kummer's formula [8],

$$e^x {}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| -x\right) = {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix} \middle| x\right), \tag{4.7}$$

we obtain the following generating function.

Theorem 4.1. *The (r,β)-Bell polynomials satisfy the following generating function:*

$$\sum_{n\geq 0} G_{n,\beta,r}(x)t^n = \frac{-1}{rt-1}\cdot\frac{1}{e^{x/\beta}}\cdot {}_1F_1\left(\begin{matrix} \dfrac{rt-1}{\beta t} \\ \dfrac{\beta t+rt-1}{\beta t} \end{matrix} \middle| \frac{x}{\beta}\right). \tag{4.8}$$

It will be interesting if one can also obtain a generating function of this form for $S_n(x)$.

Now, using the integral identity in [9],

$$\operatorname{Im}\int_0^{\pi} e^{je^{i\theta}}\sin(n\theta)d\theta = \frac{\pi}{2}\frac{j^n}{n!}, \tag{4.9}$$

and the explicit formula in (1.14), we get

$$\begin{aligned}
\frac{\pi}{2n!}S(n,k;\beta,r) &= \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\operatorname{Im}\int_0^{\pi} e^{(\beta j+r)e^{i\theta}}\sin(n\theta)d\theta \\
&= \frac{1}{\beta^k k!}\operatorname{Im}\int_0^{\pi}\left[\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(e^{\beta e^{i\theta}}\right)^j\right]e^{re^{i\theta}}\sin(n\theta)d\theta \\
&= \operatorname{Im}\int_0^{\pi}\frac{\left[\left(e^{\beta e^{i\theta}}-1\right)/\beta\right]^k}{k!}e^{re^{i\theta}}\sin(n\theta)d\theta.
\end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty}S(n,k;\beta,r)x^k &= \frac{2n!}{\pi}\operatorname{Im}\int_0^{\pi}\left\{\sum_{k=0}^{\infty}\frac{\left[\left(e^{\beta e^{i\theta}}-1\right)/\beta\right]^k}{k!}x^k\right\}e^{re^{i\theta}}\sin(n\theta)d\theta \\
&= \frac{2n!}{\pi}\operatorname{Im}\int_0^{\pi} e^{x(e^{\beta e^{i\theta}}-1)/\beta}e^{re^{i\theta}}\sin(n\theta)d\theta.
\end{aligned} \tag{4.11}$$

Thus,

$$G_{n,\beta,r}(x) = \frac{2n!}{\pi e^{x/\beta}}\operatorname{Im}\int_0^{\pi} e^{x\beta^{-1}e^{\beta e^{i\theta}}}e^{re^{i\theta}}\sin(n\theta)d\theta, \tag{4.12}$$

where $\beta\neq 0$. By simple algebraic manipulation, this can further be expressed as follows.

Theorem 4.2. *The (r, β)-Bell polynomials have the following integral representation:*

$$G_{n,\beta,r}(x) = \frac{2n!}{\pi e^{x/\beta}} \int_0^{\pi} e^{J_1(\theta)} \sin(J_2(\theta)) \sin(n\theta) d\theta, \tag{4.13}$$

where

$$J_1(\theta) = r\cos\theta + \frac{xe^{\beta\cos\theta}\cos(\beta\sin\theta)}{\beta},$$
$$J_2(\theta) = r\sin\theta + \frac{xe^{\beta\cos\theta}\sin(\beta\sin\theta)}{\beta}. \tag{4.14}$$

It will also be compelling to establish such integral representation for $S_n(x)$.

The Bell polynomials $B_n(\lambda)$ are known to be connected to the Poisson distribution. More precisely, $B_n(\lambda)$ can be expressed in terms of the moment of the Poisson random variable Z with parameter $\lambda > 0$ as

$$B_n(\lambda) = E_\lambda[Z^n]. \tag{4.15}$$

The exponential generating function for the (r, β)-Bell polynomials in (1.12) can be written as follows:

$$\begin{aligned} e^{(r/\beta)\beta t} e^{(x/\beta)(e^{\beta t}-1)} &= e^{(r/\beta)\beta t} E_{x/\beta}\left[e^{(\beta t)Z}\right] \\ &= \sum_{n\geq 0} \left\{ \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k E_{x/\beta}\left[Z^k\right] \right\} \frac{t^n}{n!}. \end{aligned} \tag{4.16}$$

Hence, we can also express the (r, β)-Bell polynomials in terms of the following moment:

$$G_{n,\beta,r}(x) = E_{x/\beta}\left[(\beta Z + r)^n\right]. \tag{4.17}$$

Now,

$$\begin{aligned} G_{n,\beta,r}(x) &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k E_{x/\beta}\left[Z^k\right] \\ &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k B_k\left(\frac{x}{\beta}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k \sum_{j=0}^{k} S(k, j) \left(\frac{x}{\beta}\right)^j. \end{aligned} \tag{4.18}$$

Thus, we have the following theorem.

Theorem 4.3. *The (r,β)-Bell polynomials equal*

$$G_{n,\beta,r}(x) = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}\sum_{j=0}^{k}\beta^{k-j}S(k,j)x^{j}. \tag{4.19}$$

An extension of the Bell polynomials $B_n(y,\lambda)$, defined by Privault [10] as

$$\sum_{n=0}^{\infty}B_n(y,\lambda)\frac{t^n}{n!} = e^{yt-\lambda(e^t-t-1)}, \tag{4.20}$$

can be expressed in terms of the (r,β)-Bell polynomials as

$$B_n(y,\lambda) = G_{n,1,\lambda+y}(-\lambda). \tag{4.21}$$

Using Theorem 4.3, we obtain

$$B_n(y,-\lambda) = G_{n,1,-\lambda+y}(\lambda) = \sum_{k=0}^{n}\binom{n}{k}(y-\lambda)^{n-k}\sum_{j=0}^{k}S(k,j)\lambda^{j}. \tag{4.22}$$

This is exactly the identity obtained by Privault in [10].

5. An Asymptotic Approximation for $G_{n,\beta,r}$

Using the exponential generating function for $G_{n,r,\beta}$ in (1.12) with $x = 1$ and Cauchy's theorem for integrals, we obtain the integral representation

$$G_{n,r,\beta} = \frac{n!}{2\pi i}\int_{\gamma}\frac{\exp\left[rz + \left(e^{\beta z-1}/\beta\right)\right]}{z^{n+1}}dz, \tag{5.1}$$

where γ is the circle $z = Re^{i\theta}, -\pi \le \theta \le \pi$. Contour integration yields

$$G_{n,r,\beta} = \frac{n!}{2\pi i R^n}\int_{-\pi}^{\pi}\exp\left(\beta^{-1}e^{\beta Re^{i\theta}} + rRe^{i\theta} - in\theta - \beta^{-1}\right)d\theta, \tag{5.2}$$

which can be written into the compact form

$$G_{n,r,\beta} = A\int_{-\pi}^{\pi}\exp(F(\theta))d\theta, \tag{5.3}$$

where

$$A = \frac{n! \exp\left(rR + \beta^{-1}e^{\beta R} - \beta^{-1}\right)}{2\pi R^n},$$
$$F(\theta) = \beta^{-1}e^{\beta R e^{i\theta}} + rRe^{i\theta} - in\theta - \left(rR + \beta^{-1}e^{\beta R}\right). \quad (5.4)$$

Define $\epsilon = e^{-3R/8}$ and let

$$J_1 = \int_{-\pi}^{\epsilon} \exp(F(\theta))d\theta, \qquad J_2 = \int_{\epsilon}^{\pi} \exp(F(\theta))d\theta. \quad (5.5)$$

Thus (5.3) can be written as

$$G_{n,r,\beta} = AJ_1 + A\int_{\epsilon}^{\epsilon} \exp(F(\theta))d\theta + AJ_2. \quad (5.6)$$

Lemma 5.1. *There exists a constant $k > 0$ such that*

$$|J_2| < e^{-k\beta^{-1}e^{\beta R}}(\pi - \epsilon). \quad (5.7)$$

Proof. It can be shown that

$$\left|\exp(F(\theta))\right| = e^{-[(rR+\beta^{-1}e^{\beta R})+\beta^{-1}\cos(\beta R\sin\theta)e^{\beta R\cos\theta}]}. \quad (5.8)$$

Since $\cos\theta < 1$ for $0 < \epsilon < \theta \leq \pi$, we have

$$\left|\exp(F(\theta))\right| = e^{-\beta^{-1}e^{\beta R}}\left[1 - \cos\left(\beta R\sin\theta\right)\right]. \quad (5.9)$$

Since $[1 - \cos(\beta R\sin\theta)] > 0$ for $\cos\theta < 1$ for $0 < \epsilon < \theta \leq \pi$, there exists a constant $k > 0$ such that $[1 - \cos(\beta R\sin\theta)] < k$. Hence

$$|J_2| < e^{-k\beta^{-1}e^{\beta R}}(\pi - \epsilon). \quad (5.10)$$

□

It will be seen later that $R \to \infty$ as $n \to \infty$. With the result in Lemma 5.1 we see that J_1 and J_2 will tend to zero as $n \to \infty$. Hence

$$G_{n,r,\beta} \sim A\int_{-\epsilon}^{\epsilon} \exp(F(\theta))d\theta. \quad (5.11)$$

Observe that $F(\theta)$ is analytic at $\theta = 0$.Thus $F(\theta)$ has a Maclaurin series expansion about $\theta = 0$. This Maclaurin expansion can be written in the form

$$\begin{aligned} F(\theta) &= \left(Re^{\beta R} + rR - n\right)i\theta + \frac{1}{2}\left(\beta R^2 + Re^{\beta R} + rR\right)i^2\theta \\ &+ \sum_{k=3}^{\infty}\left[\beta^{-1}\rho^k\left(e^{\beta R}\right) + rR\right](i\theta)^k, \end{aligned} \tag{5.12}$$

where we define ρ to be the operator $\rho = R(d\theta/dR)$. Choose R such that $Re^{\beta R} + rR - n = 0$; that is, R satisfies the equation $xe^{\beta R} + rx - n = 0$. This R is shown to exist in the following lemma.

Lemma 5.2. *There exists a unique positive real solution to the equation* $xe^{\beta R} + rx - n = 0$.

Proof. We can rewrite the given equation in the form

$$\frac{x}{n - rx} = e^{-\beta x}. \tag{5.13}$$

The desired solution is the x-coordinate of the intersection of the functions $h(x) = x/(n - rx)$ and $g(x) = e^{-\beta x}$. □

It can be seen from the preceding lemma that $R \to \infty$ as $n \to \infty$. With this choice of R, we have

$$F(\theta) = -\frac{1}{2}\left(\beta R^2 + Re^{\beta R} + rR\right)\theta + \sum_{k=3}^{\infty}\left[\beta^{-1}\rho^k\left(e^{\beta R}\right) + rR\right](i\theta)^k. \tag{5.14}$$

We now introduce the following notations:

$$\begin{aligned} \phi &= \left[(1/2)\left(\beta R^2 e^{\beta R} + Re^{\beta R} + rR\right)^{1/2}\right]\theta, \\ a_k &= \frac{[\beta^{-1}e^{-\beta R}\rho^{k+2}(e^{\beta R}) + rRe^{-\beta R}]\,(i\phi)^{k+2}}{(k+1)!\,[1/2(\beta R^2 + R + rRe^{-\beta R})]^{k+2/2}}, \\ z &= e^{-\beta R/2}, \\ f(z) &= \sum_{k=1}^{\infty} a_k z^k. \end{aligned} \tag{5.15}$$

Then $F(\theta) = -\phi^2 + f(z)$ and

$$G_{n,r,\beta} \sim C\int_{-h}^{h} \exp\left[-\phi^2 + f(z)\right]dz, \tag{5.16}$$

where $h = (1/2)(\beta R^2 e^{\beta R} + Re^{\beta R} + rR)^{1/2}e^{-3R/8}$ and $C = A/[(1/2)(\beta R^2 e^{\beta R} + rR)]^{1/2}$.

We have defined z as a function of R. However, for the moment we consider z to be an independent variable and expand $e^{f(z)}$ into a convergent Maclaurin series expansion of the form

$$e^{f(z)} = \sum_{k=0}^{\infty} b_k z^k, \tag{5.17}$$

where $b_0 = e^{f(0)} = 1$, $b_1 = e^{f(0)} f'(0) = a_1$, and $b_2 = a_2 + (a_1^2/2)$.

Lemma 5.3. *There is a constant R_o such that for all $R > R_o$,*

$$|a_k| < |2\phi|^{k+2}. \tag{5.18}$$

Proof. We see that

$$|a_k| = \frac{R^{k+2}\left[1 + o\left(R^{k+2}\right)\right](2)^{(k+2)/2}}{(k+2)!\left(\beta R^2\right)^{(k+2)/2}\left[1 + o\left(R^2\right)\right]} |\phi|^{k+2} \tag{5.19}$$

which tends to

$$\frac{2^{k+2/2}}{(k+2)!} < 2^{k+2} |\phi|^{k+2} \tag{5.20}$$

as $R \to \infty$. From this, it follows that there is a constant R_o satisfying (5.18). □

Now, it will follow from Lemma 5.3 that the radius of convergence of (5.17) becomes large when θ is near zero. Thus, $z = e^{-\beta R/2}$ is within the domain of convergence.

With $z = e^{-\beta R/2}$,

$$G_{n,r,\beta} \sim C \sum_{k=0}^{s-1} \left(\int_{-h}^{h} e^{-\phi^2} b_k \, d\phi \right) z^k + Q_s, \tag{5.21}$$

where

$$Q_s = \int_{-h}^{h} \left(\sum_{k=s}^{\infty} e^{-\phi^2} b_k z^k \right) d\phi. \tag{5.22}$$

Note that $R \to \infty$ as $n \to \infty$. Furthermore with

$$\begin{aligned} h &= \frac{1}{2}\left(\beta R^2 e^{\beta R} + R e^{\beta R} + rR\right)^{1/2} e^{-3R/8} \\ &= \frac{1}{2}\left(\beta R^2 + R + rRe^{-\beta R}\right)^{1/2} e^{(R(4\beta-3))/8}, \end{aligned} \tag{5.23}$$

$h \to \infty$ as $R \to \infty$. From these facts and the known asymptotic expansion of the function of the form

$$\int_{-h}^{h} e^{-\phi^2} (\text{polynomial in } |\phi|) \, d\phi, \tag{5.24}$$

the replacement of h by ∞ in (5.16) is easily justified (see [11]). Hence

$$G_{n,r,\beta} \sim C\sum_{k=0}^{s-1}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k \, d\phi\right) z^k + Q_s. \tag{5.25}$$

It remains to show that $Q_s = o(|z|^s)$ as $R \to \infty$, that is, $z \to 0$. From a lemma in [12], $|b_k| \le |2\phi|^{k+2}(1+|2\phi|^2)^{k-1}$. Thus,

$$\left|\sum_{k=s}^{\infty} b_k z^k\right| \le \left[|2\phi|^{s+2}\left(1+|2\phi|^2\right)^{s-1}|z|^s\right]\left[1+\mu+\mu^2+\cdots\right], \tag{5.26}$$

where $\mu = |2\phi|(1+|2\phi|^2)|z|$.

Now, for $\mu < 1$, we have

$$\left|\sum_{k=s}^{\infty} b_k z^k\right| \le \frac{|2\phi|^{s+2}\left(1+|2\phi|^2\right)^{s-1}|z|^s}{1-|z||2\phi|\left(1+|2\phi|^2\right)}. \tag{5.27}$$

Let M and $P_s(|\phi|)|z|^s$ denote the denominator and the numerator, respectively, in (5.27). Since $|\phi| \le h$ and $z = e^{-\beta R/2}$, we have

$$\left|\phi^3\right||z| \le \frac{1}{8}\left(\beta R^2 + R + rRe^{-\beta R}\right)^{3/2} e^{-3R/8} \longrightarrow 0 \text{ as } R \longrightarrow \infty. \tag{5.28}$$

Hence for sufficiently large R, $M \ge 1/2$. Moreover,

$$\int_{-\infty}^{\infty} e^{-\phi^2} P_s(|\phi|)\, d\phi \tag{5.29}$$

exists and tends to zero as $R \to \infty$. Therefore,

$$\frac{|Q_s|}{|z|^s} \le \int_{-\infty}^{\infty} \frac{e^{-\phi^2} P_s(|\phi|)}{M} d\phi. \tag{5.30}$$

Thus, $|Q_s| = o(|z|^s)$. Consequently,

$$G_{n,r,\beta} \sim C\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k \, d\phi\right) e^{(-k\beta R)/2}. \tag{5.31}$$

Since $\int_{-\infty}^{\infty} e^{-x^2} x^n = 0$ for odd n, and b_{2k+1}, as a polynomial in ϕ, contain only odd powers of ϕ, it follows that

$$G_{n,r,\beta} \sim C\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_{2k} d\phi\right) e^{-k\beta R}. \tag{5.32}$$

Calculation yields

$$a_1 = \frac{\beta R^3 + 3R^2 + \beta^{-1}R + rRe^{-\beta R}}{3!\left[(1/2)\left(\beta R^2 + R + rRe^{-\beta R}\right)\right]^{3/2}}(i\phi)^3,$$
$$a_2 = \frac{\beta R^4 + 6\beta R^3 + 7R^2 + \beta^{-1}R + rRe^{-\beta R}}{4!\left[(1/2)\left(\beta R^2 + R + rRe^{-\beta R}\right)\right]^{2}}(i\phi)^4. \tag{5.33}$$

Taking the first two terms of the asymptotic expansion of (5.32), we have

$$G_{n,r,\beta} \sim C\int_{-\infty}^{\infty} e^{-\phi^2} b_o\, d\phi + Cz^2\int_{-\infty}^{\infty} e^{-\phi^2} b_2 d\phi. \tag{5.34}$$

Since $b_2 = a_2 + a_1^2/2$ and $b_o = 1$,

$$G_{n,r,\beta} \sim C\int_{-\infty}^{\infty} e^{-\phi^2} d\phi + Cz^2\int_{-\infty}^{\infty} a_2 e^{-\phi^2} d\phi + C\frac{z^2}{2}\int_{-\infty}^{\infty} e^{-\phi^2}\left(a_1^2\right) d\phi. \tag{5.35}$$

Let I_1, I_2, and I_3 denote, respectively, the integrals in (5.35). Then evaluating the last two integrals by parts and since $\int_{-\infty}^{\infty} e^{-\phi^2} d\phi = \sqrt{\pi}$, we obtain

$$I_1 = C\sqrt{\pi},$$
$$I_2 = \frac{Ce^{-R}\sqrt{\pi}\left(\beta R^3 + 6\beta R^2 + \beta^{-1} + re^{-\beta R}\right)}{8R\left(\beta R + 1 + re^{-\beta R}\right)^2},$$
$$I_3 = \frac{-5Ce^{-R}\sqrt{\pi}\left(\beta R^2 + 3\beta^{-1}R^2 + re^{-\beta R}\right)^2}{24R\left(\beta R + 1 + re^{-\beta R}\right)^3}. \tag{5.36}$$

Substituting the results in (5.35) and simplifying, we obtain

$$G_{n,r,\beta} \sim C\sqrt{\pi}\left(1 + \frac{D+E}{F}\right), \tag{5.37}$$

where

$$D = \left(3\beta^2 R^3 + 8\beta R^3 + 3\beta R + 3 - 10\beta^{-1} - 2re^{-\beta R}\right)re^{-\beta R},$$
$$E = \left(3\beta^3 - 5\beta^2\right)R^4 + \left(21\beta^2 - 30\beta\right)R^3 + (39\beta - 55)R^2 + \left(24 - 30\beta^{-1}\right)R + \left(3\beta^{-1} - 5\beta^{-2}\right),$$
$$F = 24Re^{\beta R}\left(\beta R + 1 + re^{-\beta R}\right)^3. \tag{5.38}$$

Since $Re^{\beta R} = (n - rR)\beta^{-1}$ and $R^n = n^n(\beta e^{\beta R} + r)^{-n}$,

$$C = \frac{n! \exp\left(rR + \beta^{-1}e^{\beta R} - \beta\right)}{\pi\left[n^n\left(\beta e^{\beta R} + r\right)^{-n}\right]\left[2(n - rR)\beta^{-1}\right]^{1/2}\left(\beta R + 1 + re^{-\beta R}\right)^{1/2}}. \tag{5.39}$$

Using Stirling's approximation for $n!$, viz,

$$n! \sim (2\pi)e^{-n}n^{n+(1/2)}\left(1 + \frac{1}{12n}\right), \tag{5.40}$$

we obtain

$$C \sim \frac{n^{1/2}(1 + (1/12n))\exp\left(rR + \beta^{-1}e^{\beta R} - \beta\right)\left(\beta^{\beta R} + r\right)^n}{\pi^{1/2}\left[(n - rR)\beta^{-1}\right]^{1/2}\left(\beta R + 1 + re^{-\beta R}\right)^{1/2}e^n}. \tag{5.41}$$

Using (5.37), we obtain

$$G_{n,r,\beta} \sim \frac{n^{1/2}(1 + (1/12n))\exp\left(rR + \beta^{-1}e^{\beta R} - \beta - n\right)\left(\beta^{\beta R} + r\right)^n}{\left[(n - rR)\beta^{-1}\right]^{1/2}\left(\beta R + 1 + re^{-\beta R}\right)^{1/2}}\left(1 + \frac{D + E}{F}\right). \tag{5.42}$$

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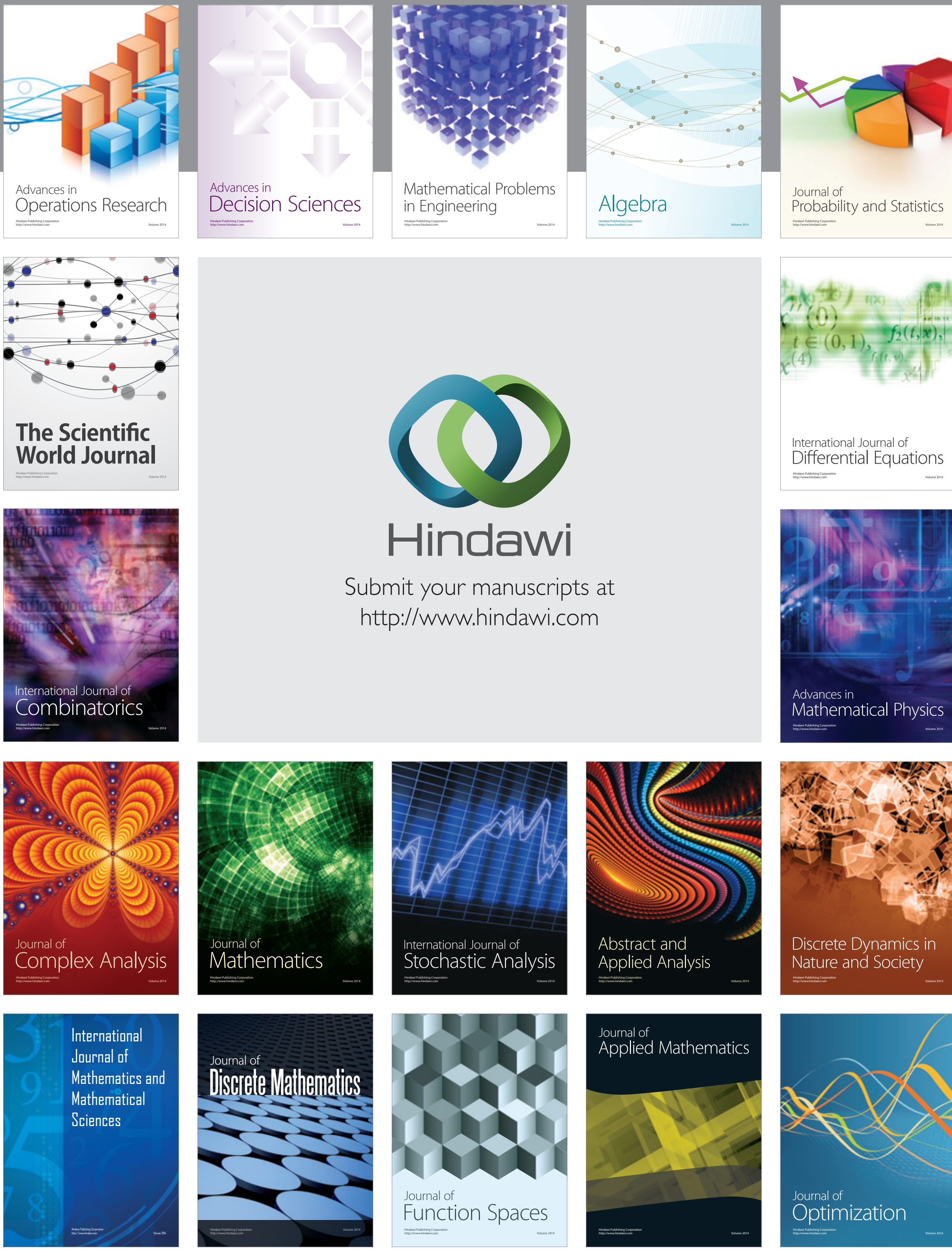

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