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Research Article

Identities on the Weighted q-Bernoulli Numbers of Higher Order

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We give a new construction of the weighted q-Bernoulli numbers and polynomials of higher order by using multivariate p-adic q-integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm of $\mathbb{C}_p$ is defined by $|x|_p = 1/p^r$ where $x = p^r s/t$ with $(p,s) = (p,t) = (s,t) = 1, r \in \mathbb{Q}$. In this paper, we assume $\alpha \in \mathbb{Q}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-1/(p-1)}$ and let $[x]_q = (1-q^x)/(1-q)$. Note that $\lim_{q\to 1}[x]_q = x$ (see [1–13]). Recently, the q-Bernoulli numbers with weight α are defined by

$$\widetilde{\beta}_{0,q}^{(\alpha)} = 1, \qquad q\left(q^{\alpha}\widetilde{\beta}_q^{\alpha} + 1\right)^n - \widetilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.1}$$

with the usual convention about replacing $(\widetilde{\beta}^{(\alpha)})^n$ by $\widetilde{\beta}_{n,q}^{(\alpha)}$, (see [4]).

The q-Bernoulli polynomials with weight α are also defined by as

$$\begin{aligned}\widetilde{\beta}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}q^{\alpha l x}\widetilde{\beta}_{l,q}^{(\alpha)} \\ &= \left([x]_{q^\alpha} + q^{\alpha x}\widetilde{\beta}_{q}^{(\alpha)}\right)^n, \quad \text{for } n \geq 0.\end{aligned} \tag{1.2}$$

For $f \in UD(\mathbb{Z}_p)$ = the space of uniformly differentiable functions on $\mathbb{Z}_p$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.3}$$

(see [4–12]). From (1.3), we note that

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{1.4}$$

where $f_n(x) = f(x+n)$, (see [1–12]).

We have the Witt formula for the q-Bernoulli numbers and polynomials with weight α as follows (see [4, 5, 12]):

$$\widetilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x), \qquad \widetilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(x). \tag{1.5}$$

From (1.4) and (1.5), we have

$$\widetilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n}\sum_{l=0}^{n}(-1)^l q^{\alpha l x}\frac{\alpha l + 1}{[\alpha l + 1]_q}, \tag{1.6}$$

(see [4]). By (1.6), we easily get $\lim_{q\to 1}\widetilde{\beta}_{n,q}^{(\alpha)}(x) = B_n(x)$, where $B_n(x)$ are the Bernoulli polynomials of degree n.

To give the new construction of the weighted q-Bernoulli numbers and polynomials of higher order, we first use the multivariate p-adic q-integral on $\mathbb{Z}_p$. The purpose of this paper is to give the higher-order q-Bernoulli numbers and polynomials with weight α and to derive a new explicit formulas by these numbers and polynomials.

2. On the Higher Order q-Bernoulli Numbers with Weight α

For h_i $(i = 1, 2, \ldots, k) \in \mathbb{Z}_+$, we consider a sequence of p-adic rational numbers as expansion of the weighted q-Bernoulli numbers and polynomials of order k as follows:

$$\widetilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \ldots, h_k) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_q(x_1) \cdots d\mu_q(x_k), \tag{2.1}$$

$$\widetilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \ldots, h_k \mid x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{2.2}$$

From (2.1) and (2.2), we can derive the following equations:

$$\widetilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \ldots, h_k) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q}, \tag{2.3}$$

$$\begin{aligned} &\widetilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \ldots, h_k \mid x) \\ &\quad = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q}, \\ &\quad = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{\beta}_{l,q}^{(k,\alpha)}(h_1, h_2, \ldots, h_k). \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we get

$$\begin{aligned} &\sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q}, \\ &\quad = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} (1-q)^{n-l} [\alpha]_q^{n-l} \sum_{s=0}^l \binom{l}{s} (-1)^s \frac{(\alpha s + h_1) \cdots (\alpha s + h_k)}{[\alpha s + h_1]_q \cdots [\alpha s + h_k]_q}. \end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. *For h_i $(i = 1, 2, \ldots, k) \in \mathbb{Z}_+$, and $k \in \mathbb{N}$, we have*

$$\begin{aligned} &\sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q} \\ &\quad = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} (1-q)^{n-l} [\alpha]_q^{n-l} \sum_{s=0}^l \binom{l}{s} (-1)^s \frac{(\alpha s + h_1) \cdots (\alpha s + h_k)}{[\alpha s + h_1]_q \cdots [\alpha s + h_k]_q}. \end{aligned} \tag{2.6}$$

From (1.3) and (1.4), we note that

$$\begin{aligned} &q^{h_1}\int_{\mathbb{Z}_p}[x_1+x+1]_{q^\alpha}^n q^{x_1(h_1-1)}d\mu_q(x_1) \\ &\quad=\int_{\mathbb{Z}_p}[x_1+x]_{q^\alpha}^n q^{x_1(h_1-1)}d\mu_q(x_1)+[x]_{q^\alpha}^n h_1(q-1)+n[x]_{q^\alpha}^{n-1}\frac{\alpha}{[\alpha]_q}. \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $n\in\mathbb{Z}_+$, we have*

$$q^{h_1}\widetilde{\beta}_{n,q}^{(1,\alpha)}(h_1\mid x+1)-\widetilde{\beta}_{n,q}^{(1,\alpha)}(h_1\mid x)=h_1(q-1)[x]_{q^\alpha}^n+n[x]_{q^\alpha}^{n-1}\frac{\alpha}{[\alpha]_q}. \tag{2.8}$$

From (2.2) and (2.3), we have

$$\begin{aligned} &q^{\alpha x}\widetilde{\beta}_{n,q}^{(k,\alpha)}(\alpha h_1+\alpha,\alpha h_2+\alpha,\ldots,\alpha h_k+\alpha\mid x) \\ &\quad=q^{\alpha x}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k+x]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(\alpha h_j+\alpha-1)}d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &\quad=(q^\alpha-1)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k+x]_{q^\alpha}^{n+1} q^{\sum_{j=1}^k x_j(\alpha h_j-1)}d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &\qquad+\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k+x]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(\alpha h_j-1)}d\mu_q(x_1)\cdots d\mu_q(x_k). \end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following proposition.

Proposition 2.3. *For $n\in\mathbb{Z}_+$, we have*

$$\begin{aligned} &q^{\alpha x}\widetilde{\beta}_{n,q}^{(k,\alpha)}(\alpha h_1+\alpha,\alpha h_2+\alpha,\ldots,\alpha h_k+\alpha\mid x) \\ &\quad=(q^\alpha-1)\widetilde{\beta}_{n+1,q}^{(k,\alpha)}(\alpha h_1,\ldots,\alpha h_k\mid x)+\widetilde{\beta}_{n,q}^{(k,\alpha)}(\alpha h_1,\ldots,\alpha h_k\mid x). \end{aligned} \tag{2.10}$$

By (2.2), we get

$$\begin{aligned} &\widetilde{\beta}_{n,q}^{(k,\alpha)}(h_1,h_2,\ldots,h_k\mid x) \\ &\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)}d\mu_q(x_1)\cdots d\mu_q(x_k) \end{aligned}$$

$$= \frac{[d]_{q^{\alpha}}^{n}}{[d]_{q}^{k}} \sum_{s_1,\ldots,s_k=0}^{d-1} q^{\sum_{j=1}^{k} h_j s_j} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{s_1+\cdots+s_k+x}{d} + x_1+\cdots+x_k\right]_{q^{d\alpha}}^{n} \times q^{d\sum_{j=1}^{k} x_j(h_j-1)} d\mu_{q^d}(x_1)\cdots d\mu_{q^d}(x_k)$$

$$= \frac{[d]_{q^{\alpha}}^{n}}{[d]_{q}^{k}} \sum_{s_1,\ldots,s_k=0}^{d-1} q^{\sum_{j=1}^{k} h_j s_j} \widetilde{\beta}_{n,q^d}^{(k,\alpha)}\left(h_1,\ldots,h_k \mid \frac{s_1+\cdots+s_k+x}{d}\right), \tag{2.11}$$

where $d \in \mathbb{N}$.

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, we have*

$$\widetilde{\beta}_{n,q}^{(k,\alpha)}(h_1,h_2,\ldots,h_k \mid x) = \frac{[d]_{q^{\alpha}}^{n}}{[d]_{q}^{k}} \sum_{s_1,\ldots,s_k=0}^{d-1} q^{\sum_{j=1}^{k} h_j s_j} \widetilde{\beta}_{n,q^d}^{(k,\alpha)}\left(h_1,\ldots,h_k \mid \frac{s_1+\cdots+s_k+x}{d}\right). \tag{2.12}$$

Let $d(>0)$ be a fixed integer. For $N \in \mathbb{N}$, we set

$$X = X_d = \lim_{\overleftarrow{N}} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \quad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \tag{2.13}$$
$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\text{mod } dp^N\right)\right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Then we consider the generalized q-Bernoulli numbers with weight α of order k as follows:

$$\widetilde{\beta}_{n,\chi,q}^{(k,\alpha)}(h_1,h_2,\ldots,h_k) = \int_X \cdots \int_X \prod_{i=1}^{k} \chi(x_i) [x_1+\cdots+x_k]_{q^{\alpha}}^{n} q^{\sum_{i=1}^{k} x_i(h_i-1)} d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{2.14}$$

From (2.14), we have

$$\widetilde{\beta}_{n,\chi,q}^{(k,\alpha)}(h_1,h_2,\ldots,h_k) = \frac{[d]_{q^{\alpha}}^{n}}{[d]_{q}^{k}} \sum_{s_1,\ldots,s_k=0}^{d-1} q^{\sum_{j=1}^{k} h_j s_j} \left(\prod_{j=1}^{k} \chi(s_j)\right) \widetilde{\beta}_{n,q^d}^{(k,\alpha)}\left(h_1,\ldots,h_k \mid \frac{s_1+\cdots+s_k}{d}\right). \tag{2.15}$$

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