

Research Article

Identities on the Weighted q -Bernoulli Numbers of Higher Order

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We give a new construction of the weighted q -Bernoulli numbers and polynomials of higher order by using multivariate p -adic q -integral on \mathbb{Z}_p .

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and \mathbb{C}_p will, respectively, denote the ring of rational integers, the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field, and the completion of the algebraic closure of \mathbb{Q}_p . The p -adic norm of \mathbb{C}_p is defined by $|x|_p = 1/p^r$ where $x = p^r s/t$ with $(p, s) = (p, t) = (s, t) = 1, r \in \mathbb{Q}$. In this paper, we assume $\alpha \in \mathbb{Q}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-1/(p-1)}$ and let $[x]_q = (1 - q^x)/(1 - q)$. Note that $\lim_{q \rightarrow 1} [x]_q = x$ (see [1–13]). Recently, the q -Bernoulli numbers with weight α are defined by

$$\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q \left(q^\alpha \tilde{\beta}_q^\alpha + 1 \right)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (1.1)$$

with the usual convention about replacing $(\tilde{\beta}^{(\alpha)})^n$ by $\tilde{\beta}_{n,q}^{(\alpha)}$ (see [4]).

The q -Bernoulli polynomials with weight α are also defined by as

$$\begin{aligned}\tilde{\beta}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(\alpha)} \\ &= \left([x]_{q^\alpha} + q^{\alpha x} \tilde{\beta}_q^{(\alpha)} \right)^n, \quad \text{for } n \geq 0.\end{aligned}\tag{1.2}$$

For $f \in UD(\mathbb{Z}_p)$ = the space of uniformly differentiable functions on \mathbb{Z}_p , the p -adic q -integral on \mathbb{Z}_p is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,\tag{1.3}$$

(see [4–12]). From (1.3), we note that

$$q^n I_q(f_n) = I_q(f) + (q-1) \sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l),\tag{1.4}$$

where $f_n(x) = f(x+n)$, (see [1–12]).

We have the Witt formula for the q -Bernoulli numbers and polynomials with weight α as follows (see [4, 5, 12]):

$$\tilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x), \quad \tilde{\beta}_{l,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^l d\mu_q(x).\tag{1.5}$$

From (1.4) and (1.5), we have

$$\tilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n (-1)^l q^{\alpha l x} \frac{\alpha l + 1}{[\alpha l + 1]_q},\tag{1.6}$$

(see [4]). By (1.6), we easily get $\lim_{q \rightarrow 1} \tilde{\beta}_{n,q}^{(\alpha)}(x) = B_n(x)$, where $B_n(x)$ are the Bernoulli polynomials of degree n .

To give the new construction of the weighted q -Bernoulli numbers and polynomials of higher order, we first use the multivariate p -adic q -integral on \mathbb{Z}_p . The purpose of this paper is to give the higher-order q -Bernoulli numbers and polynomials with weight α and to derive a new explicit formulas by these numbers and polynomials.

2. On the Higher Order q -Bernoulli Numbers with Weight α

For h_i ($i = 1, 2, \dots, k$) $\in \mathbb{Z}_+$, we consider a sequence of p -adic rational numbers as expansion of the weighted q -Bernoulli numbers and polynomials of order k as follows:

$$\tilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \dots, h_k) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_q(x_1) \cdots d\mu_q(x_k), \quad (2.1)$$

$$\tilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \dots, h_k | x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_q(x_1) \cdots d\mu_q(x_k). \quad (2.2)$$

From (2.1) and (2.2), we can derive the following equations:

$$\tilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \dots, h_k) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q}, \quad (2.3)$$

$$\begin{aligned} \tilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \dots, h_k | x) &= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q}, \\ &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(k,\alpha)}(h_1, h_2, \dots, h_k). \end{aligned} \quad (2.4)$$

By (2.3) and (2.4), we get

$$\begin{aligned} &\sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q}, \\ &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} (1-q)^{n-l} [\alpha]_q^{n-l} \sum_{s=0}^l \binom{l}{s} (-1)^s \frac{(\alpha s + h_1) \cdots (\alpha s + h_k)}{[\alpha s + h_1]_q \cdots [\alpha s + h_k]_q}. \end{aligned} \quad (2.5)$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. For h_i ($i = 1, 2, \dots, k$) $\in \mathbb{Z}_+$, and $k \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h_1)(\alpha l + h_2) \cdots (\alpha l + h_k)}{[\alpha l + h_1]_q [\alpha l + h_2]_q \cdots [\alpha l + h_k]_q} \\ &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} (1-q)^{n-l} [\alpha]_q^{n-l} \sum_{s=0}^l \binom{l}{s} (-1)^s \frac{(\alpha s + h_1) \cdots (\alpha s + h_k)}{[\alpha s + h_1]_q \cdots [\alpha s + h_k]_q}. \end{aligned} \quad (2.6)$$

From (1.3) and (1.4), we note that

$$\begin{aligned} & q^{h_1} \int_{\mathbb{Z}_p} [x_1 + x + 1]_{q^\alpha}^n q^{x_1(h_1-1)} d\mu_q(x_1) \\ &= \int_{\mathbb{Z}_p} [x_1 + x]_{q^\alpha}^n q^{x_1(h_1-1)} d\mu_q(x_1) + [x]_{q^\alpha}^n h_1 (q-1) + n[x]_{q^\alpha}^{n-1} \frac{\alpha}{[\alpha]_q}. \end{aligned} \quad (2.7)$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$, we have

$$q^{h_1} \tilde{\beta}_{n,q}^{(1,\alpha)}(h_1 | x+1) - \tilde{\beta}_{n,q}^{(1,\alpha)}(h_1 | x) = h_1(q-1)[x]_{q^\alpha}^n + n[x]_{q^\alpha}^{n-1} \frac{\alpha}{[\alpha]_q}. \quad (2.8)$$

From (2.2) and (2.3), we have

$$\begin{aligned} & q^{\alpha x} \tilde{\beta}_{n,q}^{(k,\alpha)}(\alpha h_1 + \alpha, \alpha h_2 + \alpha, \dots, \alpha h_k + \alpha | x) \\ &= q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(\alpha h_j + \alpha - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= (q^\alpha - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^{n+1} q^{\sum_{j=1}^k x_j(\alpha h_j - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &\quad + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(\alpha h_j - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k). \end{aligned} \quad (2.9)$$

Therefore, by (2.9), we obtain the following proposition.

Proposition 2.3. For $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} & q^{\alpha x} \tilde{\beta}_{n,q}^{(k,\alpha)}(\alpha h_1 + \alpha, \alpha h_2 + \alpha, \dots, \alpha h_k + \alpha | x) \\ &= (q^\alpha - 1) \tilde{\beta}_{n+1,q}^{(k,\alpha)}(\alpha h_1, \dots, \alpha h_k | x) + \tilde{\beta}_{n,q}^{(k,\alpha)}(\alpha h_1, \dots, \alpha h_k | x). \end{aligned} \quad (2.10)$$

By (2.2), we get

$$\begin{aligned} & \tilde{\beta}_{n,q}^{(k,\alpha)}(h_1, h_2, \dots, h_k | x) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_q(x_1) \cdots d\mu_q(x_k) \end{aligned}$$

$$\begin{aligned}
 &= \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{s_1, \dots, s_k=0}^{d-1} q^{\sum_{j=1}^k h_j s_j} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{s_1 + \cdots + s_k + x}{d} + x_1 + \cdots + x_k \right]_{q^{d\alpha}}^n \\
 &\quad \times q^{d \sum_{j=1}^k x_j (h_j - 1)} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_k) \\
 &= \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{s_1, \dots, s_k=0}^{d-1} q^{\sum_{j=1}^k h_j s_j} \tilde{\beta}_{n, q^d}^{(k, \alpha)} \left(h_1, \dots, h_k \mid \frac{s_1 + \cdots + s_k + x}{d} \right),
 \end{aligned} \tag{2.11}$$

where $d \in \mathbb{N}$.

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, we have

$$\tilde{\beta}_{n, q}^{(k, \alpha)}(h_1, h_2, \dots, h_k \mid x) = \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{s_1, \dots, s_k=0}^{d-1} q^{\sum_{j=1}^k h_j s_j} \tilde{\beta}_{n, q^d}^{(k, \alpha)} \left(h_1, \dots, h_k \mid \frac{s_1 + \cdots + s_k + x}{d} \right). \tag{2.12}$$

Let $d(> 0)$ be a fixed integer. For $N \in \mathbb{N}$, we set

$$\begin{aligned}
 X &= X_d = \lim_{\substack{\leftarrow \\ N}} \frac{\mathbb{Z}}{dp^N \mathbb{Z}}, \quad X_1 = \mathbb{Z}_p, \\
 X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} (a + dp\mathbb{Z}_p), \\
 a + dp^N \mathbb{Z}_p &= \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\},
 \end{aligned} \tag{2.13}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Then we consider the generalized q -Bernoulli numbers with weight α of order k as follows:

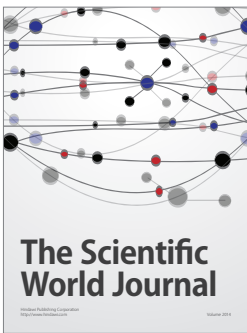
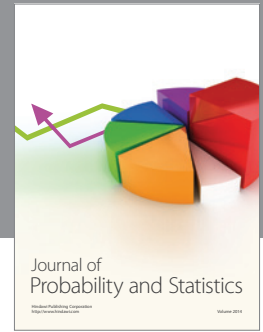
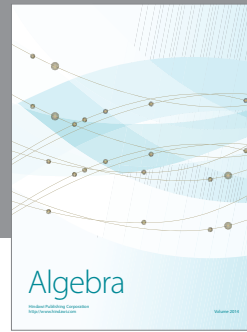
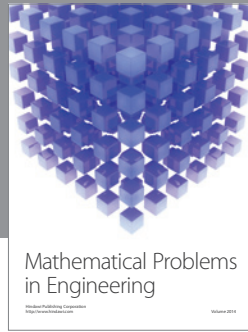
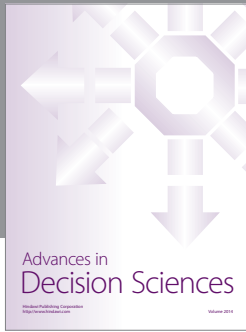
$$\tilde{\beta}_{n, \chi, q}^{(k, \alpha)}(h_1, h_2, \dots, h_k) = \int_X \cdots \int_X \prod_{i=1}^k \chi(x_i) [x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{i=1}^k x_i (h_i - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{2.14}$$

From (2.14), we have

$$\tilde{\beta}_{n, \chi, q}^{(k, \alpha)}(h_1, h_2, \dots, h_k) = \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{s_1, \dots, s_k=0}^{d-1} q^{\sum_{j=1}^k h_j s_j} \left(\prod_{j=1}^k \chi(s_j) \right) \tilde{\beta}_{n, q^d}^{(k, \alpha)} \left(h_1, \dots, h_k \mid \frac{s_1 + \cdots + s_k}{d} \right). \tag{2.15}$$

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