Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 980315, 19 pages
doi:10.1155/2011/980315

Research Article

Fuzzy-Fated Filters of R_0-Algebras

Jeong Soon Han,[1] Young Bae Jun,[2] and Hee Sik Kim[3]

[1] *Department of Applied Mathematics, Hanyang University, Ansan 426-791, Republic of Korea*
[2] *Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Republic of Korea*
[3] *Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea*

Correspondence should be addressed to Hee Sik Kim, heekim@hanyang.ac.kr

Received 5 January 2011; Revised 18 April 2011; Accepted 30 May 2011

Academic Editor: Jinde Cao

Fuzzy set theory of fated filters in R_0-algebras is considered. A characterization of a fuzzy-fated filter is established, and conditions for a fuzzy filter to be a fuzzy-fated filter are provided. The notion of an $(\in, \in \vee q)$-fuzzy-fated filter is introduced. Characterizations of an $(\in, \in \vee q)$-fuzzy-fated filter are provided. Implication-based fuzzy-fated filters are discussed.

1. Introduction

One important task of artificial intelligence is to make the computers simulate beings in dealing with certainty and uncertainty in information. Logic appears in a "sacred" (resp., a "profane") form which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold—as a tool for applications in both areas—and a technique for laying the foundations. Nonclassical logic including many-valued logic and fuzzy logic, takes the advantage of classical logic to handle information with various facets of uncertainty (see [1] for generalized theory of uncertainty), such as fuzziness randomness and Nonclassical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. The concept of R_0-algebras was first introduced by Wang in [2] by providing an algebraic proof of the completeness theorem of a formal deductive system [3]. Obviously, R_0 -algebras are different from the BL-algebras. Jun and Lianzhen [4] studied (fated) filters of R_0-algebras. Lianzhen and Kaitai [5] discussed the fuzzy set theory of filters in R_0-algebras. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [6]. Murali [7] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [8], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [9, 10] initiated the concepts of (α,β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasicoincident with" relation (q) between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroup. In particular, an $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. As a generalization of the notion of fuzzy filters in R_0-algebras, Ma et al. [11] dealt with the notion of $(\in,\in\vee q)$-fuzzy filters in R_0-algebras.

In this paper, we deal with the fuzzy set theory of fated filters in R_0-algebras. We provide conditions for a fuzzy filter to be a fuzzy-fated filter. We also introduce the notion of $(\in,\in\vee q)$-fuzzy-fated filters and investigate related properties. We establish a relation between an $(\in,\in\vee q)$-fuzzy filter and an $(\in,\in\vee q)$-fuzzy-fated filter and provide conditions for an $(\in,\in\vee q)$-fuzzy filter to be an $(\in,\in\vee q)$-fuzzy-fated filter. We deal with characterizations of an $(\in,\in\vee q)$-fuzzy-fated filter. Finally, we discuss the implication-based fuzzy-fated filters of an R_0-algebra.

2. Preliminaries

Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\to$, then $(L,\wedge,\vee,\neg,\to)$ is called an *R_0-algebra* (see [2]) if it satisfies the following axioms:

(R1) $x\to y=\neg y\to\neg x$,

(R2) $1\to x=x$,

(R3) $(y\to z)\wedge((x\to y)\to(x\to z))=y\to z$,

(R4) $x\to(y\to z)=y\to(x\to z)$,

(R5) $x\to(y\vee z)=(x\to y)\vee(x\to z)$,

(R6) $(x\to y)\vee((x\to y)\to(\neg x\vee y))=1$.

Let L be an R_0-algebra. For any $x,y\in L$, we define $x\odot y=\neg(x\to\neg y)$ and $x\oplus y=\neg x\to y$. It is proved that $\odot$ and $\oplus$ are commutative, associative, and $x\oplus y=\neg(\neg x\odot\neg y)$, and $(L,\wedge,\vee,\odot,\to,0,1)$ is a residuated lattice.

For any elements x, y, and z of an R_0-algebra L, we have the following properties (see [12]):

(a1) $x\le y$ if and only if $x\to y=1$,

(a2) $x\le y\to x$,

(a3) $\neg x=x\to 0$,

(a4) $(x\to y)\vee(y\to x)=1$,

(a5) $x\le y$ implies $y\to z\le x\to z$,

(a6) $x\le y$ implies $z\to x\le z\to y$,

(a7) $((x\to y)\to y)\to y=x\to y$,

(a8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(a9) $x \odot \neg x = 0$ and $x \oplus \neg x = 1$,

(a10) $x \odot y \leq x \wedge y$ and $x \odot (x \to y) \leq x \wedge y$,

(a11) $(x \odot y) \to z = x \to (y \to z)$,

(a12) $x \leq y \to (x \odot y)$,

(a13) $x \odot y \leq z$ if and only if $x \leq y \to z$,

(a14) $x \leq y$ implies $x \odot z \leq y \odot z$,

(a15) $x \to y \leq (y \to z) \to (x \to z)$,

(a16) $(x \to y) \odot (y \to z) \leq x \to z$.

A nonempty subset A of an R_0-algebra L is called a *filter* of L if it satisfies the following two conditions:

(b1) $1 \in A$,

(b2) $(\forall x \in A)(\forall y \in L)(x \to y \in A \Longrightarrow y \in A)$.

It can be easily verified that a nonempty subset A of an R_0-algebra L is a filter of L if and only if it satisfies the following conditions:

(b3) $(\forall x, y \in A)\ (x \odot y \in A)$,

(b4) $(\forall y \in L)\ (\forall x \in A)\ (x \leq y \Rightarrow y \in A)$.

Definition 2.1. A fuzzy subset μ of an R_0-algebra L is called a *fuzzy filter* of L if it satisfies:

(c1) $(\forall x, y \in L)(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$,

(c2) μ is order preserving, that is, $(\forall x, y \in L)(x \leq y \Rightarrow \mu(x) \leq \mu(y))$.

Denote by $F(L)$ the set of all filters of L, and by $\mathcal{F}_F(L)$ the set of all fuzzy filters of L.

Theorem 2.2. *A fuzzy subset μ of an R_0-algebra L is a fuzzy filter of L if and only if it satisfies the following:*

(c3) $(\forall x \in L)\ (\mu(1) \geq \mu(x))$,

(c4) $(\forall x, y \in L)\ (\mu(y) \geq \min\{\mu(x \to y), \mu(x)\})$.

For any fuzzy subset μ of L and $t \in (0, 1]$, the set $U(\mu; t) = \{x \in L \mid \mu(x) \geq t\}$ is called a *level subset* of L. It is well known that a fuzzy subset μ of L is a fuzzy filter of L if and only if the nonempty level subset $U(\mu; t),\ t \in (0, 1]$, of μ is a filter of L.

A fuzzy subset μ of a set L of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

3. Fuzzy-Fated Filters

In what follows, L is an R_0-algebra unless otherwise specified. In [4], the notion of a fated filter of L is introduced as follows.

A nonempty subset A of L is called a *fated filter* of L (see [4]) if it satisfies (b1) and

$$(\forall x, y \in L)(\forall a \in A) \quad (a \longrightarrow ((x \longrightarrow y) \longrightarrow x) \in A \Longrightarrow x \in A). \tag{3.1}$$

Denote by $FF(L)$ the set of all fated filters of L. Note that $FF(L)$ is a complete lattice under the set inclusion with the largest element L and the least element $\{1\}$. Now, we consider the fuzzy form of a fated filter of L.

Lemma 3.1 (see [4]). *A filter F of L is fated if and only if the following assertion is valid:*

$$(\forall x, y, z \in L) \quad (x \longrightarrow (y \longrightarrow z) \in F, x \longrightarrow y \in F \Longrightarrow x \longrightarrow z \in F). \tag{3.2}$$

Lemma 3.2 (see [4]). *A filter F of L is fated if and only if the following assertion is valid:*

$$(\forall x, y \in L) \quad ((x \longrightarrow y) \longrightarrow x \in F \Longrightarrow x \in F). \tag{3.3}$$

Definition 3.3. A fuzzy subset μ of L is called a *fuzzy-fated filter* of L if it satisfies the following assertion:

$$(\forall t \in [0,1]) \quad (U(\mu; t) \in FF(L) \cup \{\emptyset\}). \tag{3.4}$$

Denote by $\mathcal{F}_{FF}(L)$ the set of all fuzzy-fated filters of L.

Example 3.4. Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables which are given in Table 1, then $(L, \wedge, \vee, \neg, \rightarrow, 0, 1)$ is an R_0-algebra (see [5]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Define a fuzzy subset μ of L by

$$\mu : L \longrightarrow [0,1], \qquad x \longmapsto \begin{cases} 0.7 & \text{if } x \in \{c, d, 1\}, \\ 0.2 & \text{otherwise}. \end{cases} \tag{3.5}$$

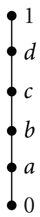
Then

$$U(\mu; t) = \begin{cases} \emptyset & \text{if } 0.7 < t \leq 1, \\ \{c, d, 1\} & \text{if } 0.2 < t \leq 0.7, \\ L & \text{if } 0 \leq t \leq 0.2, \end{cases} \tag{3.6}$$

which is a fated filter of L. Therefore, μ is a fuzzy-fated filter of L.

We provide a characterization of a fuzzy-fated filter.

Table 1: Hasse diagram and Cayley tables.



x	$\neg x$
0	1
a	d
b	c
c	b
d	a
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Theorem 3.5. *For a fuzzy subset μ of L, $\mu \in \mathcal{F}_{FF}(L)$ if and only if it satisfies the following conditions:*

(1) $(\forall x \in L)\ (\mu(1) \geq \mu(x))$,

(2) $(\forall x, a, y \in L)\ (\mu(x) \geq \min\{\mu(a \to ((x \to y) \to x)), \mu(a)\})$.

Proof. Suppose that μ is a fuzzy-fated filter of L. For any $x \in L$, let $\mu(x) = t$, then $x \in U(\mu; t)$, that is, $U(\mu; t) \neq \emptyset$, and so $U(\mu; t)$ is a fated filter of L. Thus, $1 \in U(\mu; t)$, and hence $\mu(1) \geq t = \mu(x)$ for all $x \in L$. For any $x, a, y \in L$, let

$$t_a := \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a)\}. \tag{3.7}$$

Then $a \in U(\mu; t_a)$ and $a \to ((x \to y) \to x) \in U(\mu; t_a)$. Since $U(\mu; t_a)$ is a fated filter of L, it follows from (3.1) that $x \in U(\mu; t_a)$, so that

$$\mu(x) \geq t_a = \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a)\}, \tag{3.8}$$

for all $x, a, y \in L$.

Conversely, let μ satisfy two conditions (1) and (2), and let $t \in [0, 1]$ be such that $U(\mu; t) \neq \emptyset$, then there exists $x_0 \in U(\mu; t)$ such that $\mu(x_0) \geq t$. Using (1), we have $\mu(1) \geq \mu(x_0) \geq t$, and so $1 \in U(\mu; t)$. Let $x, a, y \in L$ be such that $a \in U(\mu; t)$ and $a \to ((x \to y) \to x) \in U(\mu; t)$, then $\mu(a) \geq t$ and $\mu(a \to ((x \to y) \to x)) \geq t$. It follows from (2) that

$$\mu(x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a)\} \geq t, \tag{3.9}$$

so that $x \in U(\mu; t)$. Hence, $U(\mu; t)$ is a fated filter of L, and therefore μ is a fuzzy-fated filter of L. □

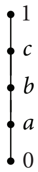
Theorem 3.6. *For any fuzzy subset μ of L, one has $\mathcal{F}_{FF}(L) \subseteq \mathcal{F}_F(L)$.*

Proof. Let μ be a fuzzy-fated filter of L. Replacing a and x by x and y, respectively, in Theorem 3.5(2) and using (R2), we have

$$\begin{aligned} \mu(y) &\geq \min\{\mu(x \longrightarrow ((y \longrightarrow y) \longrightarrow y)), \mu(x)\} \\ &= \min\{\mu(x \longrightarrow (1 \longrightarrow y)), \mu(x)\} \\ &= \min\{\mu(x \longrightarrow y), \mu(x)\}, \end{aligned} \tag{3.10}$$

for all $x, y \in L$. Using Theorem 2.2, μ is a fuzzy filter of L. □

Table 2: Hasse diagram and Cayley tables.



x	$\neg x$
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

The following example shows that the converse of Theorem 3.6 may not be true.

Example 3.7. Let $L = \{0, a, b, c, 1\}$ be a set with Hasse diagram and Cayley tables which are given in Table 2, then $(L, \wedge, \vee, \neg, \to, 0, 1)$ is an R_0-algebra (see [5]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Define a fuzzy subset μ of L by

$$\mu : L \longrightarrow [0,1], \qquad x \longmapsto \begin{cases} 0.6 & \text{if } x \in \{c, 1\}, \\ 0.2 & \text{otherwise.} \end{cases} \tag{3.11}$$

Then μ is a fuzzy filter of L, but it is not a fuzzy-fated filter of L since

$$\mu(b) = 0.2 \ngeq 0.6 = \min\{\mu(c \longrightarrow ((b \longrightarrow a) \longrightarrow b)), \mu(c)\}. \tag{3.12}$$

Theorem 3.8. *For any fuzzy filter μ of L, the following assertions are equivalent:*

(1) *μ is a fuzzy-fated filter of L,*

(2) *μ satisfies the following inequality:*

$$(\forall x, y \in L) \quad (\mu(x) \geq \mu((x \longrightarrow y) \longrightarrow x)). \tag{3.13}$$

Proof. Assume that μ is a fuzzy- fated filter of L. Putting $a = 1$ in Theorem 3.5(2) and using (R2) and Theorem 3.5(1), we have

$$\begin{aligned} \mu(x) &\geq \min\{\mu(1 \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(1)\} \\ &= \min\{\mu((x \longrightarrow y) \longrightarrow x), \mu(1)\} \\ &= \mu((x \longrightarrow y) \longrightarrow x), \end{aligned} \tag{3.14}$$

for all $x, y \in L$.

Conversely, suppose that μ satisfies (3.13), it follows from (c4) that

$$\mu(x) \geq \mu((x \longrightarrow y) \longrightarrow x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a)\}, \tag{3.15}$$

for all $x, a, y \in L$, so from Theorem 3.5 that μ is a fuzzy-fated filter of L. □

Theorem 3.9. *Let μ be a fuzzy filter of L, then $\mu \in \mathcal{F}_{FF}(L)$ if and only if it satisfies*

$$\mu(x \longrightarrow z) \geq \min\{\mu(x \longrightarrow (y \longrightarrow z)), \mu(x \longrightarrow y)\}, \tag{3.16}$$

for all $x, y, z \in L$.

Proof. Assume that $\mu \in \mathcal{F}_{FF}(L)$. If $U(\mu; t) \neq \emptyset$ for all $t \in [0,1]$, then $U(\mu; t) \in FF(L)$. Suppose that

$$\mu(x \longrightarrow z) < \min\{\mu(x \longrightarrow (y \longrightarrow z)), \mu(x \longrightarrow y)\}, \tag{3.17}$$

for some $x, y, z \in L$, then there exists $t \in (0,1]$ such that

$$\mu(x \longrightarrow z) < t \leq \min\{\mu(x \longrightarrow (y \longrightarrow z)), \mu(x \longrightarrow y)\}. \tag{3.18}$$

It follows that $x \to (y \to z) \in U(\mu; t)$ and $x \to y \in U(\mu; t)$, so from Lemma 3.1 that $x \to z \in U(\mu; t)$, that is, $\mu(x \to z) \geq t$. This is a contradiction, and so $\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y)\}$ for all $x, y, z \in L$.

Conversely, let μ be a fuzzy filter of L that satisfies (3.16). Let $t \in [0,1]$ be such that $U(\mu; t) \neq \emptyset$, then $U(\mu; t) \in F(L)$ by Theorem 3.5 in [5]. Assume that $x \to (y \to z) \in U(\mu; t)$ and $x \to y \in U(\mu; t)$ for all $x, y, z \in L$, then $\mu(x \to (y \to z)) \geq t$ and $\mu(x \to y) \geq t$. Using (3.16), we have

$$\mu(x \longrightarrow z) \geq \min\{\mu(x \longrightarrow (y \longrightarrow z)), \mu(x \longrightarrow y)\} \geq t, \tag{3.19}$$

and so $x \to z \in U(\mu; t)$. Therefore, $U(\mu; t) \in FF(L)$, and thus μ is a fuzzy-fated filter of L. □

Remark 3.10. Based on Theorem 3.9 and [5, Definition 4.1], we know that the notion of a fuzzy-fated filter is equivalent to the notion of a fuzzy implicative filter.

4. Fuzzy-Fated Filters Based on Fuzzy Points

For a fuzzy point (x, t) and a fuzzy subset μ of L, Pu and Liu [8] introduced the symbol $(x, t)\alpha\mu$, where $\alpha \in \{\in, q, \in \vee q\}$. We say that

(i) (x, t) *belong to* μ, denoted by $(x, t) \in \mu$ if $\mu(x) \geq t$,

(ii) (x, t) is *quasicoincident with* μ, denoted by $(x, t)q\mu$, if $\mu(x) + t > 1$,

(iii) $(x, t) \in \vee q\mu$ if $(x, t) \in \mu$ or $(x, t)q\mu$,

(iv) $(x, t)\overline{\alpha}\mu$ if $(x, t)\alpha\mu$ does not hold for $\alpha \in \{\in, q, \in \vee q\}$.

Definition 4.1 (see [11]). A fuzzy subset μ of L is said to be an $(\in, \in \vee q)$-*fuzzy filter* of L if it satisfies

(1) $(x, t) \in \mu$ and $(y, r) \in \mu \Rightarrow (x \odot y, \min\{t, r\}) \in \vee q\mu$,

(2) $(x, t) \in \mu$ and $x \leq y \Rightarrow (y, t) \in \vee q\mu$,

for all $x, y \in L$ and $t, r \in (0,1]$.

Theorem 4.2 (see [11]). *A fuzzy subset μ of L is an $(\in, \in\vee q)$-fuzzy filter of L if and only if the following conditions are valid:*

(1) $(\forall x \in L)\ (\mu(1) \geq \min\{\mu(x), 0.5\})$,

(2) $(\forall x, y \in L)\ (\mu(y) \geq \min\{\mu(x), \mu(x \to y), 0.5\})$.

Definition 4.3. A fuzzy subset μ of L is said to be an $(\in, \in\vee q)$*-fuzzy-fated filter* of L if it satisfies

(1) $(x, t) \in \mu \Rightarrow (1, t) \in \vee\ q\ \mu$,

(2) $(a \to ((x \to y) \to x), t) \in \mu$ and $(a, r) \in \mu \Rightarrow (x, \min\{t, r\}) \in \vee q\mu$,

for all $x, a, y \in L$ and $t, r \in (0, 1]$.

If a fuzzy subset μ of L satisfies (c3) and Definition 4.3(2), then we say that μ is a *strong* $(\in, \in\vee q)$*-fuzzy-fated filter* of L.

Example 4.4. Consider an R_0-algebra $L = \{0, a, b, c, d, 1\}$ which appeared in Example 3.4. Define a fuzzy subset μ of L by

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.3 & 0.3 & 0.3 & 0.7 & 0.6 & 0.8 \end{pmatrix}. \tag{4.1}$$

It is routine to verify that μ is a strong $(\in, \in\vee q)$-fuzzy-fated filter of L. A fuzzy subset ν of L defined by

$$\nu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.4 & 0.4 & 0.4 & 0.8 & 0.8 & 0.7 \end{pmatrix} \tag{4.2}$$

is an $(\in, \in\vee q)$-fuzzy-fated filter of L, but it is not a strong $(\in, \in\vee q)$-fuzzy-fated filter of L.

Obviously, every strong $(\in, \in\vee q)$-fuzzy-fated filter is an $(\in, \in\vee q)$-fuzzy-fated filter, but not converse as seen in Example 4.4.

We provided characterizations of an $(\in, \in\vee q)$-fuzzy-fated filter.

Theorem 4.5. *A fuzzy subset μ of L is an $(\in, \in\vee q)$-fuzzy-fated filter of L if and only if it satisfies the following inequalities:*

(1) $(\forall x \in L)\ (\mu(1) \geq \min\{\mu(x), 0.5\})$,

(2) $(\forall x, a, y \in L)\ (\mu(x) \geq \min\{\mu(a \to ((x \to y) \to x)), \mu(a), 0.5\})$.

Proof. Let μ be an $(\in, \in\vee q)$-fuzzy-fated filter of L. Assume that there exists $a \in L$ such that $\mu(1) < \min\{\mu(a), 0.5\}$, then $\mu(1) < t \leq \min\{\mu(a), 0.5\}$ for some $t \in (0, 0.5]$, and so $(a, t) \in \mu$. It follows from Definition 4.3(1) that $(1, t) \in \vee q\mu$, that is, $(1, t) \in \mu$ or $(1, t)q\mu$, so that $\mu(1) \geq t$ or $\mu(1) + t > 1$. This is a contradiction. Hence, $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$. Suppose that there exist $a, b, c \in L$ such that

$$\mu(b) < \min\{\mu(a \longrightarrow ((b \longrightarrow c) \longrightarrow b)), \mu(a), 0.5\}, \tag{4.3}$$

then $\mu(b) < t_b \leq \min\{\mu(a \to ((b \to c) \to b)), \mu(a), 0.5\}$ for some $t_b \in (0, 0.5]$. Thus $(a \to ((b \to c) \to b), t_b) \in \mu$ and $(a, t_b) \in \mu$. Using Definition 4.3(2), we have $(b, t_b) = (b, \min\{t_b, t_b\}) \in \vee q\mu$, which implies that $\mu(b) \geq t_b$ or $\mu(b) + t_b > 1$. This is a contradiction, and therefore

$$\mu(x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}, \tag{4.4}$$

for all $x, a, y \in L$. Conversely, let μ be a fuzzy subset of L that satisfies two conditions (1) and (2). Let $x \in L$ and $t \in (0,1]$ be such that $(x, t) \in \mu$, then $\mu(x) \geq t$, which implies from (1) that $\mu(1) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\}$. If $t \leq 0.5$, then $\mu(1) \geq t$, that is, $(1, t) \in \mu$. If $t > 0.5$, then $\mu(1) \geq 0.5$ and so $\mu(1) + t > 0.5 + 0.5 = 1$, that is, $(1, t) q\mu$. Hence, $(1, t) \in \vee q\mu$. Let $x, a, y \in L$ and $t, r \in (0, 1]$ be such that $(a \to ((x \to y) \to x), t) \in \mu$ and $(a, r) \in \mu$, then $\mu(a \to ((x \to y) \to x)) \geq t$ and $\mu(a) \geq r$. It follows from (2) that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\} \\ &\geq \min\{t, r, 0.5\}. \end{aligned} \tag{4.5}$$

If $\min\{t, r\} \leq 0.5$, then $\mu(x) \geq \min\{t, r\}$, which shows that $(x, \min\{t, r\}) \in \mu$. If $\min\{t, r\} > 0.5$, then $\mu(x) \geq 0.5$, and thus $\mu(x) + \min\{t, r\} > 1$, that is, $(x, \min\{t, r\}) q\mu$. Hence, $(x, \min\{t, r\}) \in \vee q\mu$. Consequently, μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L. □

Corollary 4.6. *Every strong $(\in, \in \vee q)$-fuzzy-fated filter μ of L satisfies the following inequalities:*

(1) $(\forall x \in L)\ (\mu(1) \geq \min\{\mu(x), 0.5\})$,

(2) $(\forall x, a, y \in L)\ (\mu(x) \geq \min\{\mu(a \to ((x \to y) \to x)), \mu(a), 0.5\})$.

Theorem 4.7. *A fuzzy subset μ of L is an $(\in, \in \vee q)$-fuzzy-fated filter of L if and only if it satisfies the following assertion:*

$$(\forall t \in (0, 0.5])\quad (U(\mu; t) \in FF(L) \cup \{\emptyset\}). \tag{4.6}$$

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L. Let $t \in (0, 0.5]$ be such that $U(\mu; t) \neq \emptyset$, then there exists $x \in U(\mu; t)$, and so $\mu(x) \geq t$. Using Theorem 4.5(1), we get

$$\mu(1) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t, \tag{4.7}$$

that is, $1 \in U(\mu; t)$. Assume that $a \to ((x \to y) \to x) \in U(\mu; t)$ for all $x, y \in L$ and $a \in U(\mu; t)$, then $\mu(a \to ((x \to y) \to x)) \geq t$ and $\mu(a) \geq t$. It follows from Theorem 4.5 (2) that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\} \\ &\geq \min\{t, 0.5\} = t, \end{aligned} \tag{4.8}$$

so that $x \in U(\mu; t)$. Therefore, $U(\mu; t)$ is a fated filter of L.

Conversely, let μ be a fuzzy subset of L satisfying the assertion (4.6). Assume that $\mu(1) < \min\{\mu(a), 0.5\}$ for some $a \in L$. Putting $t_a := \min\{\mu(a), 0.5\}$, we have $a \in U(\mu; t_a)$ and so $U(\mu; t_a) \neq \emptyset$. Hence, $U(\mu; t_a)$ is a fated filter of L by (4.6), which implies that $1 \in U(\mu; t_a)$. Thus, $\mu(1) \geq t_a$, which is a contradiction. Therefore, $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$. Suppose that

$$\mu(x) < \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}, \tag{4.9}$$

for some $x, a, y \in L$. Taking $t_x := \min\{\mu(a \to ((x \to y) \to x)), \mu(a), 0.5\}$, we get $a \to ((x \to y) \to x) \in U(\mu; t_x)$ and $a \in U(\mu; t_x)$. It follows from (3.1) that $x \in U(\mu; t_x)$, that is, $\mu(x) \geq t_x$. This is a contradiction. Hence,

$$\mu(x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}, \tag{4.10}$$

for all $x, a, y \in L$. Using Theorem 4.5, we conclude that μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L. □

Proposition 4.8. *Every $(\in, \in \vee q)$-fuzzy-fated filter μ of L satisfies the following inequalities:*

(1) $\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y), 0.5\}$,

(2) $\mu(x) \geq \min\{\mu((x \to y) \to x), 0.5\}$,

for all $x, y, z \in L$.

Proof. (1) Suppose that there exist $a, b, c \in L$ such that

$$\mu(a \longrightarrow c) < \min\{\mu(a \longrightarrow (b \longrightarrow c)), \mu(a \longrightarrow b), 0.5\}. \tag{4.11}$$

Taking $t := \min\{\mu(a \to (b \to c)), \mu(a \to b), 0.5\}$ implies that $t \in (0, 0.5]$,$a \to (b \to c) \in U(\mu; t)$ and $a \to b \in U(\mu; t)$. Since $U(\mu; t) \in FF(L)$ by Theorem 4.7, it follows from Lemma 3.1 that $a \to c \in U(\mu; t)$, that is, $\mu(a \to c) \geq t$. This is a contradiction, and therefore μ satisfies (1).

(2) If μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L, then $U(\mu; t) \in FF(L) \cup \{\emptyset\}$ for all $t \in (0, 0.5]$ by Theorem 4.7. Hence, $U(\mu; t) \in F(L) \cup \{\emptyset\}$ for all $t \in (0, 0.5]$. Suppose that

$$\mu(x) < t \leq \min\{\mu((x \longrightarrow y) \longrightarrow x), 0.5\}, \tag{4.12}$$

for some $x, y \in L$ and $t \in (0, 0.5]$ then $(x \to y) \to x \in U(\mu; t)$, which implies from Lemma 3.2 that $x \in U(\mu; t)$, that is, $\mu(x) \geq t$. This is a contradiction. Hence, $\mu(x) \geq \min\{\mu((x \to y) \to x), 0.5\}$ for all $x, y \in L$. □

Theorem 4.9. *If F is a fated filter of L, then a fuzzy subset μ of L defined by*

$$\mu : L \longrightarrow [0, 1], \qquad x \longmapsto \begin{cases} t_1 & \text{if } x \in F, \\ t_2 & \text{otherwise,} \end{cases} \tag{4.13}$$

where $t_1 \in [0.5, 1]$ and $t_2 \in (0, 0.5)$ is an $(\in, \in \vee q)$-fuzzy-fated filter of L.

Proof. Note that

$$U(\mu;r)=\begin{cases}F & \text{if } r\in(t_2,0.5],\\ L & \text{if } r\in(0,t_2],\end{cases}\tag{4.14}$$

which is a fated filter of L. It follows from Theorem 4.7 that μ is an $(\in,\in\vee q)$-fuzzy-fated filter of L. □

Theorem 4.10. *Let μ be an $(\in,\in\vee q)$-fuzzy-fated filter of L. If $\mu(1)<0.5$, then μ is a fuzzy-fated filter of L.*

Proof. It is straightforward.

For any fuzzy subset μ of L and any $t\in(0,1]$, we consider two subsets:

$$Q(\mu;t):=\{x\in L\mid (x,t)q\mu\},\quad [\mu]_t:=\{x\in L\mid (x,t)\in\vee q\mu\}.\tag{4.15}$$

It is clear that $[\mu]_t=U(\mu;t)\cup Q(\mu;t)$. □

Theorem 4.11. *If μ is an $(\in,\in\vee q)$-fuzzy-fated filter of L, then*

$$(\forall t\in(0.5,1])\quad (Q(\mu;t)\in FF(L)\cup\{\emptyset\}).\tag{4.16}$$

Proof. Assume that $Q(\mu;t)\neq\emptyset$ for all $t\in(0.5,1]$, then there exists $x\in Q(\mu;t)$, and so $\mu(x)+t>1$. Using Theorem 4.5(1), we have

$$\begin{aligned}\mu(1)&\geq\min\{\mu(x),0.5\}\\&=\begin{cases}0.5 & \text{if } \mu(x)\geq 0.5,\\ \mu(x) & \text{if } \mu(x)<0.5\end{cases}\\&>1-t,\end{aligned}\tag{4.17}$$

which implies that $1\in Q(\mu;t)$. Assume that $a\to((x\to y)\to x)\in Q(\mu;t)$ and $a\in Q(\mu;t)$ for all $x,a,y\in L$, then $(a\to((x\to y)\to x),t)q\mu$ and $(a,t)q\mu$, that is, $\mu(a\to((x\to y)\to x))>1-t$ and $\mu(a)>1-t$. Using Theorem 4.5(2), we get

$$\mu(x)\geq\min\{\mu(a\longrightarrow((x\longrightarrow y)\longrightarrow x)),\mu(a),0.5\}.\tag{4.18}$$

Thus, if $\min\{\mu(a\to((x\to y)\to x)),\mu(a)\}<0.5$, then

$$\mu(x)\geq\min\{\mu(a\longrightarrow((x\longrightarrow y)\longrightarrow x)),\mu(a)\}>1-t.\tag{4.19}$$

If $\min\{\mu(a\to((x\to y)\to x)),\mu(a)\}\geq 0.5$, then $\mu(x)\geq 0.5>1-t$. It follows that $(x,t)q\mu$ so that $x\in Q(\mu;t)$. Therefore, $Q(\mu;t)$ is a fated filter of L. □

Corollary 4.12. *If μ is a strong $(\in, \in \vee q)$-fuzzy-fated filter of L, then*

$$(\forall t \in (0.5, 1]) \quad (Q(\mu; t) \in FF(L) \cup \{\emptyset\}). \tag{4.20}$$

The converse of Corollary 4.12 is not true as shown by the following example.

Example 4.13. Consider the $(\in, \in \vee q)$-fuzzy-fated filter ν of L which is given in Example 4.4, then

$$Q(\nu; t) = \begin{cases} \{c, d, 1\} & \text{if } t \in (0.5, 0.6], \\ L & \text{if } t \in (0.6, 1] \end{cases} \tag{4.21}$$

is a fated filter of L. But ν is not a strong $(\in, \in \vee q)$-fuzzy-fated filter of L.

Theorem 4.14. *For a fuzzy subset μ of L, the following assertions are equivalent:*

(1) *μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L,*

(2) $(\forall t \in (0, 1])$ $([\mu]_t \in FF(L) \cup \{\emptyset\})$.

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L, and let $t \in (0, 1]$ be such that $[\mu]_t \neq \emptyset$, then there exists $x \in [\mu]_t = U(\mu; t) \cup Q(\mu; t)$, and so $x \in U(\mu; t)$ or $x \in Q(\mu; t)$. If $x \in U(\mu; t)$, then $\mu(x) \geq t$. It follows from Theorem 4.5(1) that

$$\begin{aligned} \mu(1) &\geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} \\ &= \begin{cases} t & \text{if } t \leq 0.5, \\ 0.5 > 1 - t & \text{if } t > 0.5, \end{cases} \end{aligned} \tag{4.22}$$

so that $1 \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$. If $x \in Q(\mu; t)$, then $(x, t) q \mu$, that is, $\mu(x) + t > 1$. Thus,

$$\begin{aligned} \mu(1) &\geq \min\{\mu(x), 0.5\} \geq \min\{1 - t, 0.5\} \\ &= \begin{cases} 1 - t & \text{if } t > 0.5, \\ 0.5 \geq t & \text{if } t \leq 0.5, \end{cases} \end{aligned} \tag{4.23}$$

and so $1 \in Q(\mu; t) \cup U(\mu; t) = [\mu]_t$. Let $x, a, y \in L$ be such that $a \in [\mu]_t$ and $a \to ((x \to y) \to x) \in [\mu]_t$, then $\mu(a) \geq t$ or $\mu(a) + t > 1$, and $\mu(a \to ((x \to y) \to x)) \geq t$ or $\mu(a \to ((x \to y) \to x)) + t > 1$. We can consider four cases:

$$\mu(a) \geq t \qquad \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)) \geq t, \tag{4.24}$$

$$\mu(a) \geq t \qquad \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)) + t > 1, \tag{4.25}$$

$$\mu(a) + t > 1 \qquad \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)) \geq t, \tag{4.26}$$

$$\mu(a) + t > 1 \qquad \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)) + t > 1. \tag{4.27}$$

For the first case, Theorem 4.5(2) implies that

$$\mu(x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}$$
$$\geq \min\{t, 0.5\} = \begin{cases} 0.5 & \text{if } t > 0.5, \\ t & \text{if } t \leq 0.5, \end{cases} \tag{4.28}$$

so that $x \in U(\mu; t)$ or $\mu(x) + t > 0.5 + 0.5 = 1$, that is, $x \in Q(\mu; t)$. Hence, $x \in [\mu]_t$. Case (4.25) implies that

$$\mu(x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}$$
$$\geq \min\{1-t, t, 0.5\} = \begin{cases} 1-t & \text{if } t > 0.5, \\ t & \text{if } t \leq 0.5. \end{cases} \tag{4.29}$$

Thus, $x \in Q(\mu; t) \cup U(\mu; t) = [\mu]_t$. Similarly, $x \in [\mu]_t$ for the case (4.26). The final case implies that

$$\mu(x) \geq \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}$$
$$\geq \min\{1-t, 0.5\} = \begin{cases} 1-t & \text{if } t > 0.5, \\ 0.5 & \text{if } t \leq 0.5, \end{cases} \tag{4.30}$$

so that $x \in Q(\mu; t) \cup U(\mu; t) = [\mu]_t$. Consequently, $[\mu]_t$ is a fuzzy-fated filter of L.

Conversely, let μ be a fuzzy subset of L such that $[\mu]_t$ is a fated filter of L whenever it is nonempty for all $t \in (0,1]$. If there exists $a \in L$ such that $\mu(1) < \min\{\mu(a), 0.5\}$, then $\mu(1) < t_a \leq \min\{\mu(a), 0.5\}$ for some $t_a \in (0, 0.5]$. It follows that $a \in U(\mu; t_a)$ but $1 \notin U(\mu; t_a)$. Also, $\mu(1) + t_a < 2t_a \leq 1$ and so $1 \notin Q(\mu; t_a)$. Hence, $1 \notin U(\mu; t_a) \cup Q(\mu; t_a) = [\mu]_{t_a}$, which is a contradiction. Therefore, $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$. Suppose that

$$\mu(x) < \min\{\mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), \mu(a), 0.5\}, \tag{4.31}$$

for some $x, a, y \in L$. Taking $t := \min\{\mu(a \to ((x \to y) \to x)), \mu(a), 0.5\}$ implies that $t \in (0, 0.5]$, $a \in U(\mu; t) \subseteq [\mu]_t$, and $a \to ((x \to y) \to x) \in U(\mu; t) \subseteq [\mu]_t$. Since $[\mu]_t \in FF(L)$, it follows that $x \in [\mu]_t = U(\mu; t) \cup Q(\mu; t)$. But (4.31) induces $x \notin U(\mu; t)$ and $\mu(x) + t < 2t \leq 1$, that is, $x \notin Q(\mu; t)$. This is a contradiction, and thus $\mu(x) \geq \min\{\mu(a \to ((x \to y) \to x)), \mu(a), 0.5\}$ for all $x, a, y \in L$. Using Theorem 4.5, we conclude that μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L. □

5. Implication-Based Fuzzy-Fated Filters

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\to$ in fuzzy logic, are also defined by using truth tables, and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \mu] = \mu(x), \tag{5.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}, \tag{5.2}$$

$$[\Phi \longrightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{5.3}$$

$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{5.4}$$

$$\vDash \Phi \quad \text{if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{5.5}$$

The truth valuation rules given in (5.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a, b) = \begin{cases} 1 & \text{if } a \le b, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

(b) Gödel implication operator (I_G):

$$I_G(a, b) = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{otherwise.} \end{cases} \tag{5.7}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{cG}(a, b) = \begin{cases} 1 & \text{if } a \le b, \\ 1 - a & \text{otherwise.} \end{cases} \tag{5.8}$$

Ying [13] introduced the concept of fuzzifying topology. We can expand his/her idea to R_0-algebras, and we define a fuzzifying fated filter as follows.

Definition 5.1. A fuzzy subset μ of L is called a *fuzzifying fated filter* of L if it satisfies the following conditions:

(1) for all $x \in L$, we have

$$\vDash [x \in \mu] \longrightarrow [1 \in \mu], \tag{5.9}$$

(2) for all $x, a, y \in L$, we get

$$\vDash [a \in \mu] \wedge [a \longrightarrow ((x \rightarrow y) \longrightarrow x) \in \mu] \longrightarrow [x \in \mu]. \tag{5.10}$$

Obviously, conditions (5.9) and (5.10) are equivalent to Theorem 3.5(1) and Theorem 3.5(2), respectively. Therefore, a fuzzifying fated filter is an ordinary fuzzy-fated filter.

In [14], the concept of t-tautology is introduced, that is,

$$\vDash_t \Phi \quad \text{if and only if } [\Phi] \geq t \text{ for all valuations.} \tag{5.11}$$

Definition 5.2. Let μ be a fuzzy subset of L and $t \in (0, 1]$, then μ is called a *t-implication-based fuzzy-fated filter* of L if it satisfies the following conditions:

(1) for all $x \in L$, we have

$$\vDash_t [x \in \mu] \longrightarrow [1 \in \mu], \tag{5.12}$$

(2) for all $x, a, y \in L$, we get

$$\vDash_t [a \in \mu] \wedge [a \rightarrow ((x \longrightarrow y) \longrightarrow x) \in \mu] \longrightarrow [x \in \mu]. \tag{5.13}$$

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy-fated filter of L if and only if it satisfies

(1) $(\forall x \in L)\ (I(\mu(x), \mu(1)) \geq t)$,
(2) $(\forall x, y \in L)\ (I(\min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x))\}, \mu(x)) \geq t)$.

Theorem 5.3. *For any fuzzy subset μ of L, one has the following:*

(1) *if $I = I_{GR}$, then μ is a 0.5-implication-based fuzzy-fated filter of L if and only if μ is a fuzzy-fated filter of L,*
(2) *if $I = I_G$, then μ is a 0.5-implication-based fuzzy-fated filter of L if and only if μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L,*
(3) *if $I = I_{cG}$, then μ is a 0.5-implication-based fuzzy-fated filter of L if and only if μ satisfies the following conditions:*

(3.1) $\max\{\mu(1), 0.5\} \geq \min\{\mu(x), 1\}$,
(3.2) $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), 1\}$

for all $x, a, y \in L$.

Proof. (1) It is Straightforward.

(2) Assume that μ is a 0.5-implication-based fuzzy-fated filter of L, then

(i) $(\forall x \in L)\ (I_G(\mu(x), \mu(1)) \geq 0.5)$,
(ii) $(\forall x, y \in L)\ (I_G(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) \geq 0.5)$.

From (i), we have $\mu(1) \geq \mu(x)$ or $\mu(x) \geq \mu(1) \geq 0.5$, and so $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$. The second case implies that

$$\mu(x) \geq \min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\} \tag{5.14}$$

or $\min\{\mu(a), \mu(a \to ((x \to y) \to x))\} > \mu(x) \geq 0.5$. It follows that

$$\mu(x) \geq \min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), 0.5\}, \tag{5.15}$$

for all $x, a, y \in L$. Using Theorem 4.5, we know that μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L.

Conversely, suppose that μ is an $(\in, \in \vee q)$-fuzzy-fated filter of L. From Theorem 4.5(1), if $\min\{\mu(x), 0.5\} = \mu(x)$, then $I_G(\mu(x), \mu(1)) = 1 \geq 0.5$. Otherwise, $I_G(\mu(x), \mu(1)) \geq 0.5$. From Theorem 4.5(2), if

$$\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), 0.5\} = \min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \tag{5.16}$$

then $\mu(x) \geq \min\{\mu(a), \mu(a \to ((x \to y) \to x))\}$, and so

$$I_G(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) = 1 \geq 0.5. \tag{5.17}$$

If $\min\{\mu(a), \mu(a \to ((x \to y) \to x)), 0.5\} = 0.5$, then $\mu(x) \geq 0.5$, and thus

$$I_G(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) \geq 0.5. \tag{5.18}$$

Consequently, μ is a 0.5-implication-based fuzzy-fated filter of L.

(3) Suppose that μ satisfies (3.1) and (3.2). In (3.1), if $\mu(x) = 1$, then $\max\{\mu(1), 0.5\} = 1$, and hence $I_{cG}(\mu(x), \mu(1)) = 1 \geq 0.5$. If $\mu(x) < 1$, then

$$\max\{\mu(1), 0.5\} \geq \mu(x). \tag{5.19}$$

If $\max\{\mu(1), 0.5\} = \mu(1)$ in (5.19), then $\mu(1) \geq \mu(x)$. Hence,

$$I_{cG}(\mu(x), \mu(1)) = 1 \geq 0.5. \tag{5.20}$$

If $\max\{\mu(1), 0.5\} = 0.5$ in (5.19), then $\mu(x) \leq 0.5$ which implies that

$$I_{cG}(\mu(x), \mu(1)) = \begin{cases} 1 \geq 0.5 & \text{if } \mu(1) \geq \mu(x), \\ 1 - \mu(x) \geq 0.5 & \text{otherwise.} \end{cases} \tag{5.21}$$

In (3.2), if $\min\{\mu(a), \mu(a \to ((x \to y) \to x)), 1\} = 1$, then

$$\max\{\mu(x), 0.5\} = 1, \tag{5.22}$$

and so $\mu(x) = 1 \geq \min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x))\}$. Therefore,

$$I_{\mathrm{cG}}(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) = 1 \geq 0.5. \tag{5.23}$$

If

$$\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x)), 1\} = \min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \tag{5.24}$$

then

$$\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(a \longrightarrow ((x \rightarrow y) \longrightarrow x))\}. \tag{5.25}$$

Thus, if $\max\{\mu(x), 0.5\} = 0.5$ in (5.25), then $\mu(x) \leq 0.5$ and

$$\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\} \leq 0.5. \tag{5.26}$$

Therefore,

$$I_{\mathrm{cG}}(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) = 1 \geq 0.5, \tag{5.27}$$

whenever $\mu(x) \geq \min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x))\}$,

$$\begin{aligned} &I_{\mathrm{cG}}(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) \\ &\quad = 1 - \min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\} \geq 0.5, \end{aligned} \tag{5.28}$$

whenever $\mu(x) < \min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x))\}$. Now, if

$$\max\{\mu(x), 0.5\} = \mu(x) \tag{5.29}$$

in (5.25), then $\mu(x) \geq \min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x))\}$, and so

$$I_{\mathrm{cG}}(\min\{\mu(a), \mu(a \longrightarrow ((x \longrightarrow y) \longrightarrow x))\}, \mu(x)) = 1 \geq 0.5. \tag{5.30}$$

Consequently, μ is a 0.5-implication-based fuzzy-fated filter of L.

Conversely, assume that μ is a 0.5-implication-based fuzzy-fated filter of L, then

(iii) $I_{\mathrm{cG}}(\mu(x), \mu(1)) \geq 0.5$,

(iv) $I_{\mathrm{cG}}(\min\{\mu(a), \mu(a \rightarrow ((x \rightarrow y) \rightarrow x))\}, \mu(x)) \geq 0.5$,

for all $x, a, y \in L$. The case (iii) implies that $I_{\mathrm{cG}}(\mu(x), \mu(1)) = 1$, that is, $\mu(x) \leq \mu(1)$, or $1 - \mu(x) \geq 0.5$ and so $\mu(x) \leq 0.5$. It follows that

$$\max\{\mu(1), 0.5\} \geq \mu(x) = \min\{\mu(x), 1\}. \tag{5.31}$$

From (iv), we have

$$I_{cG}\left(\min\{\mu(a),\mu(a\longrightarrow((x\longrightarrow y)\longrightarrow x))\},\mu(x)\right)=1, \tag{5.32}$$

that is, $\min\{\mu(a),\mu(a\to((x\to y)\to x))\}\le\mu(x)$, or

$$1-\min\{\mu(a),\mu(a\longrightarrow((x\longrightarrow y)\longrightarrow x))\}\ge 0.5. \tag{5.33}$$

Hence,

$$\begin{aligned}\max\{\mu(x),0.5\} &\ge \min\{\mu(a),\mu(a\longrightarrow((x\longrightarrow y)\longrightarrow x))\}\\ &= \min\{\mu(a),\mu(a\longrightarrow((x\longrightarrow y)\longrightarrow x)),1\},\end{aligned} \tag{5.34}$$

for all $x,y,z\in L$. This completes the proof. □

6. Conclusion

Using the "belongs to" relation ($\in$) and quasicoincidence with relation (q) between fuzzy points and fuzzy sets, we introduced the notion of an $(\in,\in\vee q)$-fuzzy-fated filter, this is a generalization of a fuzzy implicative filter. In fuzzy logic, one can see that various implication operators have been defined. We used Gaines-Rescher implication operator (I_{GR}), Gödel implication operator (I_G), and the contraposition of Gödel implication operator (I_{cG}) to study t-implication-based fuzzy-fated filters.

There are also other situations concerning the relations between this kind of results, another type of structures (e.g., $(\overline{\in},\overline{\in\vee q})$-fuzzy-fated filter), and (fuzzy) soft and rough set theory. How to deal with these situations will be one of our future topics. We will also try to study the intuitionistic fuzzy version of several type of filters in R_0-algebras related to the intuitionistic "belongs to" relation ($\in$) and intuitionistic quasicoincidence with relation (q) between intuitionistic fuzzy points and intuitionistic fuzzy sets.

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable comments and suggestions for improving this paper.

References

[1] L. A. Zadeh, "Toward a generalized theory of uncertainty (GTU)—an outline," *Information Sciences*, vol. 172, no. 1-2, pp. 1–40, 2005.
[2] G. J. Wang, *Non-Classical Mathematical Logic and Approximate Reasoning*, Science Press, Beijing, China, 2000.
[3] G.-J. Wang, "On the logic foundation of fuzzy reasoning," *Information Sciences*, vol. 117, no. 1-2, pp. 47–88, 1999.
[4] Y. B. Jun and L. Lianzhen, "Filters of R_0-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 93249, 9 pages, 2006.

[5] L. Lianzhen and L. Kaitai, "Fuzzy implicative and Boolean filters of R_0-algebras," *Information Sciences*, vol. 171, no. 1–3, pp. 61–71, 2005.
[6] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.
[7] V. Murali, "Fuzzy points of equivalent fuzzy subsets," *Information Sciences*, vol. 158, pp. 277–288, 2004.
[8] P. M. Pu and Y. M. Liu, "Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 2, pp. 571–599, 1980.
[9] S. K. Bhakat and P. Das, "On the definition of a fuzzy subgroup," *Fuzzy Sets and Systems*, vol. 51, no. 2, pp. 235–241, 1992.
[10] S. K. Bhakat and P. Das, "$(\in, \in \vee q)$-fuzzy subgroup," *Fuzzy Sets and Systems*, vol. 80, no. 3, pp. 359–368, 1996.
[11] X. Ma, J. Zhan, and Y. B. Jun, "On $(\in, \in \vee q)$-fuzzy filters of R_0-algebras," *Mathematical Logic Quarterly*, vol. 55, no. 5, pp. 493–508, 2009.
[12] D. W. Pei and G. J. Wang, "The completeness and application of formal systems $\mathfrak{L}$," *Science in China E*, vol. 32, no. 1, pp. 56–64, 2002.
[13] M. S. Ying, "A new approach for fuzzy topology (I)," *Fuzzy Sets and Systems.*, vol. 39, no. 3, pp. 303–321, 1991.
[14] M. S. Ying, "On standard models of fuzzy modal logics," *Fuzzy Sets and Systems*, vol. 26, no. 3, pp. 357–363, 1988.