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Research Article

On the Dynamics of the Recursive Sequence $x_{n+1} = \alpha + x_{n-k}^p / x_n^q$

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We investigate the boundedness character, the oscillatory, and the periodic character of positive solutions of the difference equation $x_{n+1} = \alpha + x_{n-k}^p / x_n^q$, $n = 0, 1, \ldots$, where $k \in \{2, 3 \ldots\}$, α, p, $q \in (0, \infty)$ and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary positive numbers. We investigate the boundedness character for $p \in (0, \infty)$. Also, we investigate the existence of a prime two periodic solution for k is odd. Moreover, when k is even, we prove that there are no prime two periodic solutions of the equation above.

1. Introduction

Our aim in this paper is to study the boundedness character, the oscillatory, and the periodic character of positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^q}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $k \in \{2, 3, \ldots\}$, α is a positive, $p, q \in (0, \infty)$ and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary positive numbers. Equation (1.1) was studied by many authors for different cases of k, p, q.

In [1] the authors studied the global stability, the boundedness character, and the periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where α is positive, and the initial values x_{-1}, x_0 are positive numbers (see also [2–4] for more results on this equation).

In [5] the authors studied the boundedness, the global attractivity, the oscillatory behaviour, and the periodicity of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where α, p are positive, and the initial values x_{-1}, x_0 are positive numbers (see also [6–8] for more results on this equation).

In [9] the authors studied general properties, the boundedness, the global stability, and the periodic character of the solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where α, p are positive, $k \in \{2, 3, \ldots\}$, and the initial values $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive numbers.

In [10, 11] the authors studied the boundedness, the persistence, the attractivity, the stability, and the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \ldots, \tag{1.5}$$

where α, p, q are positive, and the initial values x_{-1}, x_0 are positive numbers.

Finally in [12, 13] the authors studied the oscillatory, the behaviour of semicycle, and the periodic character of the positive solution of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}, \quad n = 0, 1, \ldots, \tag{1.6}$$

where $k \in \{2, 3, \ldots\}$ and $\alpha, p > 0$ under the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive numbers.

There exist many other papers related with (1.1) and on its extensions (see [14–16]).

Motivated by the above papers, we study of the boundedness character, the oscillatory, and the periodic character of positive solutions of (1.1).

In this paper, also we investigate the case $p = 1$, $k = 1$ and $q \in (0, \infty)$ of (1.1) and we give a correction for [2, Theorem 2.5].

We say that the equilibrium point $\overline{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.7}$$

is the point that satisfies the condition

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.8}$$

A solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) is called nonoscillatory if there exists $N \geq -k$ such that either

$$x_n > \overline{x} \quad \forall n \geq N \tag{1.9}$$

or

$$x_n < \overline{x} \quad \forall n \geq N. \tag{1.10}$$

A solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) is called oscillatory if it is not nonoscillatory. We say that a solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) is bounded and persists if there exist positive constant P and Q such that $P \leq x_n \leq Q$ for $n = -k, -k+1, \ldots$.

The linearized equation for (1.1) about the positive equilibrium $\overline{x}$ is

$$y_{n+1} + q\overline{x}^{p-q-1} y_n - p\overline{x}^{p-q-1} y_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{1.11}$$

2. Semicycle Analysis

A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to $\overline{x}$, with $l \geq -k$ and $m \leq \infty$, such that

$$\begin{gathered} \text{either } l = -k \text{ or } l > -k,\ x_{l-1} < \overline{x}, \\ \text{either } m = \infty \text{ or } m < \infty,\ x_{m+1} < \overline{x}. \end{gathered} \tag{2.1}$$

A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than $\overline{x}$, with $l \geq -k$ and $m \leq \infty$, such that

$$\begin{gathered} \text{either } l = -k \text{ or } l > -k,\ x_{l-1} \geq \overline{x}, \\ \text{either } m = \infty \text{ or } m < \infty,\ x_{m+1} \geq \overline{x}. \end{gathered} \tag{2.2}$$

Lemma 2.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). Then either $\{x_n\}_{n=-k}^{\infty}$ consists of a single semicycle or $\{x_n\}_{n=-k}^{\infty}$ oscillates about equilibrium $\overline{x}$ with semicycles having at most k terms.*

Proof. Suppose that $\{x_n\}_{n=-k}^{\infty}$ has at least two semicycles. Then there exists $N \geq -k$ such that either $x_N < \overline{x} \leq x_{N+1}$ or $x_{N+1} < \overline{x} \leq x_N$. We assume that the former case holds. The latter case is similar and will be omitted. Now suppose that the positive semicycle beginning with the term x_{N+1} has k terms. Then $x_N < \overline{x} \leq x_{N+k}$ and so the case

$$x_{N+k+1} = \alpha + \frac{x_N^p}{x_{N+k}^q} < \alpha + \frac{\overline{x}^p}{\overline{x}^q} = \overline{x} \tag{2.3}$$

holds for every $p, q \in (0, \infty)$, from which the result follows. □

3. Boundedness and Global Stability of (1.1)

In this section, we consider the case $p \in (0,1)$ with no restriction on other parameters and we consider the case $p > 1$ with some specified conditions. For these cases, we have the following results which give a complete picture as regards the boundedness character of positive solutions of (1.1).

Theorem 3.1. *Suppose that*

$$p \in (0,1), \tag{3.1}$$

then every positive solution of (1.1) *is bounded.*

Proof. We have

$$x_{N+1} = \alpha + \frac{x_{N-k}^p}{x_N^q} \leq \alpha + \frac{x_{N-k}^p}{\alpha^q}, \quad N \geq -k. \tag{3.2}$$

Hence we will prove that $\{x_N\}$ is bounded. If

$$f(x) = \alpha + \frac{x^p}{\alpha^q}, \quad x > 0, \tag{3.3}$$

then we have $f'(x) > 0$, $f''(x) < 0$. Hence, the function f is increasing and concave. Thus, we get that there is a unique fixed point $\overline{x}$ of the equation $f(x) = x$. Also the function f satisfies the condition

$$\left(f(x) - x\right) \cdot (x - \overline{x}) < 0, \quad x > 0. \tag{3.4}$$

Using [15, 2.6.2] we obtain that $\overline{x}$ is a global attractor of all positive solutions of (1.1) and so $\{x_N\}$ is bounded, from which the result follows. □

Now we study the boundedness of (1.1) for the case $p > 1$. We give better result than Theorem 3.1 for the boundedness of (1.1) and we prove that in this case, there exist unbounded solutions of (1.1).

Theorem 3.2. *Consider* (1.1) *and assume that* $p > 1$, $p \nrightarrow \infty$, $\alpha > 1$ *and* $q \to \infty$. *Then every positive solution of* (1.1) *is bounded and* $\lim_{n\to\infty} x_n = \alpha$.

Proof. Let

$$f(x) = \alpha + \frac{x^p}{x^q} \quad \text{for } x \in (0,\infty). \tag{3.5}$$

Suppose on the contrary that every positive solution of (1.1) is unbounded. Then, from (1.1), we obtain $x_n \geq \alpha > 1$ for $n \geq 1$. Therefore we get

$$\lim_{x\to\infty} f(x) = \alpha. \tag{3.6}$$

Thus the proof is complete. We note that in here $f(x)$ is a continuous function for x, p, $q \in (0, \infty)$. □

Theorem 3.3. *Consider* (1.1) *when the case k is odd and suppose that*

$$p > 1, \tag{3.7}$$

then there exists unbounded solutions of (1.1).

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1) with initial values $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$ such that

$$x_{-k}, x_{-k+2}, \dots, x_{-1} > \max\left\{(\alpha+1)^{p/q}, (\alpha+1)^{q/p-1}\right\}, \tag{3.8}$$

$$x_{-k+1}, x_{-k+3}, \dots, x_0 < \alpha + 1. \tag{3.9}$$

Then from (1.1), (3.7), and (3.8) we have

$$\begin{aligned} x_1 &= \alpha + \frac{x_{-k}^p}{x_0^q} > \alpha + \frac{x_{-k}^p}{(\alpha+1)^q} - x_{-k} + x_{-k} \\ &= \alpha + x_{-k}\left(\frac{x_{-k}^{p-1}}{(\alpha+1)^q} - 1\right) + x_{-k} > \alpha + x_{-k}, \end{aligned} \tag{3.10}$$

$$x_2 = \alpha + \frac{x_{-k+1}^p}{x_1^q} < \alpha + \frac{(\alpha+1)^p}{x_{-k}^q} < \alpha + 1, \tag{3.11}$$

$$x_3 = \alpha + \frac{x_{-k+2}^p}{x_2^q} > \alpha + \frac{x_{-k+2}^p}{(\alpha+1)^q} - x_{-k+2} + x_{-k+2} > \alpha + x_{-k+2}, \tag{3.12}$$

$$x_4 = \alpha + \frac{x_{-k+3}^p}{x_3^q} < \alpha + \frac{(\alpha+1)^p}{x_{-k+2}^q} < \alpha + 1, \tag{3.13}$$

$$x_k = \alpha + \frac{x_{-1}^p}{x_{k-1}^q} > \alpha + \frac{x_{-1}^p}{(\alpha+1)^q} - x_{-1} + x_{-1} > \alpha + x_{-1}. \tag{3.14}$$

Also, from (3.8) and (3.10)–(3.14), it is clear that

$$x_1, x_3, \ldots, x_k > \max\left\{(\alpha+1)^{p/q}, (\alpha+1)^{q/p-1}\right\},$$
$$x_{k+1} = \alpha + \frac{x_0^p}{x_k^q} < \alpha + \frac{(\alpha+1)^p}{x_{-1}^q} < \alpha + 1. \tag{3.15}$$

Moreover from (1.1) and (3.8)–(3.14) and arguing as above we get

$$x_{k+2} = \alpha + \frac{x_1^p}{x_{k+1}^q} > \alpha + \frac{x_1^p}{(\alpha+1)^q} - x_1 + x_1 > \alpha + x_1. \tag{3.16}$$

Therefore working inductively we can prove that for $n = 0, 1, 2, \ldots$,

$$x_{2n+1} > \alpha + x_{2n-k}, \quad x_{2n} < \alpha + 1, \tag{3.17}$$

which implies that

$$\lim_{n\to\infty} x_{2n+1} = \infty. \tag{3.18}$$

So $\{x_n\}_{n=-k}^{\infty}$ is unbounded. From which the result follows. □

Now, in the next theorem, we will provide an alternative proof for the theorem above when $1 \le p < \infty$ and k is odd, whose proof can be used for some practical applications.

Theorem 3.4. *Consider* (1.1) *when the case* k *is odd and suppose that* $1 \le p < \infty$. *If* $0 \le \alpha < 1$, *then there exists solutions of* (1.1) *that are unbounded.*

Proof. We assume that $0 < \alpha < 1$ and choose the initial conditions such that

$$x_{-k}, x_{-k+2}, \ldots, x_{-1} > \frac{1}{(1-\alpha)^{1/q}},$$
$$\alpha < x_{-k+1}, x_{-k+3}, \ldots, x_0 < 1. \tag{3.19}$$

So,

$$x_1 = \alpha + \frac{x_{-k}^p}{x_0^q} > \alpha + x_{-k}^p,$$
$$x_2 = \alpha + \frac{x_{-k+1}^p}{x_1^q} < \alpha + \frac{1}{x_1^q} < \alpha + \frac{1}{x_{-k}^q} < 1. \tag{3.20}$$

Therefore, we obtain $\alpha < x_{k+1} < 1$ and $x_{k+2} > 2\alpha + x^p_{-k}$. By induction, for $i = 1, 2, \ldots$, we have $\alpha < x_{(k+1)i} < 1$ and $x_{(k+1)i+1} > (i+1)\alpha + x^p_{-k}$. Thus,

$$\lim_{i\to\infty} x_{(k+1)i+1} = \infty, \qquad \lim_{i\to\infty} x_{(k+1)i} = \alpha. \tag{3.21}$$

Now, we assume that $\alpha = 0$ and choose the initial conditions such that

$$\begin{gathered} x_{-k}, x_{-k+2}, \ldots, x_{-1} > \frac{1}{(1-\varepsilon)} \quad \text{for some } \varepsilon \in (0,1), \\ 0 < x_{-k+1}, x_{-k+3}, \ldots, x_0 < 1. \end{gathered} \tag{3.22}$$

So, we have

$$\begin{gathered} x_1 = \frac{x^p_{-k}}{x^q_0} > x^p_{-k}, \\ x_2 = \frac{x^p_{-k+1}}{x^q_1} < \frac{1}{x^q_1} < 1. \end{gathered} \tag{3.23}$$

Further, we have

$$\begin{gathered} x_{k+1} = \frac{x^p_0}{x^q_k} < \frac{1}{x^q_k} < 1, \\ x_{k+2} = \frac{x^p_1}{x^q_{k+1}} > x^p_1 > \left(x^p_{-k}\right)^p. \end{gathered} \tag{3.24}$$

By induction for $i = 1, 2, \ldots$, we have $0 < x_{(k+1)i} < 1$ and $x_{(k+1)i+1} > (x^p_{-k})^{(i+1)}$. Thus,

$$\lim_{i\to\infty} x_{(k+1)i+1} = \infty, \qquad \lim_{i\to\infty} x_{(k+1)i} = 0, \tag{3.25}$$

from which the result follows. □

The following result is essentially proved in [10, 11] for $k = 1$. The result is satisfied for $k \in \{2, 3, \ldots\}$ and its proof is omitted.

Lemma 3.5. *If Either*

$$0 < q < p < 1 \tag{3.26}$$

or

$$0 < p < q, \quad q \nrightarrow \infty \tag{3.27}$$

holds, then (1.1) *has a unique equilibrium point* $\overline{x}$.

The following result is essentially proved in [10, 11] for $k = 1$. It is clear that the result is satisfied when k is odd and its proof is omitted.

Lemma 3.6. *Consider* (1.1) *when the case k is odd. Suppose that*

$$0 < p < 1 < q, \quad q \nrightarrow \infty, \quad \alpha > (p + q - 1)^{1/(q-p+1)} \tag{3.28}$$

or

$$0 < q < p < 1, \quad (p + q) \leq 1 \tag{3.29}$$

holds. Then the unique positive equilibrium of (1.1) *is globally asymptotically stable.*

4. Periodicity of the Solutions of (1.1)

In this section, we investigate the existence of a prime two periodic solution for k is odd. Moreover, when k is even, we prove that there are no positive prime two periodic solutions and lastly, we give a correction for Theorem 2.5 which was given in [2].

The following result is given when the case $k = 1$ in [10]. If k is odd, the result is still satisfied and its proof is omitted.

Lemma 4.1. *Assume that k is odd. Then,* (1.1) *has prime two periodic solutions if and only if*

$$0 < p < 1 < q \tag{4.1}$$

and there exists a sufficient small positive number ε_1, *such that*

$$\frac{1}{(\alpha + \varepsilon_1)^{q-p}} > \varepsilon_1, \quad (\alpha + \varepsilon_1)^{p/q} \varepsilon_1^{-1/q} < \alpha + \varepsilon_1^{-p/q} (\alpha + \varepsilon_1)^{p^2 - q^2/q}. \tag{4.2}$$

Now, let consider the case where k is even.

Theorem 4.2. *Consider* (1.1) *when the case k is even and the following conditions are satisfied separately:*

$$\begin{gathered} 0 < q < p < 1, \\ 0 < p < q < 1, \\ 1 < p, \quad q < p + 1, \\ 1 < q, \quad p < q + 1. \end{gathered} \tag{4.3}$$

Then, there are no positive prime two periodic solutions of (1.1).

Proof. Firstly, we consider the case $0 < q < p < 1$ and k is even of (1.1) and suppose that

$$\ldots, x, y, x, y, \ldots, \tag{4.4}$$

where $x, y \in (\alpha, \infty)$ is a prime two periodic solution of (1.1). Then it must be

$$x = \alpha + y^{p-q}, \tag{4.5}$$

$$y = \alpha + x^{p-q}. \tag{4.6}$$

Substituting (4.6) into (4.5), it follows that

$$x - \alpha = \left(\alpha + x^{p-q}\right)^{p-q}. \tag{4.7}$$

Taking logarithm on both sides of (4.7), we obtain that

$$F(x) = \ln(x - \alpha) - (p - q)\ln\left(x^{p-q} + \alpha\right). \tag{4.8}$$

So from (4.8)

$$\begin{aligned} &\lim_{x \to \alpha^+} F(x) = -\infty, \\ F(\overline{x}) &= \ln(\overline{x} - \alpha) - (p - q)\ln\left(\overline{x}^{p-q} + \alpha\right) \\ &= \ln\left(\overline{x}^{p-q}\right) - (p - q)\ln(\overline{x}) \\ &= 0, \\ F'(x) &= \frac{1}{x - \alpha} - \frac{(p - q)^2 x^{p-q-1}}{\alpha + x^{p-q}} \\ &> \frac{1}{x - \alpha} - \frac{x^{p-q-1}}{\alpha + x^{p-q}} \\ &> \frac{1}{x - \alpha} - \frac{x^{p-q-1}}{x^{p-q}} \\ &> \frac{\alpha}{x(x - \alpha)}. \end{aligned} \tag{4.9}$$

From $x > \alpha$, thus $F'(x) > 0$, which implies that $\overline{x}$ is a unique solution of (1.1).

We consider the case $0 < p < q < 1$ and k is even of (1.1). The proof of this case is similar to the first case's proof and will be omitted.

Now, suppose that $1 < p$, $q < p + 1$ and k is even of (1.1). In this case we have that

$$F'(x) = \frac{\alpha + x^{p-q} - (p - q)^2 (x - \alpha) x^{p-q-1}}{(x - \alpha)\left(\alpha + x^{p-q}\right)}. \tag{4.10}$$

Considering the numerator on the right hand side in (4.10) let

$$k(x) = \alpha + x^{p-q} - (p-q)^2(x-\alpha)x^{p-q-1}$$
$$= \left(1-(p-q)^2\right)x^{p-q} + \alpha(p-q)^2x^{p-q-1} + \alpha. \tag{4.11}$$

From $x > \alpha$ and $1 < p$, $q < p+1$

$$k(x) > \left(1-(p-q)^2\right)\alpha^{p-q} + \alpha(p-q)^2\alpha^{p-q-1} + \alpha$$
$$> \alpha^{p-q} + \alpha \tag{4.12}$$
$$> 0,$$

which implies that $\overline{x}$ is a unique solution of (1.1).

Suppose that $1 < q$, $p < q+1$ and k is even of (1.1). The proof of this case is similar to the third case's proof and will be omitted. □

The following result was given in [2, Theorem 2.5] for (1.1) when the case $p = 1$, $k = 1$ and $q \in (0,\infty)$. But the authors make some mistakes in this theorem. Now, we give a correction and a conjecture for this result.

Theorem A. *Consider* (1.1). *Let be* $p = 1$, $k = 1$, $q \in (0,\infty)$, $q \nrightarrow 0^+$ *and* $q \nrightarrow \infty$. *Suppose that*

$$\alpha > 1, \quad \alpha > q(1+q)^{(1-q)/q} \tag{4.13}$$

hold. Then the unique positive equilibrium $\overline{x}$ *of* (1.1) *is globally asymptotically stable.*

Correction B

Consider (1.1). Let be $p = 1$, $k = 1$, $q \in (1,\infty)$, and $q \nrightarrow \infty$. Suppose that

$$\alpha > q^{1/q} \tag{4.14}$$

holds. Then the unique positive equilibrium $\overline{x}$ of (1.1) is globally asymptotically stable.

Proof. It is easy to see the proof from Theorem 2.5 in [2]. □

Conjecture 4.3. *Consider* (1.1). *Let be* $p = 1$, $k = 1$, $q \in (0,1)$, *and* $q \nrightarrow 0^+$. *Suppose that*

$$\alpha > 1 \tag{4.15}$$

holds. Then the unique positive equilibrium $\overline{x}$ *of* (1.1) *is globally asymptotically stable.*

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