Research Article

Complex Dynamic Behaviors in Cellular Automata Rule 14

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Wolfram divided the 256 elementary cellular automata rules informally into four classes using dynamical concepts like periodicity, stability, and chaos. Rule 14, which is Bernoulli σ_τ-shift rule and is a member of Wolfram's class II, is said to be simple as periodic before. Therefore, it is worthwhile studying dynamical behaviors of rule 14, whether it possesses chaotic attractors or not. In this paper, the complex dynamical behaviors of rule 14 of one-dimensional cellular automata are investigated from the viewpoint of symbolic dynamics. We find that rule 14 is chaotic in the sense of both Li-Yorke and Devaney on its attractor. Then, we prove that there exist fixed points in rule 14. Finally, we use diagrams to explain the attractor of rule 14, where characteristic function is used to describe that all points fall into Bernoulli-shift map after two iterations under rule 14.

1. Introduction

Cellular automata (CA) was first introduced by von Neumann in 1951 [1]. CA is a mathematical model consisting of large numbers of simple identical components with local interactions [2]. The simple components act together to produce complex global behavior. CA performs complex computation with high degree of efficiency and robustness. Therefore, many applications of CA have been reported, especially in cryptography [3, 4] and image processing [5, 6].

Here, we will only consider Boolean automata for which the cellular state $x \in \{0, 1\}$. A cellular automata consists of a number of cells which evolve by a simple local rule (identical rule). The value of each cell in the next stage is determined by the values of the cell and its neighbor cells in the current stage under the local rule [4]. The identical rule contained in each cell is essentially a finite-state machine, usually specified in the form of a rule table, with an entry for every possible neighborhood of a cell consists of itself and the adjacent cells [7]. The cellular array is d-dimensional, where $d = 1, 2, 3$ is used in practice. In this paper,

Table 1: The truth table of Boolean function of rule 14.

$x_{i-1}x_ix_{i+1}$	$[f_{14}(x)]_i$	$x_{i-1}x_ix_{i+1}$	$[f_{14}(x)]_i$
000	0	100	0
001	1	101	0
010	1	110	0
011	1	111	0

we will concentrate on $d = 1$. For a one-dimensional CA, a cell is connected to r local neighbors (cells) on either side where r is referred to as the radius. A one-dimensional CA has n cells linked in a line or in a circle. Denote the value in the ith cell at the tth stage by $x_i[t]$. For 2-state 3-neighborhood CA ($r = 1$), the evolution of ith cell can be represented as a function of the present states of $(i - 1)$th, (i)th, and $(i + 1)$th. The local function f_i is a deterministic function to determine the next-stage value of the ith cell, $x_i[t + 1] = f_i(x_{i-1}[t], x_i[t], x_{i+1}[t])$. For example, the rule 14 is a one-dimensional CA, and its rule table is shown in Table 1. Thus, we have $f(000) = 0$, $f(001) = 1$, $f(010) = 1$, $f(011) = 1$, $f(100) = 0$, $f(101) = 0$, $f(110) = 0$, and $f(111) = 0$.

In 1980s, Wolfram proposed CA as models for physical systems which exhibit complex or even chaotic behaviors based on empirical observations, and he divided the 256 ECA (binary one-dimensional CA with radius 1) rules informally into four classes using dynamical concepts like periodicity, stability, and chaos [8–10]. However, some authors [11–14] found that some rules of Bernoulli σ_τ-shift rules are chaotic in the sense of both Li-York and Devaney, where these rules were said to be simple as periodic by Wolfram. Rule 24 is belonging to Bernoulli σ_τ-shift rules. Therefore, we need to research the rule 24 and to find its some new dynamical properties. The rest of the paper is organized as follows. In Section 2, the Boolean function of rule 14 is also presented, and the expressions of its attractors are given. In Section 3, the dynamical behaviors of rule 14 are studied. In Section 4, characteristic function is used to describe that all points fall into Bernoulli-shift map after several iterations under rule 14, and Lameray diagram is used to show clearly the iterative process of an attractor. Section 5 presents some conclusions.

2. Preliminaries

For simplicity, some notations about symbolic dynamics can refer to [11, 14].

In this paper, we will use some notations about CA as follows.

Chua et al. [15] mentioned that each rule has three globally equivalent local rules determined by three corresponding global transformations, namely, left-right transformation $T^\dagger$, global complementation $\overline{T}$, and left-right complementation T^*. Each equivalence class is identified by ε_m^κ, where κ is complexity index, and m is index of κth class. In [16], the authors presented that 112 rules of 256 local rules were Bernoulli σ_τ-shift rules. Each of the 112 Bernoulli σ_τ-shift rules has an ID code $B_N[\alpha, \beta, \tau]$, where α denotes the number of attractors of rule N, β denotes the slope of the Bernoulli σ_τ-shift map, and τ denotes the relevant forward time-τ. Hence, the space-time evolution of any one of the 112 rules on their attractors can be uniquely predicted by two parameters: $\beta = \pm 2^\sigma$ and τ. For example, the attractors of rule 14 are ($\beta = -1/2$, $\sigma = -1$, $\tau = 1$) and ($\beta = 2$, $\sigma = 1$, $\tau = 1$).

It follows from [17] that the Boolean function of rule 14 is

$$[f_{14}(x)]_i = \overline{x}_{i-1} \cdot x_i \oplus \overline{x}_{i-1} \cdot \overline{x}_i \cdot x_{i+1}, \tag{2.1}$$

for $\forall x \in S^Z$, $i \in Z$, where "·", "−", and "⊕" stand for "AND", "NOT", and "XOR" logical operation, respectively. Sometimes, "·" is omitted for simplicity. The truth table of Boolean function of rule 14 is shown in Table 1.

The subsets denoted by Λ_1^{14}, Λ_2^{14} are derived from the parameters of rule 14: $\beta = -1/2$, $\sigma = -1$, and $\tau = 1$ and $\beta = 2$, $\sigma = 1$, and $\tau = 1$, respectively, that is,

$$\Lambda_1^{14} = \left\{ x \in S^Z \mid [f_{14}(x)]_i = \overline{x}_{i-1}, \ \forall i \in Z \right\},$$

$$\Lambda_2^{14} = \left\{ x \in S^Z \mid [f_{14}(x)]_i = x_{i+1}, \ \forall i \in Z \right\}. \tag{2.2}$$

3. Dynamical Behaviors of f_{14} on Two Subsystems

In this section, we investigate the complexity and chaotic dynamic behaviors of f_{14}. In order to give our results, some definitions need be introduced.

Definition 3.1 (see [18]). A square $\{0,1\}$ matrix A is irreducible if for every pair of indices i and j, there is an n such that $A_{ij}^n > 0$.

Definition 3.2 (see [18]). A square $\{0,1\}$ matrix A is aperiodic if there exists N, such that $A_{ij}^n > 0$, $n > N$, $\forall i, j$.

Definition 3.3 (see [18]). Suppose that $g : X \to Y$ is a continuous mapping, where X is a compact topological space. g is said to be topologically mixing if, for any two open sets U, $V \subset X$, $\exists N > 0$, such that $g^n(U) \cap V \neq \emptyset$, $\forall n \geq N$.

Definition 3.4 (see [13]). Let (X, f) and (Y, g) be compact spaces, we say f and g are topologically conjugate if there is a homeomorphism $h : X \to Y$, such that $h \circ f = g \circ h$.

Based on the above definitions, we investigate the complexity behaviors of f_{14} on two subsystems as follows.

Proposition 3.5. *For rule 14, there exists a subset $\Lambda_1^{14} \subset S^Z$ which satisfies $f_{14}|_{\Lambda_1^{14}} = \varsigma_1^{14} \circ \overline{T}|_{\Lambda_1^{14}} = \Gamma_{14}|_{\Lambda_1^{14}}$ if and only if $\forall x = (\dots, x_{-1}, x_0, x_1, \dots) \in \Lambda_1^{14}$, and x_{i-1}, x_i, and x_{i+1} cannot be 0 simultaneously, $\forall i \in Z$.*

Proof. (Necessity) Suppose that there exists a subset $\Lambda_1^{14} \in S^Z$ such that $f_{14}|_{\Lambda_1^{14}} = \Gamma_{14}|_{\Lambda_1^{14}}$, then $\forall x = (\dots, x_{-1}, x_0, x_1, \dots) \in \Lambda_1^{14}$, and we have $[f_{14}(x)]_i = \overline{x}_{i-1}$, $\forall x \in Z$.

If $x_i = 1$, then $\overline{x}_{i-1}x_i \oplus \overline{x}_{i-1}\overline{x}_i x_{i+1} = 1 \oplus x_{i-1} \oplus 0 = \overline{x}_{i-1}$, so we get $x_{i-1} = 0$, $x_{i+1} = 0$; $x_{i-1} = 0$, $x_{i+1} = 1$; $x_{i-1} = 1$, $x_{i+1} = 0$; $x_{i-1} = 1$, $x_{i+1} = 1$; if $x_i = 0$, then $\overline{x}_{i-1}x_i \oplus \overline{x}_{i-1}\overline{x}_i x_{i+1} = 0 \oplus \overline{x}_{i-1}x_{i+1}$, so we get $x_{i+1} = 1$, $x_{i-2} = 1$; if $x_{i-1} = 1$, then $x_{i+1} = 1$ or $x_{i+1} = 0$.

(Sufficiency) Suppose that there exists a subset $\Lambda_1^{14} \subset S^Z$, $\forall x \in \Lambda_1^{14}$, x_{i-1}, x_i, and x_{i+1} cannot be 0 simultaneously, $\forall i \in Z$. This yields that $x_{i-1} + x_i + x_{i+1} = 1$. Thus, $(x_{i-1} + x_i + x_{i+1}) \oplus x_{i-1} = \overline{x}_{i-1}$, namely, $\overline{x}_{i-1}x_i \oplus \overline{x}_{i-1}\overline{x}_i x_{i+1} = \overline{x}_{i-1}$. Therefore, we have proven that $f_{14}(x) = \Gamma_{14}(x)$, $\forall x \in \Lambda_1^{14}$. □

Remark 3.6. A condition of bit strings according to Bernoulli σ_τ-shift evolution under f_{14} is obtained in Proposition 3.5. From Proposition 3.5, we can know that a bit string belonging

to Λ_1^{14} is evolved on the basis of shifting the bit string to the right by 1 bit and then complementing (changing 0 to 1 and changing 1 to 0) to obtain the next bit string after one iteration under f_{14}.

Remark 3.7. From the definition of subsystem, we know that (Λ_1^{14}, f_{14}) are subsystems of (S^Z, f_{14}).

Proposition 3.8. $\Gamma_{14} = \varsigma_{14} \circ \overline{T} = \overline{T} \circ \varsigma_{14}$.

Proposition 3.9. $\varsigma_{14} : S^Z \rightarrow S^Z$ *and* $\Gamma_{14} : S^Z \rightarrow S^Z$ *are not topologically conjugate.*
    $(\Lambda_1^{14}, \varsigma_{14})$ *is topologically mixing if and only if* $(\Lambda_1^{14}, \Gamma_{14})$ *is topologically mixing.*
    The topological entropy $\text{ent}(\&_{14}|_{\Lambda_1^{14}}) = \text{ent}(\Gamma_{14}|_{\Lambda_1^{14}})$.

Proof. (a) Suppose that ς_{14} and Γ_{14} are topologically conjugate, we give a proof by contradiction. Since ς_{14} and Γ_{14} are topologically conjugate, then there exists a homeomorphism h such that $h \circ \varsigma_{14} = \Gamma_{14} \circ h$, so we have $h \circ \varsigma_{14} = \overline{T} \circ \varsigma_{14} \circ h$. Because of $h \circ \varsigma_{14} = \varsigma_{14} \circ h$, then we obtain that $\overline{T}$ is identical map, which leads to a contradiction.
    (b) Let any nonempty open sets $U, V \subset \Lambda_1^{14}$.
    (Sufficiency) Assume that $\Gamma_{14}|_{\Lambda_1^{14}}$ is topologically mixing, then there exists $N_1 > 0$, such that $(\Gamma_{14}|_{\Lambda_1^{14}})^n(U) \cap V \neq \emptyset, \forall n \geq N_1$. Since $\overline{T}$ is a homeomorphism, $\overline{T}(V)$ is also a nonempty open set in Λ_1^{14}, hence there exists $N_2 > 0$, such that $(\Gamma_{14}|_{\Lambda_1^{14}})^n(U) \cap \overline{T}(V) \neq \emptyset, \forall n \geq N_2$. Therefore, $\overline{T}^{-1}((\Gamma_{14}|_{\Lambda_1^{14}})^n(U) \cap \overline{T}(V)) \neq \emptyset, \forall n \geq N_2$. Then, we show that there exists a $N > 0$ such that $(\varsigma_{14}|_{\Lambda_1^{14}})^n(U) \cap V \neq \emptyset, \forall n \geq N$. Let $N = \max(N_1, N_2)$.

(i) If $n = 2k, k \in Z^+$, then

$$
\begin{aligned}
\left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V &= \overline{T}^n \circ \left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V \\
&= \underbrace{\overline{T} \circ \varsigma\mid_{\Lambda_1^{14}} \circ \cdots \circ \overline{T} \circ \varsigma\mid_{\Lambda_1^{14}}(U) \cap V}_{n=2k} \\
&= \left(\Gamma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V \neq \emptyset.
\end{aligned}
\tag{3.1}
$$

(ii) If $n = 2k + 1, k \in Z^+$, then

$$
\begin{aligned}
\left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V &= \overline{T}^{-1} \circ \overline{T}\left(\left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V\right) \\
&= \overline{T}^{-1} \circ \left(\overline{T} \circ \left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap \overline{T}(V)\right) \\
&= \overline{T}^{-1} \circ \left(\left(\Gamma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap \overline{T}(V)\right) \neq \emptyset.
\end{aligned}
\tag{3.2}
$$

(Necessity) Suppose that $\varsigma_{14}|_{\Lambda_1^{14}}$ is topologically mixing, then there exists an $N_1 > 0$, such that $(\varsigma_{14}|_{\Lambda_1^{14}})^n(U) \cap V \neq \emptyset, \forall n \geq N_1$. Since $\overline{T}$ is a homeomorphism, $\overline{T}(U)$ is also a nonempty open set in Λ_1^{14}, hence for open sets $\overline{T}(U)$ and V, there exists an $N_2 > 0$, such that $(\varsigma_{14}|_{\Lambda_1^{14}})^n \circ \overline{T}(U) \cap V \neq \emptyset$. Let $N = \max(N_1, N_2)$.

(i) If $n = 2k$, $k \in Z^+$, then

$$\left(\Gamma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V = \overline{T}^n \circ \left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V$$

$$= \left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V \neq \emptyset. \tag{3.3}$$

(ii) If $n = 2k + 1$, $k \in Z^+$, then

$$\left(\Gamma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V = \overline{T} \circ \left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n(U) \cap V$$

$$= \left(\varsigma_{14}\mid_{\Lambda_1^{14}}\right)^n \circ \overline{T}(U) \cap V \neq \emptyset. \tag{3.4}$$

(c) $(\Gamma_{14}\mid_{\Lambda_1^{14}})^2 = (\overline{T} \circ \varsigma_{14}\mid_{\Lambda_1^{14}})^2 = (\overline{T})^2 \circ (\varsigma_{14}\mid_{\Lambda_1^{14}})^2 = (\varsigma_{14}\mid_{\Lambda_1^{14}})^2$.
Hence, $\mathrm{ent}(\varsigma_{14}\mid_{\Lambda_1^{14}}) = \mathrm{ent}(\Gamma_{14}\mid_{\Lambda_1^{14}})$. The proof is complete. $\qquad\square$

Because of $f_{14}\mid_{\Lambda_1^{14}} = \Gamma_{14}\mid_{\Lambda_1^{14}}$, the following corollaries are immediate.

Corollary 3.10. *$\varsigma_{14}\mid_{\Lambda_1^{14}}$ and $f_{14}\mid_{\Lambda_1^{14}}$ are not topologically conjugate.*

Corollary 3.11. *$(\Lambda_1^{14}, \varsigma_{14})$ is topologically mixing if and only if (Λ_1^{14}, f_{14}) is topologically mixing.*

Corollary 3.12. *The topological entropy $\mathrm{ent}(\&_{14}\mid_{\Lambda_1^{14}}) = \mathrm{ent}(f_{14}\mid_{\Lambda_1^{14}})$.*

Remark 3.13. By Proposition 3.5, we know that the determinative block system of Λ_1^{14} is $P_1 = \{r_0, r_1, r_2, r_3, r_4, r_5, r_6\}$, where $r_0 = (001)$, $r_1 = (010)$, $r_2 = (011)$, $r_3 = (100)$, $r_4 = (101)$, $r_5 = (110)$, and $r_6 = (111)$. We have $\Lambda_1^{14} = \Lambda_{P_1}$.

Let $\varpi_1 = \{(rr') \mid r = (b_0b_1b_2), r' = (b'_0b'_1b'_2) \in P_1, \exists 1 \leq j \leq 2 \text{ such that } b_j \neq b'_{j-1}\}$. ϖ_1 is considered as the excluded block system [19] Λ_{ϖ_1}, where $\Lambda_{\varpi_1} = \{r = (\ldots r_{-1}, r_0, r_1 \ldots) \in P_1 \mid r_i \in P_1, r_i r_{i+1} \nmid \varpi_1, \forall i \in Z\}$. Obviously, $\varsigma_{14} : \Lambda_{\varpi_1} \to \Lambda_{\varpi_1}, \varsigma_{14}(\ldots, r_{-1}\hat{r}_0 r_1, \ldots) \to (\ldots, \hat{r}_{-1}r_0 r_1, \ldots)$. Therefore, it is easy to calculate the transition matrix B^1 of the $\varsigma\mid_{\Lambda_{\varpi_1}}$ as

$$B^1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \tag{3.5}$$
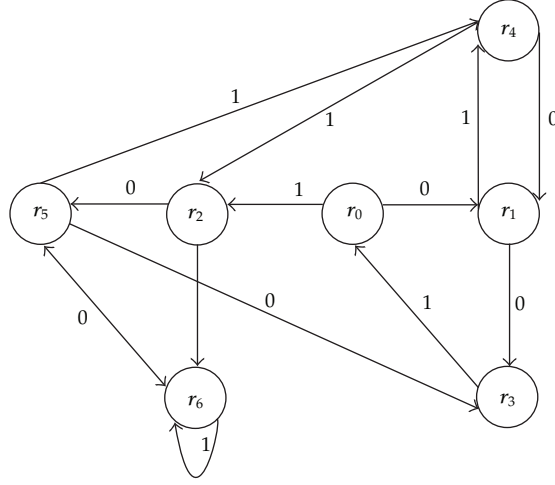
Figure 1: The corresponding graph G_1^{14} of the matrix B^1.

where $B^1 = (B_{ij}^1)_{|P_1| \times |P_1|}$

$$B_{ij}^1 = \begin{cases} 1, & r_i r_j \prec \varpi_1, \\ 0, & \text{else.} \end{cases} \tag{3.6}$$

Obviously, B^1 is a square $\{0, 1\}$ matrix. A square $\{0, 1\}$ matrix corresponds to a directed graph. The vertices of the graph are the indices for the rows and columns of B^1. There is an edge from vertex i to vertex j if $B_{ij}^1 = 1$. A square $\{0, 1\}$ matrix A is irreducible if and only if the corresponding graph is strongly connected. If Λ_A is a two-order subshift of finite type, then it is topologically mixing if and only if A is irreducible and aperiodic [18].

We give the corresponding graph G_1^{14} of matrix B^1 in Figure 1, where vertices are the elements of set P_1. It is obvious that G_1^{14} is a strongly connected graph.

Carefully observing Figure 1, we find that there are several strongly connected subgraphs: $r_0 \rightarrow r_1 \rightarrow r_3 \rightarrow r_0, r_0 \rightarrow r_1 \rightarrow r_4 \rightarrow r_1 \rightarrow r_3 \rightarrow r_0, r_0 \rightarrow r_2 \rightarrow r_5 \rightarrow r_3 \rightarrow r_0,$ $r_0 \rightarrow r_2 \rightarrow r_6 \rightarrow r_5 \rightarrow r_4 \rightarrow r_1 \rightarrow r_3 \rightarrow r_0, r_1 \rightarrow r_4 \rightarrow r_1, r_2 \rightarrow r_5 \rightarrow r_4 \rightarrow r_2,$ $r_2 \rightarrow r_6 \rightarrow r_5 \rightarrow r_4 \rightarrow r_2, r_6 \rightarrow r_6,$ and so forth. The elements of $\Lambda_{P_1}^{14}$ will be composed by all vertices of those strongly connected subgraphs, respectively. For example, if $x_1 \in \Lambda_{P_1}^{14}$, x_1 is composed by vertices of subgraph $r_1 \rightarrow r_4 \rightarrow r_1$, then we have $r_0, r_2, r_3, r_5,$ and $r_6 \nprec x_1$, and all vertices of the subgraph will occur in x_1 if $|x_1| = 2k$, $k = 1, 2, \ldots$. If $x_2 \in \Lambda_{P_1}^{142}$ and x_2 is composed by vertices of subgraph $r_2 \rightarrow r_5 \rightarrow r_4 \rightarrow r_2$, then we have $r_0, r_1, r_3,$ and $r_6 \nprec x_2$, and all vertices of the subgraph will occur in x_2 if $|x_2| = 3k$. Therefore, we can deduce the elements of set Λ_1^{14} via the corresponding strongly connected graph of matrix B^1.

Based on the above analysis, we have the following proposition.

Proposition 3.14. (a) $\varsigma_{14} : \Lambda_1^{14} \rightarrow \Lambda_1^{14}$ and $\varsigma_{14} : \Lambda_{\varpi_1} \rightarrow \Lambda_{\varpi_1}$ are topologically conjugate.

(b) $\varsigma_{14}|_{\Lambda_1^{14}}$ is topologically mixing.

(c) The topological entropy $\text{ent}(\&_{14}|_{\Lambda_1^{14}}) = 0.6094$.

Proof. (a) Define

$$h : \Lambda_1^{14} \longrightarrow \Lambda_{\varpi_1},$$

$$x = \left(\dots, x_{-1}, \overset{*}{x_0}, x_1, \dots\right) \longmapsto \left(\dots, r_{-1}, \overset{*}{r_0}, r_1, \dots\right), \tag{3.7}$$

where $r_i = (x_i x_{i+1} x_{i+2})$, $\forall i \in Z$. In fact, by the definition of Λ_{ϖ_1}, we have $h(x) \in \Lambda_{\varpi_1}$, $\forall x \in \Lambda_1^{14}$; thus, $h(\Lambda_1^{14}) \subseteq h(\Lambda_{\varpi_1})$. Then, it is easy to check that h is homeomorphism and $h \circ \varsigma_{14}|_{\Lambda_1^{14}} \to \varsigma_{14}|_{\Lambda_{\varpi_1}} \circ h$. Therefore, $\varsigma_{14}|_{\Lambda_1^{14}}$ and $\varsigma_{14}|_{\Lambda_{\varpi_1}}$ are topologically conjugate.

(b) Because $(B^1)^n > 0$, $\forall n \geq 4$, the transition matrix B^1 of subshift of finite type ς is irreducible and aperiodic. By [18, 19], $\varsigma_{14}|_{\Lambda_1^{14}}$ is topologically mixing.

(c) Because two topological conjugate systems have the same topological entropy, and the topological entropy ς_{14} on Λ_{ϖ_1} equals $\log \rho(B^1)$, where $\rho(B^1)$ is the spectral radius of the transition matrix B_1^{14} of the subshift Λ_{ϖ_1}. So $\mathrm{ent}(\varsigma_{14}|_{\Lambda_1^{14}}) = \log \rho(B^1) = 0.6094$. $\qquad\square$

Remark 3.15. By Corollaries 3.11 and 3.12, we know that (Λ_1^{14}, f_{14}) is topologically mixing and $\mathrm{ent}(f_{14}|_{\Lambda_1^{14}}) = 0.6094$.

Theorem 3.16. f_{14} *is chaotic in the sense of both Li-Yorke and Devaney on* Λ_1^{14}.

Proof. It follows from [19] that the positive topological entropy implies chaos in the sense of Li-Yorke, and a system with topologically mixing property has chaotic properties in different senses such as Devaney. Therefore, rule $N = 14$ possesses very rich and complicated dynamical properties on Λ_1^{14}. $\qquad\square$

Proposition 3.17. *For rule 14, there exists a subset* $\Lambda_2^{14} \subset S^Z$ *which satisfies* $f_{14}|_{\Lambda_2^{14}} = \upsilon_{14}|_{\Lambda_2^{14}}$ *if and only if* $\forall x = (\dots, x_{-1}, x_0, x_1, \dots) \in \Lambda_2^{14}$, *and* x_{i-1}, x_i, *and* x_{i+1} *have the following relations:*

(i) *if* $x_i = 1$, *then* $x_{i-1} = 0$, $x_{i+1} = 1$, *and* $x_{i+2} = 0$; $x_{i-1} = 1$, $x_{i+1} = 0$, *and* $x_{i+2} = 0$,

(ii) *if* $x_i = 0$, *then* $x_{i-1} = 0$, $x_{i+1} = 0$; $x_{i-1} = 1$, $x_{i+1} = 0$; $x_{i-1} = 0$, $x_{i+1} = 1$, *and* $x_{i+2} = 1$.

Let $P_2 = \{r_0, r_1, r_2, r_3, r_4\}$ be a new state set, where $r_0 = (000)$, $r_1 = (001)$, $r_2 = (011)$, $r_3 = (100)$, and $r_4 = (110)$.

Remark 3.18. From the definition of subsystem, we know that (Λ_2^{14}, f_{14}) is subsystem of (S^Z, f_{14}).

The transition matrix of subshift $(\Lambda_2^{142}, \upsilon_{14})$ is

$$B^2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.8}$$

We give the corresponding graph G_2^{14} of matrix B^2 in Figure 2, where vertices are the elements of set P_2.
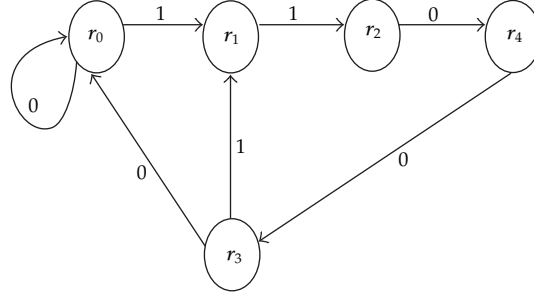
Figure 2: The corresponding graph G_2^{14} of the matrix B^2.

Proposition 3.19. (a) $f_{14}|_{\Lambda_2^{14}}$ is topologically mixing.
(b) The topological entropy $\mathrm{ent}(f_{14}|_{\Lambda_2^{14}}) = 0.3223$.

Theorem 3.20. f_{14} is chaotic in the sense of both Li-Yorke and Devaney on Λ_2^{14}.

Theorem 3.21. For rule 14, there exist fixed points of f_{14}.

Proof. A example is given as follows. Let

$$y = \left(\underbrace{\ldots 0,1,0,1,0,1}_{\text{repetition of }(0,1)}, 0, 1, \overset{*}{0}, 1, \underbrace{0,1,0,1,0,1\ldots}_{\text{repetition of }(0,1)} \right). \tag{3.9}$$

By Table 1, we have

$$f_{14}(y) = \left(\underbrace{\ldots 0,1,0,1,0,1}_{\text{repetition of }(0,1)}, 0, 1, \overset{*}{0}, 1, \underbrace{0,1,0,1,0,1\ldots}_{\text{repetition of }(0,1)} \right) = y. \tag{3.10}$$

Therefore, there exists a set of fixed points of rule 14.   □

Remark 3.22. By Theorem 3.21, we know that rule 14 has garden of Eden [20] (An bit string is said to be a garden of Eden of rule N if and only if it does not have a predecessor under the local rule transformation f_N) in the finite case. By definition of garden of Eden, the incoming and outgoing bit strings are the same in a garden of Eden, while the basin of attraction of an attractor Λ must contain, at least, one point not belonging to Λ. However, in this paper, we consider garden of Eden as a kind of special attractor. Then, we find that (000), (001), (010), (011), (100), (101), (110), or (111) $\prec \Lambda_{14} = \Lambda_1^{14} \cup \Lambda_2^{14}$. Furthermore, we guess that all initial bit strings after sufficient iterations will belong to Λ_{14} under rule f_{14}.

4. Characteristic Function of Rule 14

First, we give a definition on global characteristic function [16]. Given any local rule N, and binary configuration $x = [x_0 x_1 \cdot x_{I-1} x_I]$ for CA, where $x_i \in \{0,1\}$, then we can uniquely
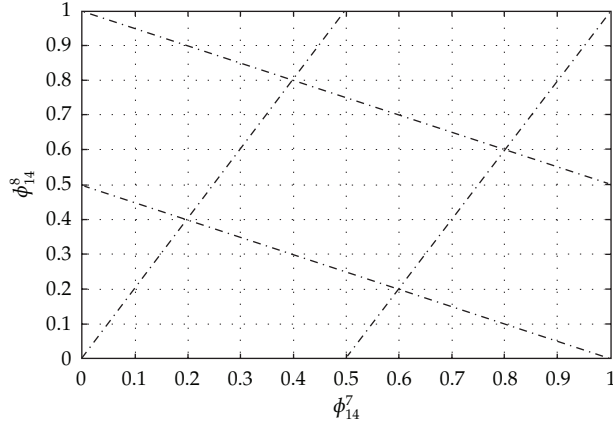
Figure 3: All points fall into Bernoulli-shift map after seven iterations under rule 14, where $I = 13$.

associate the Boolean string x with the binary expansion of a real number $0_\bullet x_0 x_1 \cdot x_{I-1} x_I$ on the unit interval $[0, 1]$,

$$x = [x_0 x_1 \cdot x_{I-1} x_I] \longmapsto \phi = 0_\bullet x_0 x_1 \cdot x_{I-1} x_I, \tag{4.1}$$

where $\phi = \sum_{i=0}^{I} 2^{-(i+1)} x_i$ is the decimal form of Boolean string $x = [x_0 x_1 \cdot x_{I-1} x_I]$. The CA' characteristic function χ_N of rule N is defined as

$$\chi_N : Q[0, 1] \longrightarrow Q[0, 1], \quad \text{i.e.} \quad \phi_n = \chi_N^1 (\phi_{n-1}), \tag{4.2}$$

where Q denotes rational numbers. By [19], the express of characteristic function of rule 14 is

$$\chi_{14}^1 = \sum_{i=0}^{I} \delta \left[-2x_{i-1} + x_i + x_{i+1} - \frac{1}{2} \right] 2^{-(i+1)}, \tag{4.3}$$

where

$$\delta\{w\} = \begin{cases} 1, & w > 0, \\ 0, & w \leq 0. \end{cases} \tag{4.4}$$

In order to transform every concrete bit string into abstract decimal digit, we use characteristic function χ_N. Next, we plot the characteristic function diagram for rule 14. We choose $I = 14$. Figure 3 describes that all points fall into Bernoulli-shift map after eight iterations under rule 14, which shows that Λ_{14} is global attractor of f_{14}. The phenomenon also shows that the prediction in Remark 3.22 is correct. Two sets Λ_1^{14} and Λ_2^{14} can also be found in Figure 3. The dots in the two straight lines, whose slope is $-1/2$, are in the set Λ_1^{14}; the dots in the two straight lines, whose slope is 2, are in the set Λ_2^{14}.

Two isles of Eden are shown in Figure 4. We find that the bit string $y = (\underbrace{\ldots, 0, 1, \ldots}_{\text{repetition of } (0,1)})$ is isle of Eden, when the length $|y|$ is even. Some attractors are shown in Figure 5. Figure 5(a) shows that the period of attractor is 1; Figure 5(b) shows that the period of attractor is
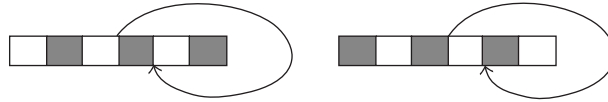
Figure 4: Two isles of Eden are shown, where $|x| = 6$.
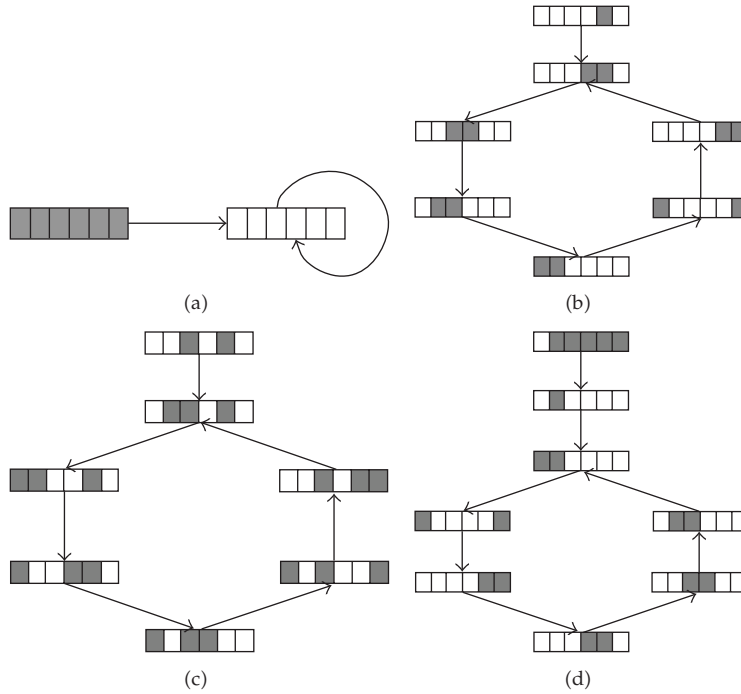

(a)

(b)

(c)

(d)

Figure 5: Some attractors of rule 14, where $I = 5$, and the white pixel stands for 0 and black for 1.

six, where $\beta = 2$, $\sigma = 1$, and $\tau = 1$; Figure 5(c) shows that the period of attractor is six, where $\beta = -1/2$, $\sigma = -1$, and $\tau = 1$; Figure 5(d) shows that after two iterations, the initial string is attracted. We find that the possible periods of attractors are 1 and 6 if $I = 5$. By Proposition 3.5, the attractors of Figures 4 and 5(c) belong to Λ_1^{14}. By Proposition 3.17, the attractors of (a), (b), and (d) belong to Λ_2^{14}. There is a interesting phenomenon, that is, Figure 5(d), where one iteration accords with Proposition 3.5; however, the two iterations accord with Proposition 3.17, and the attractor belongs to Λ_2^{14}. The phenomenon shows that these two propositions can switch in some situation. The bit string of attractor in Figure 5(c) corresponds with strongly connected subgraph $r_0 \rightarrow r_2 \rightarrow r_5 \rightarrow r_4 \rightarrow r_1 \rightarrow r_3 \rightarrow r_0$ in Figure 1.

Next, we use Lameray diagram [21] to present our attractors. The diagrams show clearly the iterative process of attractors. The evolution of characteristic function of period-5 attractor is shown in Figure 6, where $I = 4$, and the values of characteristic function are 0.0938, 0.1875, 0.3750, 0.7500, and 0.5313, respectively. The five points fall into the time-1 map ($\phi_n = \chi_N(\phi_{n-1})$) in Figure 5. Then we can associate this particular period-5 attractor of rule 14 as a period-5 point of a continuous map $f : [0, 1] \rightarrow [0, 1]$ which we know that it is chaotic because "period-5" implies chaos based on Li-York theorem [22] and Sarkovskii theorem [23]. According to Sarkovskii theorem, period-5 implies that the number of periods of f_{14} are infinite.
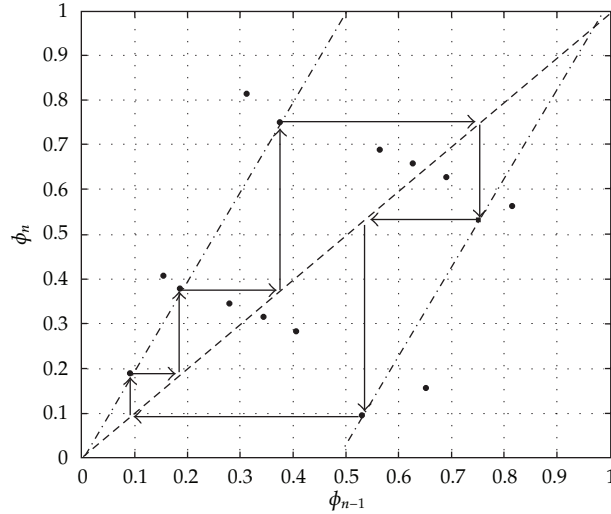
Figure 6: The evolution of characteristic function of the period-5 attractor, where the values of characteristic function of the attractor are 0.0938, 0.1875, 0.3750, 0.7500, and 0.5313, respectively.

In recent years, associative memory was researched in many papers [24, 25]. It is obvious that rule 14 can be used in associative memory. By strongly connected graph of rule 14, we can get the elements on its attractors. Then, we can choose a bit string which belongs to an attractor as memory pattern. For example, bit string 110000 in Figure 5(b) can be chosen as a memory pattern. Since there are infinite orbits, the storage capability is very large. The associative memory model provides a solution to problem where time to recognize a pattern is independent of the number of patterns stored.
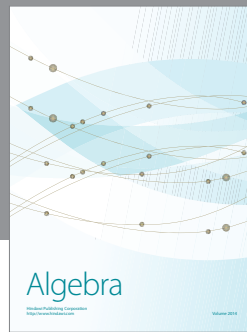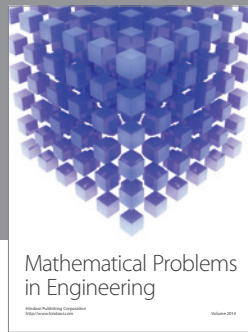
5. Conclusion

In this paper, the dynamical behaviors of rule 14 of global equivalence class ε_{13}^1 in ECA, where rule 14 is Bernoulli σ_τ-shift rule, are carefully investigated from, the viewpoint of symbolic dynamics. We derive the conditions according to Bernoulli σ_τ-shift evolution for rule 14. Then, we prove that rule 14 is chaotic in the sense of both Li-Yorke and Devaney on their attractors, respectively. We use diagrams to explain the attractor of rule 14, where characteristic function and Lameray diagram are used to describe that all points fall into Bernoulli-shift map after several iterations and to show clearly the iterative process of an attractor, respectively.

Acknowledgments

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