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Research Article

On the Dynamics of a Higher-Order Difference Equation

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This paper deals with the investigation of the following more general rational difference equation: $y_{n+1} = \alpha y_n / (\beta + \gamma \sum_{i=0}^{k} y_{n-(2i+1)}^{p} \prod_{i=0}^{k} y_{n-(2i+1)})$, $n = 0, 1, 2, \ldots$, where $\alpha, \beta, \gamma, p \in (0, \infty)$ with the initial conditions $x_0, x_{-1}, \ldots, x_{-2k}, x_{-2k-1} \in (0, \infty)$. We investigate the existence of the equilibrium points of the considered equation and then study their local and global stability. Also, some results related to the oscillation and the permanence of the considered equation have been presented.

1. Introduction

In this paper we investigate the global stability character and the oscillatory of the solutions of the following difference equation:

$$y_{n+1} = \frac{\alpha y_n}{\beta + \gamma \sum_{i=0}^{k} y_{n-(2i+1)}^{p} \prod_{i=0}^{k} y_{n-(2i+1)}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where α, β, γ, $p \in (0, \infty)$ with the initial conditions x_0, $x_{-1}, \ldots, x_{-2k}, x_{-2k-1} \in (0,\infty)$. Also we study the permanence of (1.1). The importance of permanence for biological systems was thoroughly reviewed by Huston and Schmidtt [1].

In general, there are a lot of interest in studying the global attractivity, boundedness character, and periodicity of the solutions of nonlinear difference equations. In particular there are many papers that deal with the rational difference equations and that is because many researchers believe that the results about this type of difference equations are of paramount importance in their own right, and furthermore they believe that these results offer prototype towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one.

Kulenović and Ladas [2] presented some known results and derived several new ones on the global behavior of the difference equation $x_{n+1} = (\alpha+\beta x_n+\gamma x_{n-1})/(A+Bx_n+Cx_{n-1})$ and of its special cases. Elabbasy et al. [3–5] established the solutions form and then investigated the global stability and periodicity character of the obtained solutions of the following difference equations:

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \qquad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}, \qquad x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a. \tag{1.2}$$

El-Metwally [6] gave some results about the global behavior of the solutions of the following more general rational difference equations

$$x_{n+1} = \frac{ax_{n-k_0}^{l_0}x_{n-k_1}^{l_1}\cdots x_{n-k_i}^{l_i} + bx_{n-r_0}^{s_0}x_{n-r_1}^{s_1}\cdots x_{n-r_j}^{s_j}}{cx_{n-k_0}^{l_0}x_{n-k_1}^{l_1}\cdots x_{n-k_i}^{l_i} + dx_{n-r_0}^{s_0}x_{n-r_1}^{s_1}\cdots x_{n-r_j}^{s_j}}, \qquad y_{n+1} = \frac{\alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_t y_{n-t}}{\beta_0 y_n + \beta_1 y_{n-1} + \cdots + \beta_t y_{n-t}}. \tag{1.3}$$

Çinar [7–9] obtained the solutions form of the difference equations $x_{n+1} = x_{n-1}/(1 + x_n x_{n-1})$, $x_{n+1} = x_{n-1}/(-1 + x_n x_{n-1})$ and $x_{n+1} = ax_{n-1}/(1 + bx_n x_{n-1})$. Also, Cinar et al. [10] studied the existence and the convergence for the solutions of the difference equation $x_{n+1} = x_{n-3}/(-1 + x_n x_{n-1} x_{n-2} x_{n-3})$. Simsek et al. [11] obtained the solution of the difference equation $x_{n+1} = x_{n-3}/(1 + x_{n-1})$. In [12] Yalcinkaya got the solution form of the difference equation $x_{n+1} = x_{n-(2k+1)}/(1+x_{n-k}x_{n-(2k+1)})$. In [13] Stević studied the difference equation $x_{n+1} = x_{n-1}/(1+x_n)$. Other related results on rational difference equations can be found in [14–19].

Let I be some interval of real numbers and let

$$f : I^{k+1} \longrightarrow I \tag{1.4}$$

be a continuously differentiable function. Then for every set of initial conditions x_{-k}, $x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.5}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1 (*permanence*). The difference equation (1.5) is said to be permanent if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that $m \leq x_n \leq M$ for all $n \geq N$.

Definition 1.2 (*periodicity*). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.3 (*semicycles*). A positive semicycle of a sequence $\{x_n\}_{n=-k}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$ all greater than or equal to the equilibrium point $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that either $l = -k$ or $l > -k$ and $x_{l-1} < \overline{x}$; and, either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$. A negative semicycle of a sequence $\{x_n\}_{n=-k}^{\infty}$ consists of a "string" of terms $\{x_l,\ x_{l+1}, \dots, x_m\}$ all less than the equilibrium point $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that: either $l = -k$ or $l > -k$ and $x_{l-1} \geq \overline{x}$; and, either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \overline{x}$.

Definition 1.4 (*oscillation*). A sequence $\{x_n\}_{n=-k}^{\infty}$ is called nonoscillatory about the point $\overline{x}$ if there is exists $N \geq -k$ such that either $x_n > \overline{x}$ for all $n \geq N$ or $x_n < \overline{x}$ for all $n \geq N$. Otherwise $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about $\overline{x}$.

Recall that the linearized equation of (1.5) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.6}$$

2. Dynamics of (1.1)

The change of variables $y_n = (\beta/\gamma)^{1/(p+k+1)} x_n$ reduces (1.1) to the following difference equation

$$x_{n+1} = \frac{r x_n}{1 + \sum_{i=0}^{k} x_{n-(2i+1)}^{p} \prod_{i=0}^{k} x_{n-(2i+1)}}, \quad n = 0, 1, 2, \dots, \tag{2.1}$$

where $r = \alpha/\beta$.

In this section we study the local stability character and the global stability of the equilibrium points of the solutions of (2.1). Also we give some results about the oscillation and the permanence of (2.1).

Recall that the equilibrium point of (2.1) are given by

$$\overline{x} = \frac{r\overline{x}}{1 + (k+1)\overline{x}^{p+k+1}}. \tag{2.2}$$

Then (2.1) has the equilibrium points $\overline{x} = 0$ and whenever $r > 1$, (2.1) possesses the unique equilibrium point $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$.

The following theorem deals with the local stability of the equilibrium point $\overline{x} = 0$ of (2.1).

Theorem 2.1. *The following statements are true:*

(i) *if $r < 1$, then the equilibrium point $\overline{x} = 0$ of* (2.1) *is locally asymptotically stable,*

(ii) *if $r > 1$, then the equilibrium point $\overline{x} = 0$ of* (2.1) *is a saddle point.*

Proof. The linearized equation of (2.1) about $\overline{x} = 0$ is $u_{n+1} - r u_n = 0$. Then the associated eigenvalues are $\lambda = 0$ and $\lambda = r$. Then the proof is complete. □

Theorem 2.2. *Assume that $r < 1$, then the equilibrium point $\overline{x} = 0$ of* (2.1) *is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n=-2k+1}^{\infty}$ be a solution of (2.1). It was shown by Theorem 2.1 that the equilibrium point $\overline{x} = 0$ of (2.1) is locally asymptotically stable. So, it is suffices to show that

$$\lim_{n\to\infty} x_n = 0. \tag{2.3}$$

Now it follows from (2.1) that

$$\begin{aligned} x_{n+1} &= \frac{rx_n}{1 + x_{n-1}^{p+1}x_{n-3}\cdots x_{n-2k+1} + x_{n-1}x_{n-3}^{p+1}\cdots x_{n-2k+1} + \cdots + x_{n-1}x_{n-3}\cdots x_{n-2k+1}^{p+1}} \\ &\le rx_n \\ &< x_n. \end{aligned} \tag{2.4}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and this completes the proof. □

Theorem 2.3. *Assume that $r > 1$. Then every solution of* (2.1) *is either oscillatory or tends to the equilibrium point $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$.*

Proof. Let $\{x_n\}_{n=-2k+1}^{\infty}$ be a solution of (2.1). Without loss of generality assume that $\{x_n\}_{n=-2k+1}^{\infty}$ is a nonoscillatory solution of (2.1), then it suffices to show that $\lim_{n\to\infty} x_n = \overline{x}$. Assume that $x_n \ge \overline{x}$ for $n \ge n_0$ (the case where $x_n \le \overline{x}$ for $n \ge n_0$ is similar and will be omitted). It follows from (2.1) that

$$\begin{aligned} x_{n+1} &= \frac{rx_n}{1 + x_{n-1}^{p+1}x_{n-3}\cdots x_{n-2k-1} + x_{n-1}x_{n-3}^{p+1}\cdots x_{n-2k-1} + \cdots + x_{n-1}x_{n-3}\cdots x_{n-2k-1}^{p+1}} \\ &\le x_n\left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}}\right) = x_n. \end{aligned} \tag{2.5}$$

Hence $\{x_n\}$ is monotonic for $n \ge n_0 + 2k + 1$, therefore it has a limit. Let $\lim_{n\to\infty} x_n = \mu$, and for the sake of contradiction, assume that $\mu > \overline{x}$. Then by taking the limit of both side of (2.1), we obtain $\mu = r\mu/(1 + (k+1)\mu^{p+k+1})$, which contradicts the hypothesis that $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$ is the only positive solution of (2.2). □

Theorem 2.4. *Assume that $\{x_n\}_{n=-2k+1}^{\infty}$ is a solution of* (2.1) *which is strictly oscillatory about the positive equilibrium point $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$ of* (2.1)*. Then the extreme point in any semicycle occurs in one of the first $2(k+1)$ terms of the semicycle.*

Proof. Assume that $\{x_n\}_{n=-2k-1}^{\infty}$ is a strictly oscillatory solution of (2.1). Let $N \geq 2k+2$ and let $\{x_N, x_{N+1}, \dots, x_M\}$ be a positive semicycle followed by the negative semicycle $\{x_M, x_{M+1}, \dots, x_M\}$. Now it follows from (2.1) that

$$\begin{aligned}
x_{N+2k+2} - x_N &= \frac{r x_{N+2k+1}}{1 + x_{N+2k}^{p+1} x_{N+2k-2} \cdots x_N + x_{N+2k} x_{N+2k-2}^{p+1} \cdots x_N + \cdots + x_{N+2k} \cdots x_N^{p+1}} - x_N \\
&\leq x_{N+2k+1} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_N \\
&= x_{N+2k+1} - x_N \\
&= \frac{r x_{N+2k}}{1 + x_{N+2k-1}^{p+1} x_{N+2k-3} \cdots x_{N-1} + \cdots + x_{N+2k-1} \cdots x_{N-1}^{p+1}} - x_N \\
&\leq x_{N+2k} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_N \\
&= x_{N+2k} - x_N \leq x_{N+2k-1} - x_N \leq \cdots \leq x_{N+1} - x_N \leq x_N - x_N = 0.
\end{aligned} \tag{2.6}$$

Then $x_N \geq x_{N+2(k+1)}$ for all $N \geq 2(k+1)$.

Similarly, we see from (2.1) that

$$\begin{aligned}
x_{M+2k+2} - x_M &= \frac{r x_{M+2k+1}}{1 + x_{M+2k}^{p+1} x_{M+2k-2} \cdots x_M + x_{M+2k} x_{M+2k-2}^{p+1} \cdots x_M + \cdots + x_{M+2k} \cdots x_M^{p+1}} - x_M \\
&\geq x_{M+2k+1} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_M \\
&= x_{M+2k+1} - x_M \\
&= \frac{r x_{M+2k}}{1 + x_{M+2k-1}^{p+1} x_{M+2k-3} \cdots x_{M-1} + \cdots + x_{M+2k-1} \cdots x_{M-1}^{p+1}} - x_M \\
&\geq x_{M+2k} \left(\frac{r}{1 + (k+1)\overline{x}^{p+k+1}} \right) - x_M \\
&= x_{M+2k} - x_M \leq x_{M+2k-1} - x_M \geq \cdots \geq x_{M+1} - x_M \geq x_M - x_M = 0.
\end{aligned} \tag{2.7}$$

Therefore $x_{M+2(k+1)} \geq x_M$ for all $M \geq 2(k+1)$. The proof is so complete. □

Theorem 2.5. *Equation* (2.1) *is permanent.*

Proof. Let $\{x_n\}_{n=-2k+1}^{\infty}$ be a solution of (2.1). There are two cases to consider:

(i) $\{x_n\}_{n=-2k+1}^{\infty}$ is a nonoscillatory solution of (2.1). Then it follows from Theorem 2.3 that

$$\lim_{n \to \infty} x_n = \overline{x}, \tag{2.8}$$

that is there is a sufficiently large positive integer N such that $|x_n - \overline{x}| < \varepsilon$ for all $n \geq N$ and for some $\varepsilon > 0$. So, $\overline{x} - \varepsilon < x_n < \overline{x} + \varepsilon$, this means that there are two positive real numbers, say C and D, such that

$$C \leq x_n \leq D. \tag{2.9}$$

(ii) $\{x_n\}_{n=-2k+1}^{\infty}$ is strictly oscillatory about $\overline{x} = ((r-1)/(k+1))^{1/(p+k+1)}$.

Now let $\{x_{s+1}, x_{s+2}, \ldots, x_t\}$ be a positive semicycle followed by the negative semicycle $\{x_{t+1}, x_{t+2}, \ldots, x_u\}$. If x_V and x_W are the extreme values in these positive and negative semicycle, respectively, with the smallest possible indices V and W, then by Theorem 2.4 we see that $V - s \leq 2(k+1)$ and $W - u \leq 2(k+1)$. Now for any positive indices μ and L with $\mu < L$, it follows from (2.1) for $n = \mu, \mu+1, \ldots, L-1$ that

$$\begin{aligned}
x_L &= x_{L-1}\left(\frac{r}{1 + x_{L-2}^{p+1}x_{L-4}\cdots x_{L-2k-2} + x_{L-2}x_{L-4}^{p+1}\cdots x_{L-2k-2} + \cdots + x_{L-2}x_{L-4}\cdots x_{L-2k-2}^{p+1}}\right)\\
&= \frac{r^2 x_{L-2}}{\left(1 + x_{L-3}^{p+1}\cdots x_{L-2k-3} + \cdots + x_{L-3}\cdots x_{L-2k-3}^{p+1}\right)\left(1 + x_{L-2}^{p+1}\cdots x_{L-2k-2} + \cdots + x_{L-2}\cdots x_{L-2k-2}^{p+1}\right)}\\
&\;\;\vdots\\
&= x_{L-\zeta} r^{\zeta} \prod_{\eta=1}^{\zeta}\left(\frac{1}{1 + \sum_{i=0}^{k} x_{L-(2i+1)-\eta}^{p} \prod_{i=0}^{k} x_{L-(2i+1)-\eta}}\right)\\
&= x_{\mu} r^{L-\mu} \prod_{\eta=\mu}^{L-1}\left(\frac{1}{1 + \sum_{i=0}^{k} x_{\eta-(2i+1)}^{p} \prod_{i=0}^{k} x_{\eta-(2i+1)}}\right).
\end{aligned} \tag{2.10}$$

Therefor for $V = L$ and $s = \mu$ we obtain

$$\begin{aligned}
x_V &= x_s r^{V-s} \prod_{\eta=s}^{V-1}\left(\frac{1}{1 + \sum_{i=0}^{k} x_{\eta-(2i+1)}^{p} \prod_{i=0}^{k} x_{\eta-(2i+1)}}\right)\\
&\leq \overline{x} r^{2k+1} = H.
\end{aligned} \tag{2.11}$$

Again whenever $W = L$ and $\mu = t$, we see that

$$\begin{aligned} x_W &= x_t r^{W-t} \prod_{\eta=t}^{W-1} \left(\frac{1}{1 + \sum_{i=0}^{k} x_{\eta-(2i+1)}^{p} \prod_{i=0}^{k} x_{\eta-(2i+1)}} \right) \\ &\geq \overline{x} r^{W-t} \prod_{\eta=t}^{W-1} \left(\frac{1}{1 + (k+1) H^{p+k+1}} \right) \\ &= \overline{x} r^{W-t} \left(\frac{1}{1 + (k+1) H^{p+k+1}} \right)^{W-t-1} \\ &\geq \overline{x} \left(\frac{1}{1 + (k+1) H^{p+k+1}} \right)^{2k+1} = G. \end{aligned} \tag{2.12}$$

That is, $G \leq x_n \leq H$. It follows from (i) and (ii) that

$$\min\{C, G\} \leq x_n \leq \max\{D, H\}. \tag{2.13}$$

Then the proof is complete. □

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