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Research Article

Denoising Algorithm Based on Generalized Fractional Integral Operator with Two Parameters

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In this paper, a novel digital image denoising algorithm called generalized fractional integral filter is introduced based on the generalized Srivastava-Owa fractional integral operator. The structures of $n \times n$ fractional masks of this algorithm are constructed. The denoising performance is measured by employing experiments according to visual perception and PSNR values. The results demonstrate that apart from enhancing the quality of filtered image, the proposed algorithm also reserves the textures and edges present in the image. Experiments also prove that the improvements achieved are competent with the Gaussian smoothing filter.

1. Introduction

Fractional integration and fractional differentiation are generalizations of notions of integer-order integration and differentiation and include nth derivatives and n-fold integrals as particular cases. Many applications of fractional calculus in physics amount to replace the time derivative in an evolution equation with a derivative of fractional order. Fractional calculus has been applied to a variety of physical phenomena, including anomalous diffusion, transmission line theory, problems involving oscillations, nanoplasmonics, solid mechanics, astrophysics, and viscoelasticity [1–6].

Nowadays, fractional calculus (integral and differential operators) is utilized in signal processing and image possessing. The fractional calculation enhances the quality of images, with interesting possibilities in edge detection and image restoration, to reveal faint objects in astronomical images and is devoted to astronomical images analysis [7, 8]. Furthermore, fractional calculus is employed in design problems of variables [9] and in different applications in engineering [10]. Finally, the fractional calculus (integral operators) is used in image denoising [11]. All results based on the fractional calculus operators (differential and integral) show that this method is not only effective, but also has good immunity. Therefore, the fractional calculus in the field of image processing and signal prosecuting has broad application prospect.

Many studies on fractional calculus and fractional differential equations, involving different operators, such as the Riemann-Liouville operators, the Erdélyi-Kober operators, the Weyl-Riesz operators, the Caputo operators, and the Grünwald-Letnikov operators, have evolved during the past three decades with its applications in other field. Moreover, the existence and uniqueness of holomorphic solutions for nonlinear fractional differential equations such as Cauchy problems and diffusion problems in complex domain are established and posed [12–21].

Denoising is one of the most fundamental image restoration problems in computer vision and image processing. In this paper, we have introduced an image denoising algorithm called generalized fractional integral image denoising algorithm based on the Srivastava-Owa fractional integral operator. The structures of $n \times n$ fractional masks of this algorithm are constructed. The denoising performance is measured by employing experiments according to standard of visual perception and PSNR values. This paper is organized as follows. In Section 2, we introduce the generalized integral operator. In Section 3 construction of fractional integral mask, which is the novelty of this work, is presented. The experimental results are shown in Section 4. Finally, conclusion is presented in Section 5.

2. Generalized Integral Operator

In [22], Srivastava and Owa have defined fractional operators (derivative and integral as follows.

The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta; \quad 0 \leq \alpha < 1, \tag{2.1}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0, \tag{2.2}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

In [23], Ibrahim derived a formula for the generalized fractional integral, considering for natural $n \in \mathbb{N} = \{1, 2, \dots\}$ and real μ the n-fold integral of the form

$$I_z^{\alpha,\mu} f(z) = \int_0^z \zeta_1^{\mu} d\zeta_1 \int_0^{\zeta_1} \zeta_2^{\mu} d\zeta_2 \cdots \int_0^{\zeta_{n-1}} \zeta_n^{\mu} f(\zeta_n) d\zeta_n. \tag{2.3}$$

By employing the Cauchy formula for iterated integrals yields

$$\int_0^z \zeta_1^{\mu} d\zeta_1 \int_0^{\zeta_1} \zeta^{\mu} f(\zeta) d\zeta = \int_0^z \zeta^{\mu} f(\zeta) d\zeta \int_{\zeta}^z \zeta_1^{\mu} d\zeta_1 = \frac{1}{\mu+1} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1} \right) \zeta^{\mu} f(\zeta) d\zeta. \tag{2.4}$$

Repeating the above step $n-1$ times, we have

$$\int_0^z \zeta_1^{\mu} d\zeta_1 \int_0^{\zeta_1} \zeta_2^{\mu} d\zeta_2 \cdots \int_0^{\zeta_{n-1}} \zeta_n^{\mu} f(\zeta_n) d\zeta_n = \frac{(\mu+1)^{1-n}}{(n-1)!} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1} \right)^{n-1} \zeta^{\mu} f(\zeta) d\zeta, \tag{2.5}$$

which implies the fractional operator type

$$I_z^{\alpha,\mu} f(z) = \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1} \right)^{\alpha-1} \zeta^{\mu} f(\zeta) d\zeta, \tag{2.6}$$

where α and $\mu \neq -1$ are real numbers and the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. When $\mu = 0$, we arrive at the standard Srivastava-Owa fractional integral, which is used to define the Srivastava-Owa fractional derivatives.

Corresponding to the generalized fractional integrals (2.6), we define the generalized differential operator of order α by

$$D_z^{\alpha,\mu} f(z) := \frac{(\mu+1)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^{\mu} f(\zeta)}{\left(z^{\mu+1} - \zeta^{\mu+1} \right)^{\alpha}} d\zeta, \quad 0 \leq \alpha < 1, \tag{2.7}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$.

3. Construction of Fractional Integral Mask

Using the generalized fractional integral operator defined in (2.6), we have proceeded to construct the generalized fractional integral mask which is the main contribution of this work. Since the fractional differential operator is performed if the order is positive and the fractional integral operator is performed if the order is negative; for analytic function $s(z)$, we assume that $\nu < 0$, and then

$$\begin{aligned} I_z^{\nu,\mu} s(z) &= \frac{(\mu+1)^{1+\nu}}{\Gamma(-\nu)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1} \right)^{-\nu-1} \zeta^{\mu} s(\zeta) d\zeta \\ &= \frac{(2^{\mu+1}-1)(\mu+1)^{1+\nu}}{\Gamma(-\nu)} \int_0^z \left(\zeta^{\mu+1} \right)^{-\nu-1} (z-\zeta)^{\mu} s(z-\zeta) d\zeta. \end{aligned} \tag{3.1}$$

For converting continuous integral into discrete sum. Integral section $[0, z]$ is equally transformed into the sum of the integrals of N parts, and when N is greater enough, an approximate formula can be obtained:

$$I_z^{\nu,\mu} s(z) = \frac{(2^{\mu+1} - 1)(\mu + 1)^{1+\nu}}{\Gamma(-\nu)} \sum_{k=0}^{N-1} \int_{kz/N}^{(k+1)z/N} \frac{(z-\zeta)^{\mu} s(z-\zeta)}{(\zeta^{\mu+1})^{\nu+1}} d\zeta. \tag{3.2}$$

Moreover, we can derive that

$$\begin{aligned}
\int_{kz/N}^{(k+1)z/N} \zeta^{\mu} s(\zeta) d\zeta &= \frac{s(kz/N) + s((kz+z)/N)}{2} \int_{kz/N}^{(k+1)z/N} \zeta^{\mu} d\zeta \\
&= \frac{s(kz/N) + s((kz+z)/N)}{2} \frac{\zeta^{\mu+1}}{\mu+1} \bigg|_{kz/N}^{(k+1)z/N} \\
&= \frac{s(kz/N) \ + s((kz+z)/N)}{2} \frac{((k+1)z/N)^{\mu+1} - (kz/N)^{\mu+1}}{\mu+1} \\
&= \frac{s(kz/N) + s((kz+z)/N)}{2} \left(\frac{(k+1)^{\mu+1} - k^{\mu+1}}{(\mu+1)N^{\mu+1}} \right) z^{\mu+1},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\int_{kz/N}^{(k+1)z/N} \frac{(z-\zeta)^{\mu} s(z-\zeta)}{(\zeta^{\mu+1})^{\nu+1}} d\zeta &\cong \frac{[s(z-kz/N) + s(z-(kz+z)/N)] \left[(k+1)^{\mu+1} - k^{\mu+1} \right]}{2(\mu+1)} \\
&\quad \times \frac{(\zeta^{\mu+\ 1})^{-\nu}}{-\nu} \bigg|_{kz/N}^{(k+1)z/N} \\
&= \frac{[s(z-kz/N) + s(z-((kz+z)/N))] \left[(k+1)^{\mu+1} - k^{\mu+1} \right]}{-2\nu(\mu+1)} \\
&\quad \times \left[\left(\left(\frac{kz+z}{N} \right)^{\mu+1} \right)^{-\nu} - \left((kz/N)^{\mu+1} \right)^{-\nu} \right] \\
&= \frac{(s_k + s_{k+1}) \left((k+1)^{\mu+1} - k^{\mu+1} \right)}{-2\nu(\mu+1)} \\
&\quad \times \left[\left(\left(\frac{kz+z}{N} \right)^{\mu+1} \right)^{-\nu} - \left((kz/N)^{\mu+1} \right)^{-\nu} \right].
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.2), we obtain

$$I_z^{\nu,\mu}s(z) = \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}\sum_{k=0}^{N-1}(s_k+s_{k+1})\left((k+1)^{\mu+1}-k^{\mu+1}\right)$$
$$\times\left[\left(\left(\frac{kz+z}{N}\right)^{\mu+1}\right)^{-\nu}-\left(\left(\frac{kz}{N}\right)^{\mu+1}\right)^{-\nu}\right]$$
$$= \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}s(z)+\frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}$$
$$\times\sum_{k=1}^{N-1}\left((k+1)^{\mu+1}-k^{\mu+1}\right)\left(\left((k+1)^{\mu+1}\right)^{-\nu}-\left(k^{\mu+1}\right)^{-\nu}\right)s(z-k),\quad (\nu<0,\ \mu\geq 0). \tag{3.5}$$

However, in the context of image processing, (3.5) is applied uniformly in the whole digital image and therefore should be in two directions of z and w. Thus for two variables on the negative direction of z and w coordinates, we have

$$\begin{aligned}
I_z^{\nu,\mu}s(z_-,w) &= \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}s(z,w)+\frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}\\
&\quad\times\sum_{k=1}^{N-1}\left((k+1)^{\mu+1}-k^{\mu+1}\right)\left(\left((k+1)^{\mu+1}\right)^{-\nu}-\left(k^{\mu+1}\right)^{-\nu}\right)s(z-k,w),\\
I_z^{\nu,\mu}s(z,w_-) &= \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}s(z,w)+\frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}\\
&\quad\times\sum_{k=1}^{N-1}\left((k+1)^{\mu+1}-k^{\mu+1}\right)\left(\left((k+1)^{\mu+1}\right)^{-\nu}-\left(k^{\mu+1}\right)^{-\nu}\right)s(z,w-k).
\end{aligned}\tag{3.6}$$

The next two formulae show the construction of the numerical implementation algorithm for generalized fractional integral operation in sense of the Srivastava-Owa operators

$$\begin{aligned}
I_z^{\nu,\mu}s(z_-,w) &= \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}s(z,w)+\frac{(2^{\mu+1}-1)(2^{(-\nu)(\mu+1)}-1)(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}\\
&\quad\times s(z-1,w)+\frac{(3^{\mu+1}-2^{\mu+1})(3^{(-\nu)(\mu+1)}-2^{(-\nu)(\mu+1)})(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}\\
&\quad\times s(z-2,w)+\cdots\\
&\quad+\frac{\left(n^{\mu+1}-(n-1)^{\mu+1}\right)\left(n^{(-\nu)(\mu+1)}-(n-1)^{(-\nu)(\mu+1)}\right)(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}\\
&\quad\times s(z-n+1,w),
\end{aligned}$$

$$
\begin{aligned}
I_z^{\nu,\mu} s(z, w_-) = {}& \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)} s(z,w) + \frac{(2^{\mu+1}-1)(2^{(-\nu)(\mu+1)}-1)(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)} \\
& \times s(z,w-1) + \frac{(3^{\mu+1}-2^{\mu+1})(3^{(-\nu)(\mu+1)}-2^{(-\nu)(\mu+1)})(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)} \\
& \times s(z,w-2) + \cdots \\
& + \frac{\left(n^{\mu+1}-(n-1)^{\mu+1}\right)\left(n^{(-\nu)(\mu+1)}-(n-1)^{(-\nu)(\mu+1)}\right)(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)} \\
& \times \mathrm{s}(z, w-n+1).
\end{aligned}
\tag{3.7}
$$

The nonzero values of corresponding terms in formula (3.7) are

$$
\begin{aligned}
\phi_0 &= \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}, \\
\phi_1 &= \frac{(2^{\mu+1}-1)(2^{(-\nu)(\mu+1)}-1^{-\nu(\mu+1)})(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}, \\
\phi_2 &= \frac{(3^{\mu+1}-2^{\mu+1})(3^{(-\nu)(\mu+1)}-2^{(-\nu)(\mu+1)})(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}, \\
&\vdots \\
\phi_{n-1} &= \frac{\left(n^{\mu+1}-(n-1)^{\mu+1}\right)\left(n^{(1-\nu)(\mu+1)}-(n-1)^{(1-\nu)(\mu+1)}\right)(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)}, \quad \mu \geq 0,\ \nu < 0,
\end{aligned}
\tag{3.8}
$$

which are all fractional coefficients according to the generalized Srivastava-Owa fractional integral operator. Note that when $\mu = 0$, we have the Riemann-Liouville integral operator

$$
\phi_0 = \frac{1}{\Gamma(-\nu)(-2\nu)}, \qquad \phi_1 = \frac{2^{-\nu}-1}{\Gamma(-\nu)(-2\nu)}, \qquad \phi_2 = \frac{3^{-\nu}-2^{-\nu}}{\Gamma(-\nu)(-2\nu)}, \ldots, \tag{3.9}
$$

while for two variables on the positive direction of z and w coordinates, we have obtained

$$
\begin{aligned}
I_z^{\nu,\mu} s(z_+, w) = {}& \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(-2\nu)\Gamma(-\nu)} s(z,w) + \frac{(2^{\mu+1}-1)(\mu+1)^{\nu}}{(2\nu)\Gamma(-\nu)} \\
& \times \sum_{k=1}^{N-1} \left((k+1)^{\mu+1}-k^{\mu+1}\right)\left(\left((k+1)^{\mu+1}\right)^{-\nu}-\left(k^{\mu+1}\right)^{-\nu}\right) s(z+k,w),
\end{aligned}
$$

$$I_z^{\nu,\mu} s(z, w_+) = \frac{(2^{\mu+1} - 1)(\mu + 1)^{\nu}}{(-2\nu)\Gamma(-\nu)} s(z, w) + \frac{(2^{\mu+1} - 1)(\mu + 1)^{\nu}}{(2\nu)\Gamma(-\nu)}$$
$$\times \sum_{k=1}^{N-1} \left((k+1)^{\mu+1} - k^{\mu+1}\right)\left(\left((k+1)^{\mu+1}\right)^{-\nu} - \left(k^{\mu+1}\right)^{-\nu}\right) s(z, w + k). \tag{3.10}$$

The nonzero values of corresponding terms in formula (3.10) are

$$\psi_0 = \frac{(2^{\mu+1} - 1)(\mu + 1)^{\nu}}{(-2\nu)\Gamma(-\nu)},$$
$$\psi_1 = \frac{(2^{\mu+1} - 1)(2^{(-\nu)(\mu+1)} - 1^{-\nu(\mu+1)})(2^{\mu+1} - 1)(\mu + 1)^{\nu}}{(2\nu)\Gamma(-\nu)},$$
$$\psi_2 = \frac{(3^{\mu+1} - 2^{\mu+1})(3^{(-\nu)(\mu+1)} - 2^{(-\nu)(\mu+1)})(2^{\mu+1} - 1)(\mu + 1)^{\nu}}{(2\nu)\Gamma(-\nu)},$$
$$\vdots$$
$$\psi_{n-1} = \frac{\left(n^{\mu+1} - (n-1)^{\mu+1}\right)\left(n^{(1-\nu)(\mu+1)} - (n-1)^{(1-\nu)(\mu+1)}\right)(2^{\mu+1} - 1)(\mu + 1)^{\nu}}{(2\nu)\Gamma(-\nu)}, \quad \mu \ge 0,\ \nu < 0. \tag{3.11}$$

For digital images, 2-dimensional fractional integral filter coefficients can be obtained in eight directions of 180°, 0°, 90°, 270°, 45°, 135°, 315°, 225° as shown in Figure 1. These filters are rotation invariant and are used to describe the edges present and for removing noise. They are, respectively, on the directions of negative x-coordinate, positive x-coordinate, negative y-coordinate, positive y-coordinate, right upward diagonal, left upward diagonal and right downward diagonal, and left downward diagonal, which are all implemented. In this paper, we have applied fractional mask convolution on eight directions with the gray value of corresponding digital grayscale image pixels, adding all product terms to obtain weighting sum on eight directions. For digital color images, the same algorithm which is used for gray image can be applied but it performs separately for each of the R, G, B color components (RGB).

The fractional integral filter coefficients are labeled as $f_{180}(n)$, $f_0(n)$, $f_{90}(n)$, $f_{270}(n)$, $f_{45}(n)$, $f_{135}(n)$, $f_{315}(n)$, and $f_{255}(n)$, respectively, where $n = 1, \ldots, m$ represents the location of pixel inside each mask.

The magnitude for each filter can be obtained as follows:

$$G_{180}(i, j) = \sum_{n=1}^{16} a_n(i, j) * f_{180}(n),$$
$$G_0(i, j) = \sum_{n=1}^{16} a_n(i, j) * f_0(n),$$

$$
\begin{aligned}
G_{90}(i,j) &= \sum_{n=1}^{16} a_n(i,j) * f_{90}(n), \\
G_{270}(i,j) &= \sum_{n=1}^{16} a_n(i,j) * f_{270}(n), \\
G_{45}(i,j) &= \sum_{n=1}^{16} a_n(i,j) * f_{45}(n), \\
G_{135}(i,j) &= \sum_{n=1}^{16} a_n(i,j) * f_{135}(n), \\
G_{315}(i,j) &= \sum_{n=1}^{16} a_n(i,j) * f_{315}(n), \\
G_{225}(i,j) &= \sum_{n=1}^{16} a_n(i,j) * f_{255}(n).
\end{aligned}
\tag{3.12}
$$

The proposed image denoising algorithm for grayscale images includes the following steps:

(i) set the mask window size and the values of the fractional powers μ and ν;

(ii) apply fractional mask convolution on eight directions with the gray value of corresponding image pixels, adding all product terms to obtain weighting sum on eight directions;

(iii) find the arithmetic mean of the weighting sum value on the eight directions as approximate value of fractional integral for image pixel;

(iv) repeat steps (ii) and (iii) for all image pixels;

(v) measure the PSNR for the result image.

The fractional mask convolution is performed by sliding the mask window over the image, generally starting at the top left corner of the image through all the pixels where the fractional fits entirely within the boundaries of the image. The size of the mask window and the values of the fractional powers μ and ν are chosen to achieve the requirements of image denoising. For testing we have added a Gaussian noise to the original image, and the noisy image is then used for image smoothing; therefore the PSNR can be calculated for the restored images.

4. Experimental Results

This section aims at demonstrating that the proposed image denoising algorithm using fractional integral masks has better capability than the traditional approaches for image denoising.

The test image employed here is the grayscale images "Lena," "Cameraman," "Boat" and "Peppers" with 512×512 pixels. The default Gaussian noise is added into the image with different noise variances. All filters considered operate using 3×3 processing window mask. The values of the fractional powers are taken with the range $\mu \in (10^{-3}, 10^{-2})$ and $\nu = -0.441$.

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
Ψ_{n-1}	$\cdots$	$\cdots$	$\cdots$	Ψ_2	Ψ_1	Ψ_0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(a)

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
Ψ_0	Ψ_1	Ψ_2	$\cdots$	$\cdots$	$\cdots$	Ψ_{n-1}
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(b)

0	0	0	Ψ_{n-1}	0	0	0
0	0	0	$\vdots$	0	0	0
0	0	0	$\vdots$	0	0	0
0	0	0	$\vdots$	0	0	0
0	0	0	Ψ_2	0	0	0
0	0	0	Ψ_1	0	0	0
0	0	0	Ψ_0	0	0	0

(c)

0	0	0	Ψ_0	0	0	0
0	0	0	Ψ_1	0	0	0
0	0	0	Ψ_2	0	0	0
0	0	0	$\vdots$	0	0	0
0	0	0	$\vdots$	0	0	0
0	0	0	$\vdots$	0	0	0
0	0	0	Ψ_{n-1}	0	0	0

(d)

0	0	0	0	0	0	Ψ_{n-1}
0	0	0	0	0	$\iddots$	0
0	0	0	0	$\iddots$	0	0
0	0	0	$\iddots$	0	0	0
0	0	Ψ_2	0	0	0	0
0	Ψ_1	0	0	0	0	0
Ψ_0	0	0	0	0	0	0

(e)

Ψ_{n-1}	0	0	0	0	0	0
0	$\ddots$	0	0	0	0	0
0	0	$\ddots$	0	0	0	0
0	0	0	$\ddots$	0	0	0
0	0	0	0	Ψ_2	0	0
0	0	0	0	0	Ψ_1	0
0	0	0	0	0	0	Ψ_0

(f)

Ψ_0	0	0	0	0	0	0
0	Ψ_1	0	0	0	0	0
0	0	Ψ_2	0	0	0	0
0	0	0	$\ddots$	0	0	0
0	0	0	0	$\ddots$	0	0
0	0	0	0	0	$\ddots$	0
0	0	0	0	0	0	Ψ_{n-1}

(g)

0	0	0	0	0	0	Ψ_0
0	0	0	0	0	Ψ_1	0
0	0	0	0	Ψ_2	0	0
0	0	0	$\iddots$	0	0	0
0	0	$\iddots$	0	0	0	0
0	$\iddots$	0	0	0	0	0
Ψ_{n-1}	0	0	0	0	0	0

(h)

Figure 1: Fractional integral masks on directions of 180°, 0°, 90°, 270°, 45°, 135°, 315°, and 225°.

The performance of filters was evaluated by computing the peak signal to noise ratio (PSNR) which has been wildly used in the literature [11, 24]. The value of PSNR depends totally on the size of the mask window and the values of the fractional powers μ and ν. PSNR is defined via the mean squared error (MSE) for two images I and K, where one of the images is considered the original noisy image and the other is the filtered image:

$$\begin{gathered}\mathrm{MSE} = \frac{1}{MN}\sum_{i-0}^{M-1}\sum_{j=0}^{N-1}\left[I(i,j)-K(i,j)\right]^2,\\ \mathrm{PSNR} = \frac{10\log_{10}\max(I,K)^2}{\mathrm{MSE}},\end{gathered} \tag{4.1}$$

where max is the maximum possible pixel value of the image. In grayscale image, this is equal to 255. Table 1 shows the results of PSNR values of our proposed method as compared with

Figure 2: Results of denoising by fractional integral and Gaussian smoothing filter with noise variance = 0.01.

Gaussian smoothing filter with different Gaussian variance noise values. As it can be seen, the maximum PSNR value was obtained by our proposed approach. From the human visual system effect, Figures 2, 3, 4, and 5, illustrate that the proposed denoising algorithm using fractional integral masks has good denoising performance for both testing images by different degrees of noise. The proposed algorithm not only enhances the quality of filtered image but also reserves the textures and edges present in the image. Table 2 shows the comparison of experimental results of the proposed algorithm with other denoising algorithm with the variance of noise (σ) of 10, 15, 20 and 25. The proposed algorithm for image denoising algorithm provides satisfactory results. The good PSNR of the proposed algorithm acts as one of the important parameters to judge its performance.

5. Conclusion

In this paper, a novel digital image denoising algorithm called generalized fractional integral filter based on generalized Srivastava-Owa fractional integral operator was used. Fractional

Table 1: Results of denoising by Gaussian smoothing filter and fractional integral filter with different noise variance values.

Images (512 × 512)	Gaussian with noise variance	Input PSNR (dB)	PSNR (dB) Gaussian smoothing filter	PSNR (dB) Fractional integral filter
Lena	0.01	20.0966	23.8478	31.93251
Cameraman	0.02	17.5301	21.2419	28.8437
Boats	0.03	15.6127	19.3674	27.7730
Peppers	0.05	13.74429	17.5075	25.9861

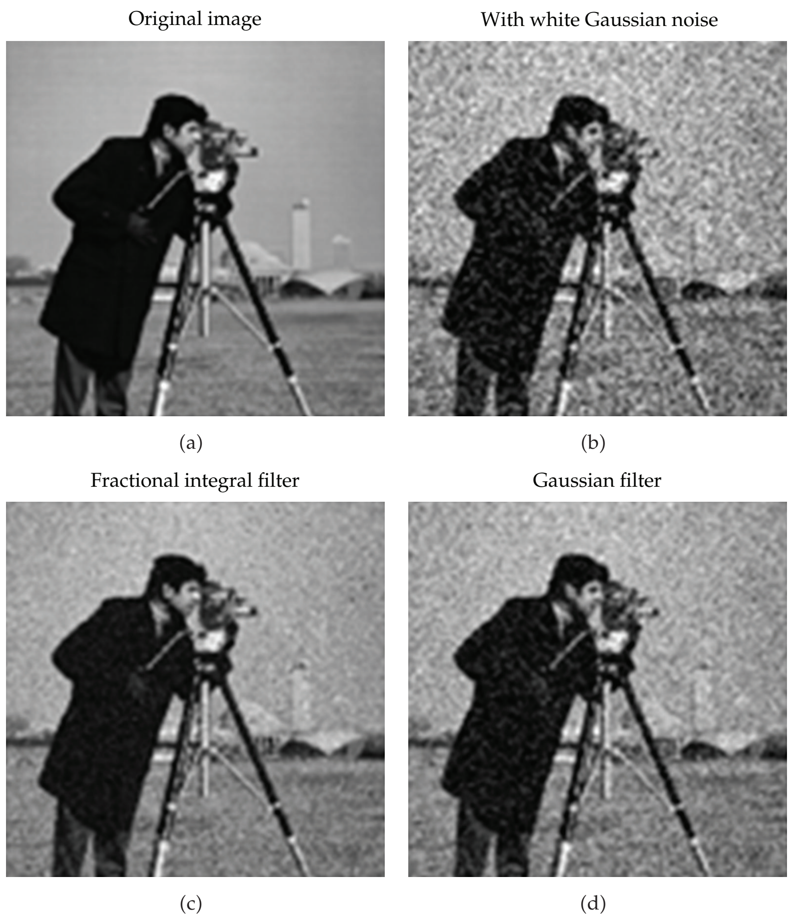

Figure 3: Results of denoising by fractional integral and Gaussian smoothing filter with noise variance = 0.02.

mask convolution on eight directions with the gray value had been applied on eight directions. The results proved that the proposed algorithm not only enhances the quality of filtered image but also reserves the textures and edges present in the image. Changing the size of the mask window or any of the values of the fractional powers μ and ν will allow adjusting the fractional integral filter coefficients to each image according to it characteristics.

Table 2: Comparison of the experimental results with other standard methods.

Boats (512 × 512)				
σ of a noise image	Input PSNR of a noise image	Proposed algorithm	[24]	[11]
10	28.49	31.23	15.74	31.2
15	25.63	29.93	14.10	29.2
20	22.15	28.01	12.65	27.91
25	20.95	27.35	11.57	26.97

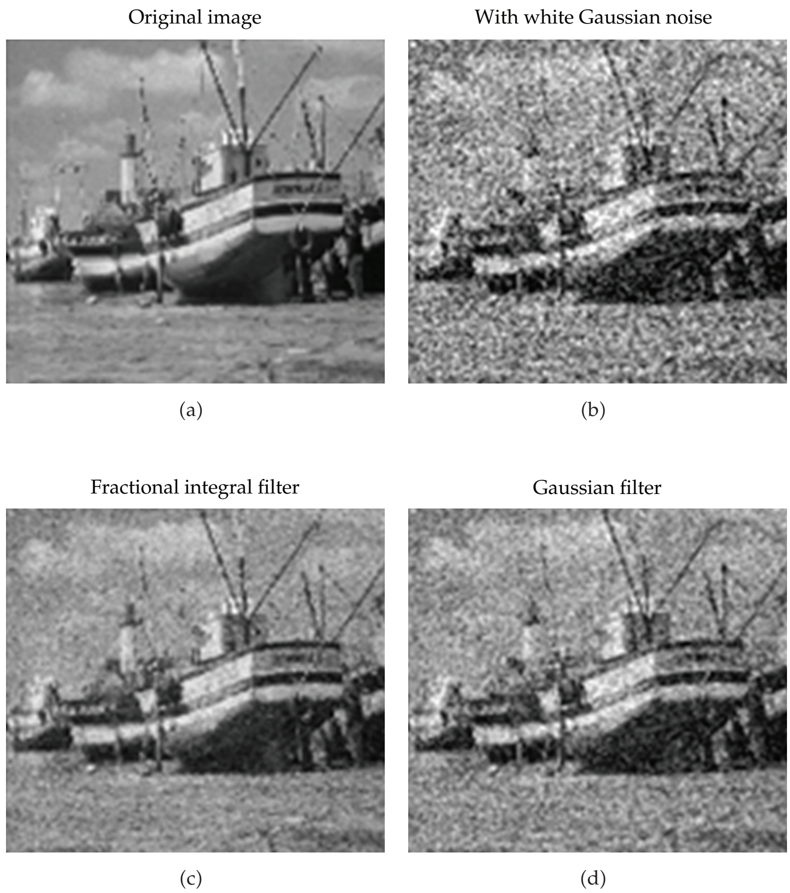

Figure 4: Results of denoising by fractional integral and Gaussian smoothing filter with noise variance = 0.03.

Several experiments proved that the improvements achieved were comparable to Gaussian smoothing filter. Besides, the proposed algorithm exhibited better PSNR than the other two standard denoising methods.

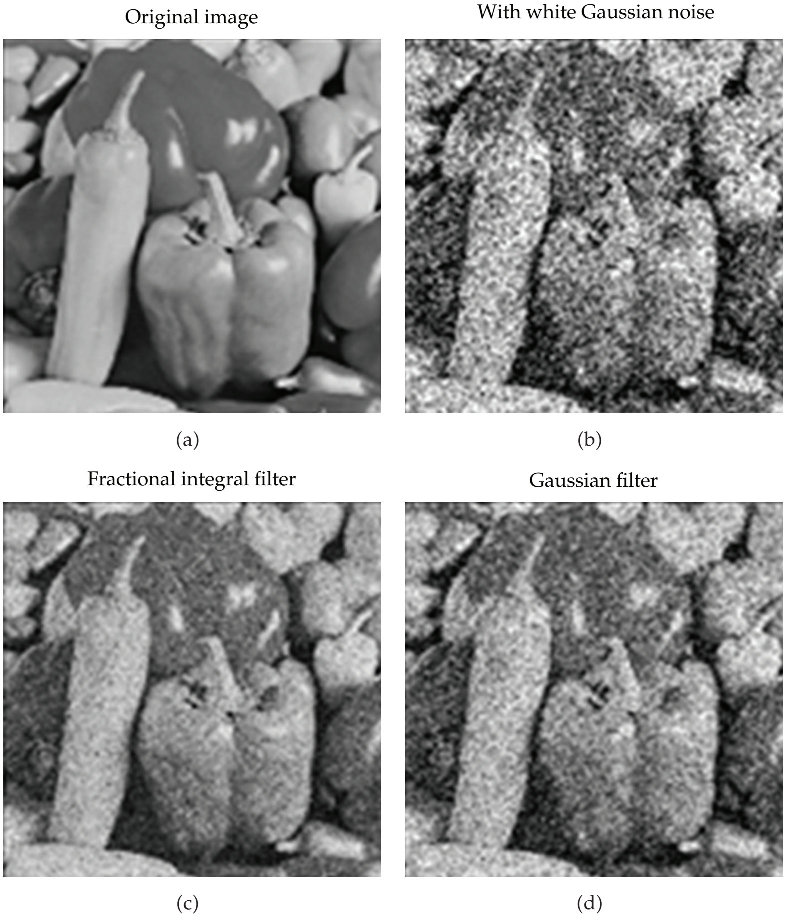

Figure 5: Results of denoising by fractional integral and Gaussian smoothing filter with noise variance = 0.05.

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