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Research Article

Some Identities on Bernoulli and Hermite Polynomials Associated with Jacobi Polynomials

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We investigate some identities on the Bernoulli and the Hermite polynomials arising from the orthogonality of Jacobi polynomials in the inner product space $\mathbf{P}_n$.

1. Introduction

For $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined as

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} 2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{\binom{n}{k}(1+\alpha+\beta+n)_k}{(\alpha+1)_k} \left(\frac{x-1}{2}\right)^k, \tag{1.1}$$

(see [1–4]), where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$.

From (1.1), we note that

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+1+n)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} \frac{\binom{n}{k}\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{x-1}{2}\right)^k. \tag{1.2}$$

By (1.2), we see that $P_n^{(\alpha,\beta)}(x)$ is polynomial of degree n with real coefficients. It is not difficult to show that the leading coefficient of $P_n^{(\alpha,\beta)}(x)$ is $2^{-n}\binom{\alpha+\beta+2n}{n}$. From (1.2), we have $P_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}$.

By (1.1), we get

$$\left(\frac{d}{dx}\right)^k P_n^{(\alpha,\beta)}(x) = 2^{-k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-k}^{(\alpha+k,\beta+k)}(x) = \frac{1}{2^k}(n+\alpha+\beta+k)(n+\alpha+\beta+k-1)\cdots(n+\alpha+\beta+1)P_{n-k}^{(\alpha+k,\beta+k)}(x), \tag{1.3}$$

where k is a positive integer (see [1–4]).

The Rodrigues' formula for $P_n^{(\alpha,\beta)}(x)$ is given by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}. \tag{1.4}$$

It is easy to show that $u = P_n^{(\alpha,\beta)}(x)$ is a solution of the following differential equation:

$$\left(1-x^2\right)u'' + \{\beta-\alpha-(\alpha+\beta+2)x\}u' + n(n+\alpha+\beta+1)u = 0. \tag{1.5}$$

As is well known, the generating function of $P_n^{(\alpha,\beta)}(x)$ is given by

$$F(x,t) = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = \frac{2^{\alpha+\beta}}{R(1-t+R)^\alpha(1+t+R)^\beta}, \tag{1.6}$$

where $R = \sqrt{1-2xt+t^2}$, (see [1–4]).

From (1.3), (1.4), and (1.6), we can derive the following identity:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{n,m}, \tag{1.7}$$

where $\delta_{n,m}$ is the Kronecker symbol.

Let $\mathbf{P}_n = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) \le n\}$. Then $\mathbf{P}_n$ is an inner product space with respect to the inner product $\langle q_1(x), q_2(x)\rangle = \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta q_1(x)q_2(x)dx$, where $q_1(x), q_2(x) \in \mathbf{P}_n$. From (1.7), we note that $\{P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_n^{(\alpha,\beta)}(x)\}$ is an orthogonal basis for $\mathbf{P}_n$.

The so-called Euler polynomials $E_n(x)$ may be defined by means of

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.8}$$

(see [5–22]), with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the Euler numbers.

The Bernoulli polynomials are also defined by the generating function to be

$$\frac{t}{e^t - 1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.9}$$

(see [11–21]), with the usual convention about replacing $B^n(x)$ by $B_n(x)$.
From (1.8) and (1.9), we note that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}x^k, \qquad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}x^k. \tag{1.10}$$

For $n \in \mathbb{Z}_+$, we have

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x), \qquad \frac{dE_n(x)}{dx} = nE_{n-1}(x) \tag{1.11}$$

(see [23–29]) By the definition of Bernoulli and Euler polynomials, we get

$$B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad E_0 = 1, \quad E_n(1) + E_n = 2\delta_{0,n}. \tag{1.12}$$

In this paper we give some interesting identities on the Bernoulli and the Hermite polynomials arising from the orthogonality of Jacobi polynomials in the inner product space $\mathbf{P}_n$.

2. Bernoulli, Euler and Jacobi Polynomials

From (1.4), we have

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}. \tag{2.1}$$

By (2.1), we have

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = \frac{1}{2\pi i}\oint \frac{(1+((x+1)/2)z)^{n+\alpha}(1+((x-1)/2)z)^{n+\beta}}{z^{n+1}}dz, \tag{2.2}$$

where we assume $x \neq \pm 1$ and circle around 0 is taken so small that $-2(x \pm 1)^{-1}$ lie neither on it nor in its interior. It is not so difficult to show that $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$.

For $q(x) \in \mathbf{P}_n$, let

$$q(x) = \sum_{k=0}^{n} C_k P_k^{(\alpha,\beta)}(x), \quad (C_k \in \mathbb{R}). \tag{2.3}$$

From (1.7), we note that

$$\begin{aligned}\left\langle q(x), P_k^{(\alpha,\beta)}(x)\right\rangle &= C_k\left\langle P_k^{(\alpha,\beta)}(x), P_k^{(\alpha,\beta)}(x)\right\rangle \\ &= C_k\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left(P_k^{(\alpha,\beta)}(x)\right)^2dx \\ &= C_k\frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}.\end{aligned} \tag{2.4}$$

Thus, by (2.4), we get

$$C_k = \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x)q(x)dx. \tag{2.5}$$

Therefore, by (1.7), (2.3), and (2.5), we obtain the following proposition.

Proposition 2.1. *For $q(x)\in\mathbf{P}_n (n\in\mathbb{N})$, one has*

$$q(x) = \sum_{k=0}^{n}C_kP_k^{(\alpha,\beta)}(x), \tag{2.6}$$

where

$$C_k = \frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x)^{k+\alpha}(1+x)^{k+\beta}\right)q(x)dx. \tag{2.7}$$

Let us take $q(x)=x^n\in\mathbf{P}_n$. First, we consider the following integral:

$$\begin{aligned}&\int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}q(x)dx \\ &\quad= \int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}x^n dx \\ &\quad= (-n)\int_{-1}^{1}\left(\frac{d}{dx}\right)^{k-1}\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}x^{n-1}dx \\ &\quad= \cdots \\ &\quad= (-1)^k\frac{n!}{(n-k)!}\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}x^{n-k}dx\end{aligned}$$

$$= \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \int_0^1 y^{k+\beta} (1-y)^{k+\alpha} (2y-1)^{n-k} dy$$

$$= \frac{(-1)^k n!}{(n-k)!} 2^{2k+\alpha+\beta+1} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \mathrm{B}(k+l+\beta+1, k+\alpha+1)$$

$$= \frac{(-1)^k n!}{(n-k)!} 2^{2k+\alpha+\beta+1} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \frac{\Gamma(k+l+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+l+2)}. \tag{2.8}$$

From (2.5) and (16), we have

$$\begin{aligned} C_k &= \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\ &\quad \times \int_{-1}^1 \left(\frac{d}{dx}\right)^k \left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\} x^n dx \\ &= \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \cdot \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \\ &\quad \times \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \frac{\Gamma(k+l+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+l+2)} \\ &= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1) n! 2^k}{\Gamma(\beta+k+1)(n-k)!} \sum_{l=0}^{n-k} \frac{(-1)^{n-k-l}\binom{n-k}{l} 2^l \Gamma(k+l+\beta+1)}{\Gamma(2k+\alpha+\beta+l+2)}. \end{aligned} \tag{2.9}$$

By Proposition 2.1, we get

$$\begin{aligned} x^n = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} &\left(\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)(n-k)!} 2^k \right) \\ &\times \left(\frac{(-1)^{n-k-l}\binom{n-k}{l} 2^l \Gamma(k+l+\beta+1)}{\Gamma(2k+\alpha+\beta+l+2)} \right) P_k^{(\alpha,\beta)}(x). \end{aligned} \tag{2.10}$$

From (1.9), we have

$$e^{xt} = \frac{1}{t} \frac{t}{e^t - 1} e^{xt}(e^t - 1) = \sum_{n=0}^{\infty} \left(\frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1} \right) \frac{t^n}{n!}. \tag{2.11}$$

By (2.11), we get

$$x^n = \frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1}, \quad (n \in \mathbb{Z}_+). \tag{2.12}$$

Therefore, by (2.10) and (2.12), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\frac{1}{(n+1)!}\{B_{n+1}(x+1)-B_{n+1}(x)\} \\
&\quad=\sum_{k=0}^{n}\left(\sum_{l=0}^{n-k}\frac{(-1)^{n-k-l}2^{k+l}(2k+\alpha+\beta+1)\binom{n-k}{l}}{\Gamma(k+\beta+1)\Gamma(2k+\alpha+\beta+l+2)(n-k)!}\right. \\
&\qquad\left.\times\Gamma(k+\alpha+\beta+1)\Gamma(k+l+\beta+1)\right)P_k^{(\alpha,\beta)}(x).
\end{aligned} \tag{2.13}$$

Let us take $q(x)=B_n(x)\in\mathbf{P}_n$. Then we evaluate the following integral:

$$\begin{aligned}
&\int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}B_n(x)dx \\
&\quad=\sum_{l=k}^{n}\binom{n}{l}B_{n-l}\frac{(-1)^k l!}{(l-k)!}2^{2k+\alpha+\beta+1}\int_0^1 y^{k+\beta}(1-y)^{k+\alpha}(2y-1)^{l-k}dy \\
&\quad=\sum_{l=k}^{n}\binom{n}{l}B_{n-l}\frac{(-1)^k l!}{(l-k)!}2^{2k+\alpha+\beta+1}\sum_{m=0}^{l-k}\binom{l-k}{m}2^m(-1)^{l-k-m} \\
&\qquad\times\frac{\Gamma(k+m+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+m+2)} \\
&\quad=\sum_{l=k}^{n}\sum_{m=0}^{l-k}\frac{\binom{n}{l}B_{n-l}(-1)^{l-m}l!2^{2k+\alpha+\beta+1}\binom{l-k}{m}2^m\Gamma(k+m+\beta+1)\Gamma(k+\alpha+1)}{(l-k)!\Gamma(2k+\alpha+\beta+m+2)}.
\end{aligned} \tag{2.14}$$

Finding (2.5) and (21), we have

$$\begin{aligned}
C_k&=\frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\quad\times\int_{-1}^{1}\left(\frac{d^k}{dx^k}(1-x)^{k+\alpha}(1+x)^{k+\beta}\right)B_n(x)dx \\
&=\sum_{l=k}^{n}\sum_{m=0}^{l-k}\frac{2^{k+m}\binom{n}{l}B_{n-l}(-1)^{l-m-k}l!(2k+\alpha+\beta+1)\binom{l-k}{m}}{\Gamma(\beta+k+1)(l-k)!\Gamma(2k+\alpha+\beta+m+2)} \\
&\quad\times\Gamma(k+m+\beta+1)\Gamma(k+\alpha+\beta+1).
\end{aligned} \tag{2.15}$$

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$B_n(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{l-k}\frac{2^{k+m}\binom{n}{l}B_{n-l}(-1)^{l-m-k}l!(2k+\alpha+\beta+1)\binom{l-k}{m}}{\Gamma(\beta+k+1)(l-k)!\Gamma(2k+\alpha+\beta+m+2)}\right.$$
$$\left.\times\Gamma(k+m+\beta+1)\Gamma(k+\alpha+\beta+1)\right)P_k^{(\alpha,\beta)}(x). \tag{2.16}$$

Let $q(x) = P_n^{(\alpha,\beta)}(x) \in \mathbf{P}_n$. From Proposition 2.1 , we firstly evaluate the following integral:

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}P_n^{(\alpha,\beta)}(x)dx$$
$$= (-1)^k\frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}P_{n-k}^{(\alpha+k,\beta+k)}(x)dx. \tag{2.17}$$

By (2.1) and (2.17), we get

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}P_n^{(\alpha,\beta)}(x)dx$$
$$= \frac{(-1)^k}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{l=0}^{n-k}\binom{n+\alpha}{n-k-l}\binom{n+\beta}{l}$$
$$\times\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}\left(\frac{x-1}{2}\right)^l\left(\frac{x+1}{2}\right)^{n-k-l}dx$$
$$= \frac{(-1)^k}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{l=0}^{n-k}\binom{n+\alpha}{n-k-l}\binom{n+\beta}{l}(-1)^l 2^{2k+\alpha+\beta+1}$$
$$\times\int_{0}^{1}(1-y)^{k+\alpha+l}y^{n+\beta-l}dy$$
$$= (-1)^k 2^{\alpha+\beta+k+1}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{l=0}^{n-k}\binom{n+\alpha}{n-k-l}\binom{n+\beta}{l}(-1)^l$$
$$\times B(k+\alpha+l+1, n+\beta-l+1)$$
$$= (-1)^k 2^{\alpha+\beta+k+1}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{l=0}^{n-k}\binom{n+\alpha}{n-k-l}\binom{n+\beta}{l}(-1)^l$$

$$\times \frac{\Gamma(\alpha+k+l+1)\Gamma(n+\beta-l+1)}{\Gamma(\alpha+\beta+k+n+2)}$$
$$= (-1)^k 2^{\alpha+\beta+k+1} \frac{1}{\Gamma(n+\alpha+\beta+1)} \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l} \binom{n+\beta}{l} (-1)^l$$
$$\times \frac{\Gamma(\alpha+k+l+1)\Gamma(n+\beta-l+1)}{(\alpha+\beta+k+n+1)}. \tag{2.18}$$

It is easy to show that

$$\begin{aligned}\frac{\Gamma(n+\beta-l+1)}{\Gamma(\beta+k+1)} &= \frac{(n+\beta-l)\cdots\beta\Gamma(\beta)}{(\beta+k)\cdots\beta\Gamma(\beta)} = (n+\beta-l)\cdots(\beta+k+1) \\ &= \binom{n+\beta-l}{n-k-l}(n-k-l)!.\end{aligned} \tag{2.19}$$

From (2.5), (2.18), and (2.19), we can derive the following equation:

$$\begin{aligned}C_k &= \frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\ &\quad\times \int_{-1}^{1} \left(\frac{d}{dx}\right)^k \left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\} P_n^{(\alpha,\beta)}(x)dx \\ &= \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)}{\Gamma(\beta+k+1)\Gamma(n+\alpha+\beta+1)} \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l}\binom{n+\beta}{l}\binom{\alpha+k+l}{l} \\ &\quad\times l!(-1)^l \frac{\Gamma(n+\beta-l+1)}{(\alpha+\beta+k+n+1)} \\ &= (2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1) \sum_{l=0}^{n-k} \binom{n+\alpha}{n-k-l}\binom{n+\beta}{l}\binom{n+\beta-l}{n-k-l} \\ &\quad\times \frac{(n-k-l)!l!}{\alpha+\beta+k+n+1}(-1)^l.\end{aligned} \tag{2.20}$$

Therefore, by Proposition 2.1, we obtain the following theorem.

Theorem 2.4. *For* $(n \in \mathbb{Z}_+)$*, one has*

$$\frac{\Gamma(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)}{\Gamma(\alpha+\beta+1)} = \sum_{k=0}^{n}\left\{\sum_{l=0}^{n-k}(2k+\alpha+\beta+1)\binom{\alpha+\beta+k}{k}\binom{n+\alpha}{n-k-l} \times\binom{n+\beta}{l}\binom{\alpha+k+l}{l}\binom{n+\beta-l}{n-k-l}\frac{(-1)^l(n-k-l)!k!l!}{\alpha+\beta+n+k+1}\right\}P_k^{(\alpha,\beta)}(x). \tag{2.21}$$

Let $H_n(x)$ be the Hermite polynomial with

$$H_n(x) = q(x) = \sum_{k=0}^{n} C_k P_k^{(\alpha,\beta)}(x), \tag{2.22}$$

where

$$C_k = \frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times\int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}H_n(x)dx. \tag{2.23}$$

Integrating by parts, one has

$$\begin{aligned}
&\int_{-1}^{1}\left(\frac{d}{dx}\right)^k\left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\right\}H_n(x)dx \\
&= \frac{2^k(-1)^k n!}{(n-k)!}\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}H_{n-k}(x)dx \\
&= \frac{2^k(-1)^k n!}{(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}H_{n-k-l}2^l\int_{-1}^{1}(1-x)^{k+\alpha}(1+x)^{k+\beta}x^l dx \\
&= \frac{2^k(-1)^k n!}{(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}H_{n-k-l}2^{2k+\alpha+\beta+l+1}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}2^m \\
&\quad\times\int_{0}^{1}(1-y)^{k+\alpha}y^{k+\beta+m}dy \\
&= \frac{2^k(-1)^k n!}{(n-k)!}\sum_{l=0}^{n-k}\sum_{m=0}^{l}\binom{n-k}{l}\binom{l}{m}H_{n-k-l}(-1)^{l-m}2^{2k+\alpha+\beta+m+l+1} \\
&\quad\times\frac{\Gamma(k+\alpha+1)\Gamma(\beta+k+m+1)}{\Gamma(2k+\alpha+\beta+m+2)}.
\end{aligned} \tag{2.24}$$

By (2.23) and (29), we get

$$C_k = \sum_{l=0}^{n-k}\sum_{m=0}^{l} \frac{\binom{n-k}{l}\binom{l}{m} H_{n-k-l}(-1)^{l-m}(2k+\alpha+\beta+1)\binom{\alpha+\beta+k}{k}k!}{(\alpha+\beta+1)\binom{2k+\alpha+\beta+m+1}{m+2k}(m+2k)!(n-k)!} \times 2^{2k+m+l} n! \binom{\beta+k+m}{m} m!. \tag{2.25}$$

Therefore, by (2.22) and (2.25), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$\frac{(\alpha+\beta+1)H_n(x)}{n!} = \sum_{k=0}^{n}\left\{\sum_{l=0}^{n-k}\sum_{m=0}^{l} \frac{\binom{n-k}{l}\binom{l}{m} H_{n-k-l}(-1)^{l-m}(2k+\alpha+\beta+1)}{\binom{2k+\alpha+\beta+m+1}{m+2k}(m+2k)!(n-k)!} \times \binom{\alpha+\beta+k}{k} k! 2^{2k+m+l} \binom{\beta+k+m}{m} m! \right\} P_k^{(\alpha,\beta)}(x), \tag{2.26}$$

where H_n is the nth Hermite number.

Remark 2.6. By the same method as Theorem 2.3 , we get

$$\frac{1}{2n!}\{E_n(x+1)+E_n(x)\} = \sum_{k=0}^{n}\left(\sum_{l=0}^{n-k} \frac{2^{k+l}(2k+\alpha+\beta+1)\binom{n-k}{l}(-1)^{n-k-l}}{\Gamma(k+\beta+1)\Gamma(2k+\alpha+\beta+l+2)(n-k)!} \times \Gamma(k+\alpha+\beta+1)\Gamma(k+l+\beta+1)\right) P_k^{(\alpha,\beta)}(x). \tag{2.27}$$

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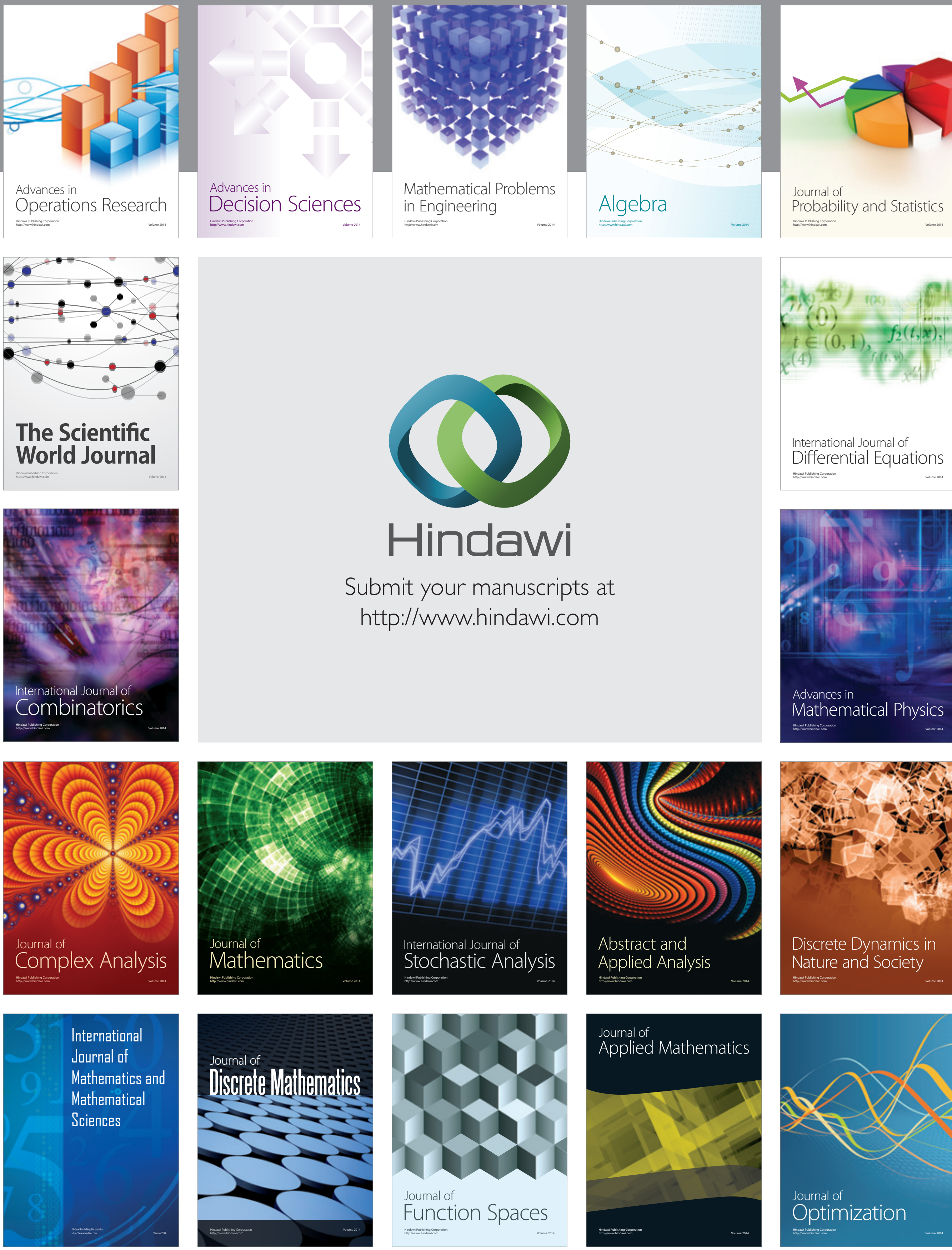
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