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Research Article

Almost Periodic Solutions of a Discrete Mutualism Model with Variable Delays

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We discuss a discrete mutualism model with variable delays of the forms $N_1(n+1) = N_1(n)\exp\{r_1(n)[(K_1(n)+\alpha_1(n)N_2(n-\mu_2(n)))/1+N_2(n-\mu_2(n))) - N_1(n-\nu_1(n))]\}$, $N_2(n+1) = N_2(n)\exp\{r_2(n)[(K_2(n)+\alpha_2(n)N_1(n-\mu_1(n)))/(1+N_1(n-\mu_1(n))) - N_2(n-\nu_2(n))]\}$. By means of an almost periodic functional hull theory, sufficient conditions are established for the existence and uniqueness of globally attractive almost periodic solution to the previous system. Our results complement and extend some scientific work in recent years. Finally, some examples and numerical simulations are given to illustrate the effectiveness of our main results.

1. Introduction

All species on the earth are closely related to other species. In a simple view, the interaction between a pair of species can be classified into three typical categories: predation (one gains and the other suffers) $(+,-)$, competition $(-,-)$, and mutualism $(+,+)$ (see [1]). In recent years, the concern for mutualism is growing, since most of the world's biomass is dependent on mutualism (see [1, 2]). For example, microbial species influence the abundances and ecological functions of related species (see [3–5]). Many bacterial species coexist in a syntrophic association (obligate mutualism); that is, one species lives off the products of another species. So far, mathematical models for mutualisms have often been neglected in many ecological textbooks.

The variation of the environment plays an important role in many biological and ecological dynamical systems. As pointed out in [6, 7], a periodically varying environment and an almost periodically varying environment are foundations for the theory of natural selection. Compared with periodic effects, almost periodic effects are more frequent. Hence, the effects of the almost periodic environment on the evolutionary theory have been the object of intensive analysis by numerous authors, and some of these results can be found in [8–12]. On the other hand, discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. In the last ten years, the dynamic behavior (the existence of positive periodic or almost periodic solutions, permanence, oscillation, and stability) of discrete biological systems has attracted much attention. We refer the reader to [13–19] and the references cited therein.

In paper [15], Wang and Li considered the following discrete mutualism model:

$$\begin{aligned} N_1(n+1) &= N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)N_2(n)}{1+N_2(n)} - N_1(n)\right]\right\}, \\ N_2(n+1) &= N_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n)+\alpha_2(n)N_1(n)}{1+N_1(n)} - N_2(n)\right]\right\}, \end{aligned} \tag{1.1}$$

where N_i $(i = 1, 2)$ are the density of ith mutualist species. By using the main result obtained by Zhang [20], they studied the existence and uniformly asymptotically stability of a unique almost periodic solution of system (1.1).

In biological phenomena, the rate of variation in the system state depends on past states. This characteristic is called a delay or a time delay. Time delay phenomena were first discovered in biological systems. They are often a source of instability and poor control performance. Time-delay systems have attracted the attention of many researchers [8, 10, 12, 16, 18, 21–23] because of their importance and widespread occurrence. Specially, in the real world, the delays in differential equations of biological phenomena are usually time-varying. Thus, it is worthwhile continuing to study the existence and stability of a unique almost periodic solution of the discrete mutualism model with time varying delays.

In this paper, we investigate a discrete mutualism model with variable delays of the form

$$\begin{aligned} N_1(n+1) &= N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)N_2(n-\mu_2(n))}{1+N_2(n-\mu_2(n))} - N_1(n-\nu_1(n))\right]\right\}, \\ N_2(n+1) &= N_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n)+\alpha_2(n)N_1(n-\mu_1(n))}{1+N_1(n-\mu_1(n))} - N_2(n-\nu_2(n))\right]\right\}, \end{aligned} \tag{1.2}$$

where all coefficients of system (1.2) are almost periodic sequences, and μ_i and ν_i are two nonnegative integer valued sequences, $i = 1, 2$.

In recent years, there are some research papers on the dynamic behavior (existence, uniqueness, and stability) of almost periodic solution of discrete biological models with constant delays (see [24–26]). However, there are few papers concerning the discrete biological models with variable delays such as system (1.2). Motivated by the previous reason, our purpose of this paper is to establish sufficient conditions for the existence and uniqueness of globally attractive almost periodic solution of system (1.2) by means of an almost periodic functional hull theory.

For any bounded sequence $\{f(n)\}$ defined on $\mathbb{Z}$, $f^u = \sup_{n\in\mathbb{Z}}\{f(n)\}$, $f^l = \inf_{n\in\mathbb{Z}}\{f(n)\}$. Let $[a,b]_{\mathbb{Z}} \overset{\text{def}}{=} [a,b]\cap\mathbb{Z}$, for all $a,b\in\mathbb{R}$.

Throughout this paper, we assume that

(H_1) $\{r_i(n)\}$, $\{\alpha_i(n)\}$, $\{K_i(n)\}$, $\{\mu_i(n)\}$, and $\{\nu_i(n)\}$ are bounded nonnegative almost periodic sequences such that

$$0 < r_i^l \le r_i(n) \le r_i^u, \quad 0 < K_i^l \le K_i(n) \le K_i^u, \quad 0 < \alpha_i^l \le \alpha_i(n) \le \alpha_i^u, \quad i = 1,2. \tag{1.3}$$

Let $\rho \overset{\text{def}}{=} \max_{i=1,2}\{\mu_i^u, \nu_i^u\}$. We consider system (1.2) together with the following initial condition:

$$N_i(\theta) = \varphi_i(\theta) \ge 0, \quad \theta \in [-\rho, 0]_{\mathbb{Z}}, \varphi_i(0) > 0, i = 1,2. \tag{1.4}$$

One can easily show that the solutions of system (1.2) with initial condition (1.4) are defined and remain positive for $n \in \mathbb{Z}^+$.

The organization of this paper is as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, global attractivity of system (1.2) is investigated. In Section 4, by means of an almost periodic functional hull theory, some sufficient conditions are established for the existence and uniqueness of almost periodic solution of system (1.2). Three illustrative examples are given in Section 5.

2. Preliminaries

Now, let us state the following definitions and lemmas, which will be useful in proving our main result.

Definition 2.1 (see [27]). A sequence $x : \mathbb{Z} \to \mathbb{R}$ is called an *almost periodic sequence* if the ϵ-translation set of x

$$E\{\epsilon, x\} = \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| < \epsilon,\ \forall n \in \mathbb{Z}\} \tag{2.1}$$

is a relatively dense set in $\mathbb{Z}$ for all $\epsilon > 0$; that is, for any given $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer $\tau \in E\{\epsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \epsilon, \quad \forall n \in \mathbb{Z}. \tag{2.2}$$

τ is called the ϵ-translation number or ϵ-almost period.

Definition 2.2 (see [27]). Let $f : \mathbb{Z}\times\mathbb{D} \to \mathbb{R}$, where $\mathbb{D}$ is an open set in $C := \{\phi : [-\tau, 0]_{\mathbb{Z}} \to \mathbb{R}\}$. $f(n,\phi)$ is said to be almost periodic in n uniformly for $\phi \in \mathbb{D}$, or uniformly almost periodic for short, if for any $\epsilon > 0$ and any compact set $\mathbb{S}$ in $\mathbb{D}$ there exists a positive integer $l(\epsilon, \mathbb{S})$ such that any interval of length $l(\epsilon, \mathbb{S})$ contains an integer τ for which

$$|f(n+\tau, \phi) - f(n, \phi)| < \epsilon, \quad \forall n \in \mathbb{Z}, \ \phi \in \mathbb{S}. \tag{2.3}$$

τ is called the ϵ-translation number of $f(n, \phi)$.

Definition 2.3 (see [27]). The hull of f, denoted by $H(f)$, is defined by

$$H(f) = \left\{ g(n,x) : \lim_{k \to \infty} f(n+\tau_k, x) = g(n,x) \text{ uniformly on } \mathbb{Z} \times \mathbb{S} \right\} \tag{2.4}$$

for some sequence $\{\tau_k\}$, where $\mathbb{S}$ is any compact set in $\mathbb{D}$.

Definition 2.4. Suppose that (N_1, N_2) is any solution of system (1.2). (N_1, N_2) is said to be a strictly positive solution on $\mathbb{Z}$ if for $n \in \mathbb{Z}$,

$$0 < \inf_{n\in\mathbb{Z}} N_i(n) \le \sup_{n\in\mathbb{Z}} N_i(n) < \infty, \quad i = 1, 2. \tag{2.5}$$

Lemma 2.5 (see [27]). *$\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{h'_k\} \subset \mathbb{Z}$ there exists a subsequence $\{h_k\} \subset \{h'_k\}$ such that $x(n+h_k)$ converges uniformly on $n \in \mathbb{Z}$ as $k \to +\infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Let

$$\begin{gathered}
F_1 \stackrel{\text{def}}{=} \frac{1}{r_1^l} \exp\{r_1^u (K_1^u + \alpha_1^u)(\nu_1^u + 1) - 1\}, \qquad F_2 \stackrel{\text{def}}{=} \frac{1}{r_2^l} \exp\{r_2^u (K_2^u + \alpha_2^u)(\nu_2^u + 1) - 1\}, \\
A_1 \stackrel{\text{def}}{=} \frac{r_1^l K_1^l}{r_1^u (1+F_2)} \exp\left\{ \frac{r_1^l K_1^l \nu_1^u}{1+F_2} - r_1^u F_1 \nu_1^u \right\}, \\
B_1 \stackrel{\text{def}}{=} A_1 \exp\left\{ \frac{r_1^l K_1^l}{1+F_2} - r_1^u F_1 \exp\left[r_1^u F_1 \nu_1^u - \frac{r_1^l K_1^l \nu_1^u}{1+F_2} \right] \right\}, \\
A_2 \stackrel{\text{def}}{=} \frac{r_2^l K_2^l}{r_2^u (1+F_1)} \exp\left\{ \frac{r_2^l K_2^l \nu_2^u}{1+F_1} - r_2^u F_2 \nu_2^u \right\}, \\
B_2 \stackrel{\text{def}}{=} A_2 \exp\left\{ \frac{r_2^l K_2^l}{1+F_1} - r_2^u F_2 \exp\left[r_2^u F_2 \nu_2^u - \frac{r_2^l K_2^l \nu_2^u}{1+F_1} \right] \right\}, \\
f_1 \stackrel{\text{def}}{=} \min\{A_1, B_1\}, \qquad f_2 \stackrel{\text{def}}{=} \min\{A_2, B_2\}.
\end{gathered} \tag{2.6}$$

In paper [28], Chen obtained the permanence of system (1.2) as follows.

Lemma 2.6 (see [28]). *Assume that* (H_1) *holds; then every solution* (N_1, N_2) *of system* (1.2) *satisfies*

$$f_i \le \lim_{n\to\infty}\inf N_i(n) \le \lim_{n\to\infty}\sup N_i(n) \le F_i, \quad i = 1,2. \tag{2.7}$$

In this section, we obtain the following permanence result of system (1.2).

Lemma 2.7. *Assume that* (H_1) *holds; then every solution* (N_1, N_2) *of system* (1.2) *satisfies*

$$g_i \le \lim_{n\to\infty}\inf N_i(n) \le \lim_{n\to\infty}\sup N_i(n) \le G_i, \tag{2.8}$$

where

$$\begin{aligned} G_i &\overset{\text{def}}{=} (K_i^u + \alpha_i^u)\exp\{r_i^u(K_i^u + \alpha_i^u)(\nu_i^u + 1)\}, \\ g_i &\overset{\text{def}}{=} \frac{K_1^l}{1+G_2}\exp\left\{r_1^u\left[\frac{K_1^l}{1+G_2} - G_1\right](\nu_1^u + 1)\right\}, \quad i = 1,2. \end{aligned} \tag{2.9}$$

Proof. Let (N_1, N_2) be any positive solution of system (1.2) with initial condition (1.4). From the first equation of system (1.2), it follows that

$$\begin{aligned} N_1(n+1) &\le N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n) + \alpha_1(n)N_2(n-\mu_2(n))}{1+N_2(n-\mu_2(n))}\right]\right\} \\ &= N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)}{1+N_2(n-\mu_2(n))} + \frac{\alpha_1(n)N_2(n-\mu_2(n))}{1+N_2(n-\mu_2(n))}\right]\right\} \\ &\le N_1(n)\exp\{r_1(n)[K_1(n) + \alpha_1(n)]\} \\ &\le N_1(n)\exp\{r_1^u(K_1^u + \alpha_1^u)\}, \end{aligned} \tag{2.10}$$

which yields that

$$N_1(n - \nu_1(n)) \ge N_1(n)\exp\{-r_1^u(K_1^u + \alpha_1^u)\nu_1^u\}, \tag{2.11}$$

which implies that

$$N_1(n+1) \le N_1(n)\exp\{r_1(n)[(K_1^u + \alpha_1^u) - N_1(n)\exp\{-r_1^u(K_1^u + \alpha_1^u)\nu_1^u\}]\}. \tag{2.12}$$

First, we present two cases to prove that

$$\lim_{n\to\infty}\sup N_1(n) \le G_1. \tag{2.13}$$

Case I. There exists $l_0 \in \mathbb{Z}^+$ such that $N_1(l_0+1) \geq N_1(l_0)$. Then, by (2.12), we have

$$\left(K_1^u + \alpha_1^u\right) - N_1(l_0)\exp\left\{-r_1^u\left(K_1^u + \alpha_1^u\right)\nu_1^u\right\} \geq 0, \tag{2.14}$$

which implies that $N_1(l_0) \leq (K_1^u + \alpha_1^u)\exp\{r_1^u(K_1^u + \alpha_1^u)\nu_1^u\} \leq G_1$. From (2.12), we get

$$N_1(l_0+1) \leq N_1(l_0)\exp\left\{r_1^u\left(K_1^u + \alpha_1^u\right)\right\} \leq \left(K_1^u + \alpha_1^u\right)\exp\left\{r_1^u\left(K_1^u + \alpha_1^u\right)\left(\nu_1^u + 1\right)\right\} \stackrel{\text{def}}{=} G_1. \tag{2.15}$$

We claim that

$$N_1(n) \leq G_1, \quad \forall n \geq l_0. \tag{2.16}$$

In fact, if there exists an integer $k_0 \geq l_0 + 2$ such that $N_1(k_0) > G_1$, and letting k_1 be the least integer between l_0 and k_0 such that $N_1(k_1) = \max_{l_0 \leq n \leq k_0}\{N_1(n)\}$, then $k_1 \geq l_0 + 2$ and $N_1(k_1) > N_1(k_1 - 1)$, which implies from the argument as that in (2.15) that

$$N_1(k_1) \leq G_1 < N_1(k_0). \tag{2.17}$$

This is impossible. This proves the claim.

Case II. $N_1(n) \geq N_1(n+1)$, for all $n \in \mathbb{Z}^+$. In particular, $\lim_{n\to\infty} N_1(n)$ exists, denoted by $\overline{N}_1$. Taking limit in the first equation of system (1.2) gives

$$\lim_{n\to\infty}\left[\frac{K_1(n) + \alpha_1(n)N_2(n - \mu_2(n))}{1 + N_2(n - \mu_2(n))} - N_1(n - \nu_1(n))\right] = 0. \tag{2.18}$$

Hence, $\overline{N}_1 \leq (K_1^u + \alpha_1^u) \leq G_1$. This proves the claim.

So, $\limsup_{n\to\infty} N_1(n) \leq G_1$. In view of the second equation of system (1.2), similar to the previous analysis, we can obtain

$$\limsup_{n\to\infty} N_2(n) \leq G_2. \tag{2.19}$$

For arbitrary $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that

$$N_i(n) \leq G_i + \epsilon \quad \text{for } n \geq n_0, \ i = 1, 2. \tag{2.20}$$

For $n > n_0 + \rho$, from the first equation of system (1.2), we have

$$\begin{aligned} N_1(n+1) &\geq N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)}{1 + N_2(n - \mu_2(n))} - N_1(n - \nu_1(n))\right]\right\} \\ &\geq N_1(n)\exp\left\{r_1^u\left[\frac{K_1^l}{1 + (G_2 + \epsilon)} - (G_1 + \epsilon)\right]\right\}. \end{aligned} \tag{2.21}$$

Here, we use the inequality $K_1^l/[1+(G_2+\epsilon)]-(G_1+\epsilon)\le K_1^l-G_1\le K_1^l-K_1^u\le 0$. So,

$$N_1(n-\nu_1(n))\le N_1(n)\exp\left\{-r_1^u\left[\frac{K_1^l}{1+(G_2+\epsilon)}-(G_1+\epsilon)\right]\nu_1^u\right\}, \tag{2.22}$$

which yields from the first equation of system (1.2) that

$$N_1(n+1)\ge N_1(n)\exp\left\{r_1(n)\left[\frac{K_1^l}{1+(G_2+\epsilon)}-N_1(n)\exp\left(-r_1^u\left[\frac{K_1^l}{1+(G_2+\epsilon)}-(G_1+\epsilon)\right]\nu_1^u\right)\right]\right\}$$
$$\forall n\ge n_0+\rho. \tag{2.23}$$

Next, we also present two cases to prove that

$$\liminf_{n\to\infty}N_1(n)\ge g_1. \tag{2.24}$$

Case I. There exists $l_0\ge n_0+\rho$ such that $N_1(l_0+1)\le N_1(l_0)$. Then, we have from (2.23) that

$$\frac{K_1^l}{1+(G_2+\epsilon)}-N_1(l_0)\exp\left(-r_1^u\left[\frac{K_1^l}{1+(G_2+\epsilon)}-(G_1+\epsilon)\right]\nu_1^u\right)\le 0, \tag{2.25}$$

which implies that

$$N_1(l_0)\ge\frac{K_1^l}{1+(G_2+\epsilon)}\exp\left(r_1^u\left[\frac{K_1^l}{1+(G_2+\epsilon)}-(G_1+\epsilon)\right]\nu_1^u\right). \tag{2.26}$$

In view of (2.21), we can easily obtain that

$$N_1(l_0+1)\ge\frac{K_1^l}{1+(G_2+\epsilon)}\exp\left\{r_1^u\left[\frac{K_1^l}{1+(G_2+\epsilon)}-(G_1+\epsilon)\right](\nu_1^u+1)\right\}\stackrel{\text{def}}{=}g_1(\epsilon). \tag{2.27}$$

We claim that

$$N_1(n)\ge g_1(\epsilon)\quad\text{for } n\ge l_0. \tag{2.28}$$

By way of contradiction, assume that there exists a $c_0\ge l_0$ such that $N_1(c_0)<g_1(\epsilon)$. Then, $c_0\ge l_0+2$. Let $c_1\ge l_0+2$ be the smallest integer such that $N_1(c_0)<g_1(\epsilon)$. Then $N_1(c_1-1)>N_1(c_1)$. The previous argument produces that $N_1(c_1)\ge g_1(\epsilon)$, a contradiction. This proves the claim.

Case II. We assume that $N_1(n) < N_1(n+1)$, for all $n \geq n_0 + \rho$. Then, $\lim_{n\to\infty} N_1(n)$ exists, denoted by $\underline{N}_1$. Taking limit in the first equation of system (1.2) gives

$$\lim_{n\to\infty}\left[\frac{K_1(n)+\alpha_1(n)N_2(n-\mu_2(n))}{1+N_2(n-\mu_2(n))} - N_1(n-\nu_1(n))\right] = 0. \tag{2.29}$$

Hence, $\underline{N}_1 \geq K_1^l/(1+(G_2+\epsilon)) \geq g_1(\epsilon)$ and $\lim_{\epsilon\to 0} g_1(\epsilon) = g_1$. This proves the claim.

So, $\liminf_{n\to\infty} N_1(n) \geq g_1$. In view of the second equation of system (1.2), similar to the previous analysis, we can obtain

$$\liminf_{n\to\infty} N_2(n) \geq g_2. \tag{2.30}$$

So, the proof of Lemma 2.7 is complete. □

Example 2.8. Consider the following discrete mutualism model with delays:

$$\begin{aligned} N_1(n+1) &= N_1(n)\exp\left\{0.1\left[\frac{0.002+0.001N_2(n)}{1+N_2(n)} - N_1(n-1)\right]\right\}, \\ N_2(n+1) &= N_2(n)\exp\left\{0.1\left[\frac{0.002+0.001N_1(n)}{1+N_1(n)} - N_2(n-1)\right]\right\}. \end{aligned} \tag{2.31}$$

Corresponding to system (1.2), $r_i \equiv 0.1$, $K_i \equiv 0.002$, $\alpha_i \equiv 0.001$, $\mu_i = 0$, $\nu_i = 1$, $i = 1, 2$. By calculation, we obtain

$$F_1 = F_2 \approx 3.68, \qquad f_1 = f_2 \approx 1.8 \times 10^{-4}. \tag{2.32}$$

By Lemma 2.6, one has

$$1.8 \times 10^{-4} \leq \liminf_{n\to\infty} N_i(n) \leq \limsup_{n\to\infty} N_i(n) \leq 3.68, \quad i = 1, 2. \tag{2.33}$$

Further, we could calculate

$$G_1 = G_2 \approx 3 \times 10^{-3}, \qquad g_1 = g_2 \approx 1.994 \times 10^{-3}. \tag{2.34}$$

By Lemma 2.7, one also has

$$1.994 \times 10^{-3} \leq \liminf_{n\to\infty} N_i(n) \leq \limsup_{n\to\infty} N_i(n) \leq 3 \times 10^{-3}, \quad i = 1, 2. \tag{2.35}$$

For system (2.31), it is easy to see that Lemma 2.7 gives a more accurate result than Lemma 2.6 (see Figure 1).

By Lemmas 2.6 and 2.7, we can easily show the following.

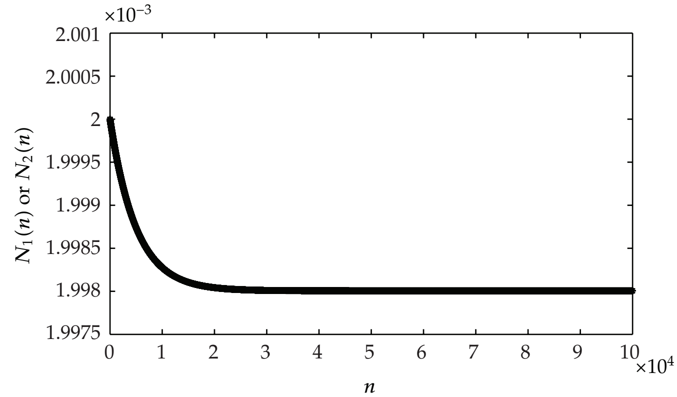

Figure 1: System (2.31) with $N_1(-1) = N_1(0) = N_2(-1) = N_2(0) = 0.002$.

Theorem 2.9. *Assume that* (H_1) *holds; then every solution* (N_1, N_2) *of system* (1.2) *satisfies*

$$\max\{f_i, g_i\} \stackrel{\text{def}}{=} m_i \leq \lim\inf_{n\to\infty} N_i(n) \leq \lim\sup_{n\to\infty} N_i(n) \leq M_i \stackrel{\text{def}}{=} \min\{F_i, G_i\}, \quad \forall n \in \mathbb{Z},\ i = 1, 2. \tag{2.36}$$

3. Global Attractivity

Define a function $\chi : [0, \infty)_{\mathbb{Z}} \to \{0, 1\}$ as follows:

$$\chi(n) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n \in [1, \infty)_{\mathbb{Z}}. \end{cases} \tag{3.1}$$

Let

$$\begin{aligned} \sigma_1 &\stackrel{\text{def}}{=} \exp\left\{\max\left(r_1^l\left[\frac{K_1^u + \alpha_1^u M_2}{1 + m_2} - m_1\right], r_1^u\left[\frac{K_1^u + \alpha_1^u M_2}{1 + m_2} - m_1\right]\right)\right\}, \\ \sigma_2 &\stackrel{\text{def}}{=} \exp\left\{\max\left(r_2^l\left[\frac{K_2^u + \alpha_2^u M_1}{1 + m_1} - m_2\right], r_2^u\left[\frac{K_2^u + \alpha_2^u M_1}{1 + m_1} - m_2\right]\right)\right\}, \\ v_1 &\stackrel{\text{def}}{=} \max\left\{\left|\frac{K_1^u + \alpha_1^u M_2}{1 + m_2} - m_1\right|, \left|\frac{K_1^l + \alpha_1^l m_2}{1 + M_2} - M_1\right|\right\}, \\ v_2 &\stackrel{\text{def}}{=} \max\left\{\left|\frac{K_2^u + \alpha_2^u M_1}{1 + m_1} - m_2\right|, \left|\frac{K_2^l + \alpha_2^l m_1}{1 + M_1} - M_2\right|\right\}. \end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that* (H_1) *holds. Suppose further that*

(H_2) *there exist two positive constants* λ_1 *and* λ_2 *such that* $\min\{\Theta_1, \Theta_2\} > 0$, *where*

$$
\begin{aligned}
\Theta_1 &\stackrel{\text{def}}{=} \lambda_1 \min\left[r_1^l, \frac{2}{M_1} - r_1^u\right] - \lambda_1\sigma_1 (r_1^u)^2 \chi(\nu_1^u)\nu_1^u\left(\nu_1^u - \nu_1^l + 1\right)M_1 - \lambda_1\sigma_1 \upsilon_1 (r_1^u)^2 \chi(\nu_1^u)\nu_1^u \\
&\quad - \frac{\lambda_2\sigma_2 (r_2^u)^2 \chi(\nu_2^u)|K_2 - \alpha_2|^u\left(\mu_2^u - \mu_2^l + 1\right)\nu_2^u M_2}{[1+m_1]^2} - \frac{\lambda_2 r_2^u\left(\mu_1^u - \mu_1^l + 1\right)|K_2 - \alpha_2|^u}{[1+m_1]^2}, \\
\Theta_2 &\stackrel{\text{def}}{=} \lambda_2 \min\left[r_2^l, \frac{2}{M_2} - r_2^u\right] - \lambda_2\sigma_2 (r_2^u)^2 \chi(\nu_2^u)\nu_2^u\left(\nu_2^u - \nu_2^l + 1\right)M_2 - \lambda_2\sigma_2 \upsilon_2 (r_2^u)^2 \chi(\nu_2^u)\nu_2^u \\
&\quad - \frac{\lambda_1\sigma_1 (r_1^u)^2 \chi(\nu_1^u)|K_1 - \alpha_1|^u\left(\mu_1^u - \mu_1^l + 1\right)\nu_1^u M_1}{[1+m_2]^2} - \frac{\lambda_1 r_1^u\left(\mu_2^u - \mu_2^l + 1\right)|K_1 - \alpha_1|^u}{[1+m_2]^2}.
\end{aligned}
\tag{3.3}
$$

Then, system (1.2) *is globally attractive, that is, for any positive solution* (N_1, N_2) *and* (W_1, W_2) *of system* (1.2),

$$
\lim_{n\to+\infty} |N_1(n) - W_1(n)| = 0, \qquad \lim_{n\to+\infty} |N_2(n) - W_2(n)| = 0. \tag{3.4}
$$

Proof. In view of condition (H_2), there exist small enough positive constants ϵ and λ such that

$$
\begin{aligned}
\Theta_1(\epsilon) &\stackrel{\text{def}}{=} \Bigg\{ \lambda_1 \min\left[r_1^l, \frac{2}{M_1+\epsilon} - r_1^u\right] - \lambda_1\sigma_1(\epsilon) (r_1^u)^2 \chi(\nu_1^u)\nu_1^u\left(\nu_1^u - \nu_1^l + 1\right)(M_1+\epsilon) \\
&\quad - \frac{\lambda_2\sigma_2(\epsilon) (r_2^u)^2 \chi(\nu_2^u)|K_2 - \alpha_2|^u\left(\mu_2^u - \mu_2^l + 1\right)\nu_2^u (M_2+\epsilon)}{[1+m_1-\epsilon]^2} \\
&\quad - \frac{\lambda_2 r_2^u\left(\mu_1^u - \mu_1^l + 1\right)|K_2 - \alpha_2|^u}{[1+m_1-\epsilon]^2} - \lambda_1\sigma_1(\epsilon)\upsilon_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)\nu_1^u \Bigg\} > \lambda, \\
\Theta_2(\epsilon) &\stackrel{\text{def}}{=} \Bigg\{ \lambda_2 \min\left[r_2^l, \frac{2}{M_2+\epsilon} - r_2^u\right] - \lambda_2\sigma_2(\epsilon) (r_2^u)^2 \chi(\nu_2^u)\nu_2^u\left(\nu_2^u - \nu_2^l + 1\right)(M_2+\epsilon) \\
&\quad - \frac{\lambda_1\sigma_1(\epsilon) (r_1^u)^2 \chi(\nu_1^u)|K_1 - \alpha_1|^u\left(\mu_1^u - \mu_1^l + 1\right)\nu_1^u (M_1+\epsilon)}{[1+m_2-\epsilon]^2} \\
&\quad - \frac{\lambda_1 r_1^u\left(\mu_2^u - \mu_2^l + 1\right)|K_1 - \alpha_1|^u}{[1+m_2-\epsilon]^2} - \lambda_2\sigma_2(\epsilon)\upsilon_2(\epsilon)(r_2^u)^2\chi(\nu_2^u)\nu_2^u \Bigg\} > \lambda,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
\sigma_1(\epsilon) &\stackrel{\text{def}}{=} \exp\left\{\max\left(r_1^l\left[\frac{K_1^u+\alpha_1^u(M_2+\epsilon)}{1+m_2-\epsilon}-(m_1-\epsilon)\right], r_1^u\left[\frac{K_1^u+\alpha_1^u(M_2+\epsilon)}{1+m_2-\epsilon}-(m_1-\epsilon)\right]\right)\right\},\\
\sigma_2(\epsilon) &\stackrel{\text{def}}{=} \exp\left\{\max\left(r_2^l\left[\frac{K_2^u+\alpha_2^u(M_1+\epsilon)}{1+m_1-\epsilon}-(m_2-\epsilon)\right], r_2^u\left[\frac{K_2^u+\alpha_2^u(M_1+\epsilon)}{1+m_1-\epsilon}-(m_2-\epsilon)\right]\right)\right\},\\
\upsilon_1(\epsilon) &\stackrel{\text{def}}{=} \max\left\{\left|\frac{K_1^u+\alpha_1^u(M_2+\epsilon)}{1+m_2-\epsilon}-(m_1-\epsilon)\right|, \left|\frac{K_1^l+\alpha_1^l(m_2-\epsilon)}{1+M_2+\epsilon}-(M_1+\epsilon)\right|\right\},\\
\upsilon_2(\epsilon) &\stackrel{\text{def}}{=} \max\left\{\left|\frac{K_2^u+\alpha_2^u(M_1+\epsilon)}{1+m_1-\epsilon}-(m_2-\epsilon)\right|, \left|\frac{K_2^l+\alpha_2^l(m_1-\epsilon)}{1+M_1+\epsilon}-(M_2+\epsilon)\right|\right\}.
\end{aligned}
\tag{3.6}
$$

Suppose that (N_1, N_2) and (W_1, W_2) are two positive solutions of system (1.2). By Theorem 2.9, there exists a constant $\mathbb{K}_0 > 0$ such that

$$
m_i-\epsilon \le N_i(n), \quad W_i(n) \le M_i+\epsilon, \quad \forall n > \mathbb{K}_0,\ i = 1,2. \tag{3.7}
$$

Let

$$
V_{11}(n) = |\ln N_1(n) - \ln W_1(n)|. \tag{3.8}
$$

In view of system (1.2), we have

$$
\begin{aligned}
V_{11}(n+1) &= |\ln N_1(n+1) - \ln W_1(n+1)|\\
&= \Bigg|[\ln N_1(n) - \ln W_1(n)] - r_1(n)[N_1(n-\nu_1(n)) - W_1(n-\nu_1(n))]\\
&\quad + r_1(n)\left[\frac{K_1(n)-\alpha_1(n)}{1+N_2(n-\mu_2(n))} - \frac{K_1(n)-\alpha_1(n)}{1+W_2(n-\mu_2(n))}\right]\Bigg|\\
&= \Bigg|[\ln N_1(n) - \ln W_1(n)] - r_1(n)[N_1(n) - W_1(n)]\\
&\quad - r_1(n)\chi(\nu_1^u)\{[W_1(n) - W_1(n-\nu_1(n))] - [N_1(n) - N_1(n-\nu_1(n))]\}\\
&\quad + r_1(n)\left[\frac{K_1(n)-\alpha_1(n)}{1+N_2(n-\mu_2(n))} - \frac{K_1(n)-\alpha_1(n)}{1+W_2(n-\mu_2(n))}\right]\Bigg|\\
&\le |[\ln N_1(n) - \ln W_1(n)] - r_1(n)[N_1(n) - W_1(n)]|\\
&\quad + r_1(n)\chi(\nu_1^u)|[W_1(n) - W_1(n-\nu_1(n))] - [N_1(n) - N_1(n-\nu_1(n))]|\\
&\quad + r_1(n)\left|\frac{K_1(n)-\alpha_1(n)}{1+N_2(n-\mu_2(n))} - \frac{K_1(n)-\alpha_1(n)}{1+W_2(n-\mu_2(n))}\right|.
\end{aligned}
\tag{3.9}
$$

Using the mean value theorem, it follows that

$$N_1(n) - W_1(n) = \exp\{\ln N_1(n)\} - \exp\{\ln W_1(n)\} = \theta_1(n)[\ln N_1(n) - \ln W_1(n)], \tag{3.10}$$

where $\theta_1(n)$ lies between $N_1(n)$ and $W_1(n)$, and

$$\begin{aligned} &\left|\frac{K_1(n) - \alpha_1(n)}{1 + N_2(n - \mu_2(n))} - \frac{K_1(n) - \alpha_1(n)}{1 + W_2(n - \mu_2(n))}\right| \\ &\quad = \left|-\frac{K_1(n) - \alpha_1(n)}{[1 + \theta_2(n)]^2}\left[N_2(n - \mu_2(n)) - W_2(n - \mu_2(n))\right]\right|, \end{aligned} \tag{3.11}$$

where $\theta_2(n)$ lies between $N_2(n - \mu_2(n))$ and $W_2(n - \mu_2(n))$.

Define

$$\begin{aligned} P_1(n) &\stackrel{\text{def}}{=} r_1(n)\left[\frac{K_1(n) + \alpha_1(n) N_2(n - \mu_2(n))}{1 + N_2(n - \mu_2(n))} - N_1(n - \nu_1(n))\right], \\ Q_1(n) &\stackrel{\text{def}}{=} r_1(n)\left[\frac{K_1(n) + \alpha_1(n) W_2(n - \mu_2(n))}{1 + W_2(n - \mu_2(n))} - W_1(n - \nu_1(n))\right]. \end{aligned} \tag{3.12}$$

By a similar argument as that in (3.9), we obtain from (3.11) that

$$\begin{aligned} &|[W_1(n) - W_1(n - \nu_1(n))] - [N_1(n) - N_1(n - \nu_1(n))]| \\ &\quad = \left|\sum_{s=n-\nu_1(n)}^{n-1}[N_1(s+1) - W_1(s+1)] - \sum_{s=n-\nu_1(n)}^{n-1}[N_1(s) - W_1(s)]\right| \\ &\quad = \left|\sum_{s=n-\nu_1(n)}^{n-1}\left[N_1(s)e^{P_1(s)} - W_1(s)e^{Q_1(s)}\right] - \sum_{s=n-\nu_1(n)}^{n-1}[N_1(s) - W_1(s)]\right| \\ &\quad = \left|\sum_{s=n-\nu_1(n)}^{n-1} N_1(s)\left[e^{P_1(s)} - e^{Q_1(s)}\right] + \sum_{s=n-\nu_1(n)}^{n-1}[N_1(s) - W_1(s)]\left[e^{Q_1(s)} - 1\right]\right| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{s=n-\nu_1^u}^{n-1} N_1(s)\xi_1(s)r_1(s)\Bigg|[N_1(s-\nu_1(s))-W_1(s-\nu_1(s))]\\
&\qquad\qquad -\left[\frac{K_1(s)-\alpha_1(s)}{1+N_2(s-\mu_2(s))}-\frac{K_1(s)-\alpha_1(s)}{1+W_2(s-\mu_2(s))}\right]\Bigg|\\
&\quad+\sum_{s=n-\nu_1^u}^{n-1}\xi_2(s)r_1(s)\left|\frac{K_1(s)+\alpha_1(s)W_2(s-\mu_2(s))}{1+W_2(s-\mu_2(s))}-W_1(s-\nu_1(s))\right||N_1(s)-W_1(s)|\\
&\le \sum_{s=n-\nu_1^u}^{n-1}\sigma_1(\epsilon)r_1^u(M_1+\epsilon)|N_1(s-\nu_1(s))-W_1(s-\nu_1(s))|\\
&\quad+\sum_{s=n-\nu_1^u}^{n-1}\frac{\sigma_1(\epsilon)r_1^u|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\big|N_2(s-\mu_2(s))-W_2(s-\mu_2(s))\big|\\
&\quad+\sum_{s=n-\nu_1^u}^{n-1}\sigma_1(\epsilon)\upsilon_1(\epsilon)r_1^u|N_1(s)-W_1(s)|\\
&\le \sigma_1(\epsilon)r_1^u(M_1+\epsilon)\sum_{s=n-\nu_1^u}^{n-1}\sum_{k=\nu_1^l}^{\nu_1^u}|N_1(s-k)-W_1(s-k)|\\
&\quad+\frac{\sigma_1(\epsilon)r_1^u|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\sum_{s=n-\nu_1^u}^{n-1}\sum_{k=\mu_1^l}^{\mu_1^u}|N_2(s-k)-W_2(s-k)|\\
&\quad+\sigma_1(\epsilon)\upsilon_1(\epsilon)r_1^u\sum_{s=n-\nu_1^u}^{n-1}|N_1(s)-W_1(s)|\\
&\le \sigma_1(\epsilon)r_1^u(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_1(s)-W_1(s)|\\
&\quad+\frac{\sigma_1(\epsilon)r_1^u|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\sum_{k=\mu_1^l}^{\mu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_2(s)-W_2(s)|\\
&\quad+\sigma_1(\epsilon)\upsilon_1(\epsilon)r_1^u\sum_{s=n-\nu_1^u}^{n-1}|N_1(s)-W_1(s)|,\quad \forall n>\mathbb{K}_0+2\rho,
\end{aligned}
\tag{3.13}
$$

where $\xi_1(s)$ lies between $e^{P_1(s)}$ and $e^{Q_1(s)}$, and $\xi_2(s)$ lies between $e^{Q_1(s)}$ and 1, $s=n-\nu_1^u,\dots,n-1$.

In view of (3.9), it follows from (3.10)–(3.13) that

$$
\begin{aligned}
\Delta V_{11}(n) \le& -\left[\frac{1}{\theta_1(n)}-\left|\frac{1}{\theta_1(n)}-r_1(n)\right|\right]|N_1(n)-W_1(n)|\\
&+\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_1(s)-W_1(s)|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\sum_{k=\mu_1^l}^{\mu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_2(s)-W_2(s)| \\
&+\sigma_1(\epsilon)\upsilon_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\sum_{s=n-\nu_1^u}^{n-1}|N_1(s)-W_1(s)| \\
&+\frac{r_1^u|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}\sum_{s=n-\mu_2^u}^{n-\mu_2^l}|N_2(s)-W_2(s)| \\
\le\ &-\min\left[r_1^l,\frac{2}{M_1+\epsilon}-r_1^u\right]|N_1(n)-W_1(n)| \\
&+\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_1(s)-W_1(s)| \\
&+\frac{\sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\sum_{k=\mu_1^l}^{\mu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_2(s)-W_2(s)| \\
&+\sigma_1(\epsilon)\upsilon_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\sum_{s=n-\nu_1^u}^{n-1}|N_1(s)-W_1(s)| \\
&+\frac{r_1^u|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}\sum_{s=n-\mu_2^u}^{n-\mu_2^l}|N_2(s)-W_2(s)|,\quad \forall n>\mathbb{K}_0+2\rho.
\end{aligned}
\tag{3.14}
$$

Let

$$
\begin{aligned}
V_{12}(n)&=\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{t=0}^{\nu_1^u-1}\sum_{s=n-\nu_1^u-k+t}^{n-1}|N_1(s)-W_1(s)|, \\
V_{13}(n)&=\sigma_1(\epsilon)\upsilon_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\sum_{t=0}^{\nu_1^u-1}\sum_{s=n-\nu_1^u+t}^{n-1}|N_1(s)-W_1(s)|, \\
V_{14}(n)&=\frac{\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\sum_{k=\mu_1^l}^{\mu_1^u}\sum_{t=0}^{\nu_1^u-1}\sum_{s=n-\nu_1^u-k+t}^{n-1}|N_2(s)-W_2(s)|, \\
V_{15}(n)&=\frac{r_1^u|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}\sum_{t=0}^{\mu_2^u-\mu_2^l}\sum_{s=n-\mu_2^u+t}^{n-1}|N_2(s)-W_2(s)|.
\end{aligned}
\tag{3.15}
$$

So,

$$
\begin{aligned}
\Delta V_{12}(n)=\ &\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{t=0}^{\nu_1^u-1}|N_1(n)-W_1(n)| \\
&-\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{t=0}^{\nu_1^u-1}\left|N_1\left(n-\nu_1^u-k+t\right)-W_1\left(n-\nu_1^u-k+t\right)\right|
\end{aligned}
$$

$$= \sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)\nu_1^u\left(\nu_1^u - \nu_1^l + 1\right)(M_1+\epsilon)|N_1(n) - W_1(n)|$$
$$- \sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)(M_1+\epsilon)\sum_{k=\nu_1^l}^{\nu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_1(s) - W_1(s)|, \tag{3.16}$$

$$\Delta V_{13}(n) = \sigma_1(\epsilon)\upsilon_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)\nu_1^u|N_1(n) - W_1(n)|$$
$$- \sigma_1(\epsilon)\upsilon_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)\sum_{s=n-\nu_1^u}^{n-1}|N_1(s) - W_1(s)|, \tag{3.17}$$

$$\Delta V_{14}(n) = \frac{\sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)|K_1-\alpha_1|^u\left(\mu_1^u - \mu_1^l + 1\right)\nu_1^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}|N_2(n) - W_2(n)|$$
$$- \frac{\sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)|K_1-\alpha_1|^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\sum_{k=\mu_1^l}^{\mu_1^u}\sum_{s=n-\nu_1^u-k}^{n-k-1}|N_2(s) - W_2(s)|, \tag{3.18}$$

$$\Delta V_{15}(n) = \frac{r_1^u\left(\mu_2^u - \mu_2^l + 1\right)|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}|N_2(n) - W_2(n)|$$
$$- \frac{r_1^u|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}\sum_{s=n-\mu_2^u}^{n-\mu_2^l}|N_2(s) - W_2(s)|. \tag{3.19}$$

Define

$$V_1(n) = V_{11}(n) + V_{12}(n) + V_{13}(n) + V_{14}(n) + V_{15}(n). \tag{3.20}$$

It follows from (3.14)–(3.19) that

$$\Delta V_1(n) \le -\min\left[r_1^l, \frac{2}{M_1+\epsilon} - r_1^u\right]|N_1(n) - W_1(n)|$$
$$+ \sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)\nu_1^u\left(\nu_1^u - \nu_1^l + 1\right)(M_1+\epsilon)|N_1(n) - W_1(n)|$$
$$+ \frac{\sigma_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)|K_1-\alpha_1|^u\left(\mu_1^u - \mu_1^l + 1\right)\nu_1^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}|N_2(n) - W_2(n)|$$
$$+ \sigma_1(\epsilon)\upsilon_1(\epsilon)(r_1^u)^2\chi(\nu_1^u)\nu_1^u|N_1(n) - W_1(n)|$$

$$
\begin{aligned}
&+\frac{r_1^u\left(\mu_2^u-\mu_2^l+1\right)|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}|N_2(n)-W_2(n)| \\
&\le -\left\{\min\left[r_1^l,\frac{2}{M_1+\epsilon}-r_1^u\right]-\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\nu_1^u\left(\nu_1^u-\nu_1^l+1\right)(M_1+\epsilon)\right. \\
&\qquad \left.-\sigma_1(\epsilon)\upsilon_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\nu_1^u\right\}|N_1(n)-W_1(n)| \\
&\quad +\left\{\frac{\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)|K_1-\alpha_1|^u\left(\mu_1^u-\mu_1^l+1\right)\nu_1^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\right. \\
&\qquad \left.+\frac{r_1^u\left(\mu_2^u-\mu_2^l+1\right)|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}\right\}|N_2(n)-W_2(n)|, \quad \forall n>\mathbb{K}_0+2\rho.
\end{aligned}
\tag{3.21}
$$

Let

$$
V_2(n)=V_{21}(n)+V_{22}(n)+V_{23}(n)+V_{24}(n)+V_{25}(n), \tag{3.22}
$$

where

$$
\begin{aligned}
V_{21}(n)&=|\ln N_2(n)-\ln W_2(n)|, \\
V_{22}(n)&=\sigma_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)(M_2+\epsilon)\sum_{k=\nu_2^l}^{\nu_2^u}\sum_{t=0}^{\nu_2^u-1}\sum_{s=n-\nu_2^u-k+t}^{n-1}|N_2(s)-W_2(s)|, \\
V_{23}(n)&=\sigma_2(\epsilon)\upsilon_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)\sum_{t=0}^{\nu_2^u-1}\sum_{s=n-\nu_2^u+t}^{n-1}|N_2(s)-W_2(s)|, \\
V_{24}(n)&=\frac{\sigma_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)|K_2-\alpha_2|^u(M_2+\epsilon)}{[1+m_1-\epsilon]^2}\sum_{k=\mu_2^l}^{\mu_2^u}\sum_{t=0}^{\nu_2^u-1}\sum_{s=n-\nu_2^u-k+t}^{n-1}|N_1(s)-W_1(s)|, \\
V_{25}(n)&=\frac{r_2^u|K_2-\alpha_2|^u}{[1+m_1-\epsilon]^2}\sum_{t=0}^{\mu_1^u-\mu_1^l}\sum_{s=n-\mu_1^u+t}^{n-1}|N_1(s)-W_1(s)|.
\end{aligned}
\tag{3.23}
$$

By a similar argument as that in (3.21), we could easily obtain that

$$
\begin{aligned}
\Delta V_2(n)\le& -\left\{\min\left[r_2^l,\frac{2}{M_2+\epsilon}-r_2^u\right]-\sigma_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)\nu_2^u\left(\nu_2^u-\nu_2^l+1\right)(M_2+\epsilon)\right. \\
&\qquad \left.-\sigma_2(\epsilon)\upsilon_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)\nu_2^u\right\}|N_2(n)-W_2(n)|
\end{aligned}
$$

$$+\left\{\frac{\sigma_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)|K_2-\alpha_2|^u\left(\mu_2^u-\mu_2^l+1\right)\nu_2^u(M_2+\epsilon)}{[1+m_1-\epsilon]^2}\right.$$
$$\left.+\frac{r_2^u\left(\mu_1^u-\mu_1^l+1\right)|K_2-\alpha_2|^u}{[1+m_1-\epsilon]^2}\right\}|N_1(n)-W_1(n)|,\quad \forall n>\mathbb{K}_0+2\rho. \tag{3.24}$$

We construct a Lyapunov functional as follows:

$$V(n)=\lambda_1V_1(n)+\lambda_2V_2(n), \tag{3.25}$$

which implies from (3.21) and (3.24) that

$$\begin{aligned}
\Delta V(n)\leq &-\left\{\lambda_1\min\left[r_1^l,\frac{2}{M_1+\epsilon}-r_1^u\right]-\lambda_1\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\nu_1^u\left(\nu_1^u-\nu_1^l+1\right)(M_1+\epsilon)\right.\\
&-\frac{\lambda_2\sigma_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)|K_2-\alpha_2|^u\left(\mu_2^u-\mu_2^l+1\right)\nu_2^u(M_2+\epsilon)}{[1+m_1-\epsilon]^2}\\
&\left.-\frac{\lambda_2r_2^u\left(\mu_1^u-\mu_1^l+1\right)|K_2-\alpha_2|^u}{[1+m_1-\epsilon]^2}-\lambda_1\sigma_1(\epsilon)\upsilon_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)\nu_1^u\right\}|N_1(n)-W_1(n)|\\
&-\left\{\lambda_2\min\left[r_2^l,\frac{2}{M_2+\epsilon}-r_2^u\right]-\lambda_2\sigma_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)\nu_2^u\left(\nu_2^u-\nu_2^l+1\right)(M_2+\epsilon)\right.\\
&-\frac{\lambda_1\sigma_1(\epsilon)\left(r_1^u\right)^2\chi\left(\nu_1^u\right)|K_1-\alpha_1|^u\left(\mu_1^u-\mu_1^l+1\right)\nu_1^u(M_1+\epsilon)}{[1+m_2-\epsilon]^2}\\
&\left.-\frac{\lambda_1r_1^u\left(\mu_2^u-\mu_2^l+1\right)|K_1-\alpha_1|^u}{[1+m_2-\epsilon]^2}-\lambda_2\sigma_2(\epsilon)\upsilon_2(\epsilon)\left(r_2^u\right)^2\chi\left(\nu_2^u\right)\nu_2^u\right\}|N_2(n)-W_2(n)|\\
\leq &-\lambda[|N_1(n)-W_1(n)|+|N_2(n)-W_2(n)|],\quad \forall n>\mathbb{K}_0+2\rho.
\end{aligned} \tag{3.26}$$

Taking $n_1\in(\mathbb{K}_0+2\rho,\infty)_{\mathbb{Z}}$ and Summing both sides of inequality (3.26) over $[n_1,n]_{\mathbb{Z}}$, we have

$$V(n+1)+\lambda\sum_{s=n_1}^{n}|N_1(s)-W_1(s)|+\lambda\sum_{s=n_1}^{n}|N_2(s)-W_2(s)|\leq V(n_1)<\infty. \tag{3.27}$$

Therefore,

$$\lim_{n\to+\infty}\sup\sum_{s=n_1}^{n}|N_1(s)-W_1(s)|+\lim_{n\to+\infty}\sup\sum_{s=n_1}^{n}|N_2(s)-W_2(s)|\le\frac{V(n_1)}{\lambda}<\infty. \tag{3.28}$$

From the previous inequality one could easily deduce that

$$\lim_{n\to+\infty}|N_1(s)-W_1(s)|=0,\qquad \lim_{n\to+\infty}|N_2(s)-W_2(s)|=0. \tag{3.29}$$

This completes the proof. □

If $\mu_1=\mu_2=\nu_1=\nu_2\equiv 0$ in system (1.2), then we obtain a discrete mutualism model without delay as follows:

$$\begin{aligned}
N_1(n+1)&=N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)N_2(n)}{1+N_2(n)}-N_1(n)\right]\right\},\\
N_2(n+1)&=N_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n)+\alpha_2(n)N_1(n)}{1+N_1(n)}-N_2(n)\right]\right\}.
\end{aligned} \tag{3.30}$$

Let

$$\overline{M}_i\overset{\text{def}}{=}M_i \text{ with } \nu_i^u=0,\quad \overline{m}_i\overset{\text{def}}{=}m_i \text{ with } \nu_1^u=\nu_2^u=0,\quad i=1,2. \tag{3.31}$$

Corollary 3.2. *Assume that* (H_1) *holds. Suppose further that*

(H_3) *there exist two positive constants* λ_1 *and* λ_2 *such that*

$$\begin{aligned}
\lambda_1\min\left[r_1^l,\frac{2}{\overline{M}_1}-r_1^u\right]&>\frac{\lambda_2 r_2^u|K_2-\alpha_2|^u}{[1+\overline{m}_1]^2},\\
\lambda_2\min\left[r_2^l,\frac{2}{\overline{M}_2}-r_2^u\right]&>\frac{\lambda_1 r_1^u|K_1-\alpha_1|^u}{[1+\overline{m}_2]^2}.
\end{aligned} \tag{3.32}$$

Then, system (3.30) *is globally attractive.*

Further, we consider the following discrete mutualism model with constant delays:

$$\begin{aligned}
N_1(n+1)&=N_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)N_2(n-\mu_2)}{1+N_2(n-\mu_2)}-N_1(n-\nu_1)\right]\right\},\\
N_2(n+1)&=N_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n)+\alpha_2(n)N_1(n-\mu_1)}{1+N_1(n-\mu_1)}-N_2(n-\nu_2)\right]\right\},
\end{aligned} \tag{3.33}$$

where $\mu_1, \mu_2, \nu_1,$ and ν_2 are nonnegative integers. Let

$$\widehat{M}_i \overset{\text{def}}{=} M_i \text{ with } \nu_i^u = \nu_i, \quad \widehat{m}_i \overset{\text{def}}{=} m_i \text{ with } \nu_1^u = \nu_1, \quad \nu_2^u = \nu_2, \quad i = 1, 2,$$
$$\widehat{\sigma}_i \overset{\text{def}}{=} \sigma_i \text{ with } M_1 = \widehat{M}_1, \quad M_2 = \widehat{M}_2, \quad m_1 = \widehat{m}_1, \quad m_2 = \widehat{m}_2, \quad i = 1, 2, \tag{3.34}$$
$$\widehat{\upsilon}_i \overset{\text{def}}{=} \upsilon_i \text{ with } M_1 = \widehat{M}_1, \quad M_2 = \widehat{M}_2, \quad m_1 = \widehat{m}_1, \quad m_2 = \widehat{m}_2, \quad i = 1, 2.$$

Corollary 3.3. *Assume that* (H_1) *holds. Suppose further that*

(H_4) *there exist two positive constants* λ_1 *and* λ_2 *such that*

$$\lambda_1 \min\left[r_1^l, \frac{2}{\widehat{M}_1} - r_1^u\right] > \lambda_1\widehat{\sigma}_1\left(r_1^u\right)^2\chi(\nu_1)\nu_1\widehat{M}_1 + \lambda_1\widehat{\sigma}_1\widehat{\upsilon}_1\left(r_1^u\right)^2\chi(\nu_1)\nu_1 + \frac{\lambda_2\widehat{\sigma}_2\left(r_2^u\right)^2\chi(\nu_2)|K_2 - \alpha_2|^u\nu_2\widehat{M}_2}{[1 + \widehat{m}_1]^2} + \frac{\lambda_2 r_2^u|K_2 - \alpha_2|^u}{[1 + \widehat{m}_1]^2}, \tag{3.35}$$

$$\lambda_2 \min\left[r_2^l, \frac{2}{\widehat{M}_2} - r_2^u\right] > \lambda_2\widehat{\sigma}_2\left(r_2^u\right)^2\chi(\nu_2)\nu_2\widehat{M}_2 + \lambda_2\widehat{\sigma}_2\widehat{\upsilon}_2\left(r_2^u\right)^2\chi(\nu_2)\nu_2 + \frac{\lambda_1\widehat{\sigma}_1\left(r_1^u\right)^2\chi(\nu_1)|K_1 - \alpha_1|^u\nu_1\widehat{M}_1}{[1 + \widehat{m}_2]^2} + \frac{\lambda_1 r_1^u|K_1 - \alpha_1|^u}{[1 + \widehat{m}_2]^2}. \tag{3.36}$$

Then, system (3.33) *is globally attractive.*

4. Almost Periodic Solution

In this section, we investigate the existence and uniqueness of a globally attractive almost periodic solution of system (1.2) by using almost periodic functional hull theory.

Let $\{\tau_p\}$ be any integer valued sequence such that $\tau_p \to \infty$ as $p \to \infty$. According to Lemma 2.5, taking a subsequence if necessary, we have

$$r_i(n + \tau_p) \longrightarrow r_i^*(n), \qquad K_i(n + \tau_p) \longrightarrow K_i^*(n), \qquad \alpha_i(n + \tau_p) \longrightarrow \alpha_i^*(n),$$
$$\mu_i(n + \tau_p) \longrightarrow \mu_i^*(n), \quad \nu_i(n + \tau_p) \longrightarrow \nu_i^*(n), \quad \text{as } p \longrightarrow \infty \tag{4.1}$$

for $n \in \mathbb{Z}$, $i = 1, 2$. Then, we get the hull equations of system (1.2) as follows:

$$N_1(n + 1) = N_1(n)\exp\left\{r_1^*(n)\left[\frac{K_1^*(n) + \alpha_1^*(n)N_2(n - \mu_2^*(n))}{1 + N_2(n - \mu_2^*(n))} - N_1(n - \nu_1^*(n))\right]\right\},$$
$$N_2(n + 1) = N_2(n)\exp\left\{r_2^*(n)\left[\frac{K_2^*(n) + \alpha_2^*(n)N_1(n - \mu_1^*(n))}{1 + N_1(n - \mu_1^*(n))} - N_2(n - \nu_2^*(n))\right]\right\}. \tag{4.2}$$

By the almost periodic theory, we can conclude that if system (1.2) satisfies (H_1)–(H_4), then the hull equations (4.2) of system (1.2) also satisfies (H_1)–(H_4).

By Theorem 3.4 in [27], it is easy to obtain the following lemma.

Lemma 4.1. *If each of the hull equations of system* (1.2) *has a unique strictly positive solution, then system* (1.2) *has a unique strictly positive almost periodic solution.*

Theorem 4.2. *If system* (1.2) *satisfies* (H_1)-(H_2)*, then system* (1.2) *admits a unique strictly positive almost periodic solution.*

Proof. By Lemma 4.1, in order to prove the existence of a unique strictly positive almost periodic solution of system (1.2), we only need to prove that each hull equations of system (1.2) has a unique strictly positive solution.

Firstly, we prove the existence of a strictly positive solution of hull equations (4.2). By the almost periodicity of $\{r_i(n)\}$, $\{K_i(n)\}$, $\{\alpha_i(n)\}$, $\{\mu_i(n)\}$, and $\{\nu_i(n)\}$, $i = 1,2$, there exists an integer-valued sequence $\{\eta_p\}$ with $\eta_p \to \infty$ as $p \to \infty$ such that

$$r_i^*(n+\eta_p) \longrightarrow r_i^*(n), \qquad K_i^*(n+\eta_p) \longrightarrow K_i^*(n), \quad \alpha_i^*(n+\eta_p) \longrightarrow \alpha_i^*(n),$$
$$\mu_i^*(n+\eta_p) \longrightarrow \mu_i^*(n), \quad \nu_i^*(n+\eta_p) \longrightarrow \nu_i^*(n), \quad \text{as } p \longrightarrow \infty,\ i = 1,2. \tag{4.3}$$

Suppose that (N_1, N_2) is any solution of hull equations (4.2). Let ϵ be an arbitrary small positive number. It follows from Theorem 2.9 that there exists a positive integer I_0 such that

$$m_i - \epsilon \le N_i(n) \le M_i + \epsilon, \quad \forall n > I_0,\ i = 1,2. \tag{4.4}$$

Write $N_i^p(n) = N_i(n+\eta_p)$ for $n \ge I_0 - \eta_p$, $p = 1,2,\ldots$, $i = 1,2$. For any positive integer q, it is easy to see that there exist sequences $\{N_1^p(n) : p \ge q\}$ and $\{N_2^p(n) : p \ge q\}$ such that the sequences $\{N_1^p(n)\}$ and $\{N_2^p(n)\}$ have subsequences, denoted by $\{N_1^p(n)\}$ and $\{N_2^p(n)\}$ again, converging on any finite interval of $\mathbb{Z}$ as $p \to \infty$, respectively. Thus, we have sequences $\{W_1(n)\}$ and $\{W_2(n)\}$ such that

$$N_i^p(n) \longrightarrow W_i(n), \quad \forall n \in \mathbb{Z} \text{ as } p \longrightarrow \infty,\ i = 1,2. \tag{4.5}$$

Combined with

$$\begin{aligned} N_1^p(n+1) &= N_1^p(n)\exp\left\{ r_1^*(n+\eta_p)\left[\frac{K_1^*(n+\eta_p) + \alpha_1^*(n+\eta_p)N_2(n+\eta_p-\mu_2^*(n+\eta_p))}{1+N_2(n+\eta_p-\mu_2^*(n+\eta_p))} \right.\right. \\ &\qquad \left.\left. -N_1(n+\eta_p-\nu_1^*(n+\eta_p)) \right]\right\}, \\ N_2^p(n+1) &= N_2^p(n)\exp\left\{ r_2^*(n+\eta_p)\left[\frac{K_2^*(n+\eta_p) + \alpha_2^*(n+\eta_p)N_1(n+\eta_p-\mu_1^*(n+\eta_p))}{1+N_1(n+\eta_p-\mu_1^*(n+\eta_p))} \right.\right. \\ &\qquad \left.\left. -N_2(n+\eta_p-\nu_2^*(n+\eta_p)) \right]\right\} \end{aligned} \tag{4.6}$$

gives

$$W_1(n+1) = W_1(n)\exp\left\{r_1^*(n)\left[\frac{K_1^*(n)+\alpha_1^*(n)W_2(n-\mu_2^*(n))}{1+W_2(n-\mu_2^*(n))} - W_1(n-\nu_1^*(n))\right]\right\},$$
$$W_2(n+1) = W_2(n)\exp\left\{r_2^*(n)\left[\frac{K_2^*(n)+\alpha_2^*(n)W_1(n-\mu_1^*(n))}{1+W_1(n-\mu_1^*(n))} - W_2(n-\nu_2^*(n))\right]\right\}. \tag{4.7}$$

We can easily see that (W_1, W_2) is a solution of hull equations (4.2) and $m_i - \epsilon \le W_i(n) \le M_i + \epsilon$ for $n \in \mathbb{Z}$, $i = 1, 2$. Since ϵ is an arbitrary small positive number, it follows that $m_i \le W_i(n) \le M_i$ for $n \in \mathbb{Z}$, $i = 1, 2$, which implies that each of the hull equations of system (1.2) has at least one strictly positive solution.

Now, we prove the uniqueness of the strictly positive solution of each of the hull equations (4.2). Suppose that the hull equations of (4.2) have two arbitrary strictly positive solutions (N_1^*, N_2^*) and (W_1^*, W_2^*) which satisfy

$$m_i \le N_i^*, \quad W_i^* \le M_i, \quad i = 1, 2. \tag{4.8}$$

Similar to Theorem 3.1, we define a Lyapunov functional

$$V^*(n) = \lambda_1 V_1^*(n) + \lambda_2 V_2^*(n), \tag{4.9}$$

where

$$V_1^*(n) = V_{11}^*(n) + V_{12}^*(n) + V_{13}^*(n) + V_{14}^*(n) + V_{15}^*(n),$$
$$V_2^*(n) = V_{21}^*(n) + V_{22}^*(n) + V_{23}^*(n) + V_{24}^*(n) + V_{25}^*(n). \tag{4.10}$$

Here,

$$V_{11}^*(n) = \left|\ln N_1^*(n) - \ln W_1^*(n)\right|,$$
$$V_{12}^*(n) = \sigma_1 (r_1^u)^2 \chi(\nu_1^u) M_1 \sum_{k=\nu_1^l}^{\nu_1^u} \sum_{t=0}^{\nu_1^u-1} \sum_{s=n-\nu_1^u-k+t}^{n-1} \left|N_1^*(s) - W_1^*(s)\right|,$$
$$V_{13}^*(n) = \sigma_1 \upsilon_1 (r_1^u)^2 \chi(\nu_1^u) \sum_{t=0}^{\nu_1^u-1} \sum_{s=n-\nu_1^u+t}^{n-1} \left|N_1^*(s) - W_1^*(s)\right|,$$

$$V_{14}^*(n) = \frac{\sigma_1 (r_1^u)^2 \chi(\nu_1^u) |K_1 - \alpha_1|^u M_1}{[1+m_2]^2} \sum_{k=\mu_1^l}^{\mu_1^u} \sum_{t=0}^{\nu_1^u-1} \sum_{s=n-\nu_1^u-k+t}^{n-1} \left|N_2^*(s) - W_2^*(s)\right|,$$

$$V_{15}^*(n) = \frac{r_1^u |K_1 - \alpha_1|^u}{[1+m_2]^2} \sum_{t=0}^{\mu_2^u-\mu_2^l} \sum_{s=n-\mu_2^u+t}^{n-1} \left|N_2^*(s) - W_2^*(s)\right|,$$

$$V_{21}^*(n) = \left|\ln N_2^*(n) - \ln W_2^*(n)\right|,$$

$$V_{22}^*(n) = \sigma_2 (r_2^u)^2 \chi(\nu_2^u) M_2 \sum_{k=\nu_2^l}^{\nu_2^u} \sum_{t=0}^{\nu_2^u-1} \sum_{s=n-\nu_2^u-k+t}^{n-1} \left|N_2^*(s) - W_2^*(s)\right|,$$

$$V_{23}^*(n) = \sigma_2 \upsilon_2 (r_2^u)^2 \chi(\nu_2^u) \sum_{t=0}^{\nu_2^u-1} \sum_{s=n-\nu_2^u+t}^{n-1} \left|N_2^*(s) - W_2^*(s)\right|,$$

$$V_{24}^*(n) = \frac{\sigma_2 (r_2^u)^2 \chi(\nu_2^u) |K_2 - \alpha_2|^u M_2}{[1+m_1]^2} \sum_{k=\mu_2^l}^{\mu_2^u} \sum_{t=0}^{\nu_2^u-1} \sum_{s=n-\nu_2^u-k+t}^{n-1} \left|N_1^*(s) - W_1^*(s)\right|,$$

$$V_{25}^*(n) = \frac{r_2^u |K_2 - \alpha_2|^u}{[1+m_1]^2} \sum_{t=0}^{\mu_1^u-\mu_1^l} \sum_{s=n-\mu_1^u+t}^{n-1} \left|N_1^*(s) - W_1^*(s)\right|. \tag{4.11}$$

Similar to the argument as that in (3.26), one has

$$\Delta V^* \le -\lambda \left|N_1^*(n) - W_1^*(n)\right| - \lambda \left|N_2^*(n) - W_2^*(n)\right|, \quad \forall n \in \mathbb{Z}. \tag{4.12}$$

Summing both sides of the previous inequality from n to 0, we have

$$\lambda \sum_{s=n}^{0} \left|N_1^*(s) - W_1^*(s)\right| + \lambda \sum_{s=n}^{0} \left|N_2^*(s) - W_2^*(s)\right| \le V^*(n) - V^*(1), \quad \forall n < 0. \tag{4.13}$$

Note that V^* is bounded. Hence, we have

$$\sum_{s=-\infty}^{0} \left|N_1^*(s) - W_1^*(s)\right| < \infty, \qquad \sum_{s=-\infty}^{0} \left|N_2^*(s) - W_2^*(s)\right| < \infty, \tag{4.14}$$

which imply that

$$\lim_{n \to -\infty} \left|N_1^*(n) - W_1^*(n)\right| = 0, \qquad \lim_{n \to -\infty} \left|N_2^*(n) - W_2^*(n)\right| = 0. \tag{4.15}$$

Let

$$P_0 \stackrel{\text{def}}{=} 2(\lambda_1+\lambda_2)\max_{1\le i,j\le 2, i\ne j}\left\{\frac{1}{m_i}+2\sigma_i\left(r_i^u\right)^2 M_i\left(\nu_i^u\right)^3+\sigma_i\upsilon_i\left(r_i^u\right)^2\left(\nu_i^u\right)^2\right.$$
$$\left.+\sigma_i\left(r_i^u\right)^2|K_i-\alpha_i|^u M_i\left(\mu_i^u+1\right)\left(\nu_i^u\right)^2+r_i^u|K_i-\alpha_i|^u\left(\mu_j^u+1\right)\mu_j^u\right\}. \tag{4.16}$$

For arbitrary $\epsilon>0$, there exists a positive integer $\mathbb{K}_1$ such that

$$\left|N_1^*(n)-W_1^*(n)\right|<\frac{\epsilon}{P_0},\quad \left|N_2^*(n)-W_2^*(n)\right|<\frac{\epsilon}{P_0},\quad \forall n<-\mathbb{K}_1. \tag{4.17}$$

Hence, for $i,j=1,2$ with $i\ne j$, one has

$$\begin{aligned}
&V_{i1}^*(n)\le\frac{\epsilon}{m_iP_0},\quad \forall n<-\mathbb{K}_1,\\
&V_{i2}^*(n)\le 2\sigma_i\left(r_i^u\right)^2M_i\left(\nu_i^u\right)^3\frac{\epsilon}{P_0},\quad \forall n<-\mathbb{K}_1,\\
&V_{i3}^*(n)\le \sigma_i\upsilon_i\left(r_i^u\right)^2\left(\nu_i^u\right)^2\frac{\epsilon}{P_0},\quad \forall n<-\mathbb{K}_1,\\
&V_{i4}^*(n)\le \sigma_i\left(r_i^u\right)^2|K_i-\alpha_i|^uM_i\left(\mu_i^u+1\right)\left(\nu_i^u\right)^2\frac{\epsilon}{P_0},\quad \forall n<-\mathbb{K}_1,\\
&V_{i5}^*(n)\le r_i^u|K_i-\alpha_i|^u\left(\mu_j^u+1\right)\mu_j^u\frac{\epsilon}{P_0},\quad \forall n<-\mathbb{K}_1,
\end{aligned} \tag{4.18}$$

which imply that

$$V^*(n)=\lambda_1V_1^*(n)+\lambda_2V_2^*(n)<\epsilon,\quad \forall n<-\mathbb{K}_1. \tag{4.19}$$

So,

$$\lim_{n\to-\infty}V^*(n)=0. \tag{4.20}$$

Note that $V^*(n)$ is a nonincreasing function on $\mathbb{Z}$ and that $V^*(n)\equiv 0$. That is,

$$N_1^*(n)=W_1^*(n),\quad N_2^*(n)=W_2^*(n),\quad \forall n\in\mathbb{Z}. \tag{4.21}$$

Therefore, each of the hull equations of system (1.2) has a unique strictly positive solution.

In view of the previous discussion, any of the hull equations of system (1.2) has a unique strictly positive solution. By Lemma 4.1, system (1.2) has a unique strictly positive almost periodic solution. The proof is completed. □

By Theorems 3.1 and 4.2, we can easily obtain the following.

Theorem 4.3. *Suppose that* (H_1)-(H_2) *hold; then system* (1.2) *admits a unique strictly positive almost periodic solution, which is globally attractive.*

By Corollaries 3.2–3.3 and Theorem 4.2, we can show the following.

Theorem 4.4. *Suppose that* (H_1) *and* (H_3) *hold; then system* (3.30) *admits a unique strictly positive almost periodic solution, which is globally attractive.*

Theorem 4.5. *Suppose that* (H_1) *and* (H_4) *hold, then system* (3.33) *admits a unique strictly positive almost periodic solution, which is globally attractive.*

5. Examples

Example 5.1. Consider the following discrete mutualism model without delay:

$$\begin{aligned} N_1(n+1) &= N_1(n)\exp\left\{\frac{0.15+0.05\cos\left(\sqrt{2}n\right)+0.4N_2(n)}{1+N_2(n)} - N_1(n)\right\}, \\ N_2(n+1) &= N_2(n)\exp\left\{\frac{0.15+0.05\sin\left(\sqrt{3}n\right)+0.4N_1(n)}{1+N_1(n)} - N_2(n)\right\}. \end{aligned} \tag{5.1}$$

Then, system (5.1) admits a unique globally attractive almost periodic solution.

Proof. Corresponding to system (3.30), $r_i \equiv 1$, $K_i^u = 0.2$, $K_i^l = 0.1$, $\alpha_i \equiv 0.4$, $i = 1,2$. By calculation, we obtain

$$\overline{M}_1 = \overline{M}_2 \leq 0.67, \tag{5.2}$$

which implies that condition (H_3) of Corollary 3.2 is satisfied with $\lambda_1 = \lambda_2 = 1$. It is easy to verify that (H_1) holds, and the result follows from Theorem 4.4. □

In paper [15], Wang and Li studied system (3.30) and obtained the following result.

Theorem 5.2 (see [15]). *Assume that* (H_1) *holds. Suppose further that*

(H_5) $\alpha_i > K_i$, $i = 1,2$,

(H_6) $0 < \min\{\Theta_{12}, \Theta_{21}\} < 1$, *where*

$$\begin{aligned} \Theta_{ij} \overset{\text{def}}{=} & 2r_i^l x_{i*} - \left[r_i^u x_j^* + \left(r_i^u\right)^2 x_i^* x_j^*\right]\left(\alpha_i^u - K_i^l\right) - \left(r_i^u x_i^*\right)^2 \\ & - \left(r_j^l x_i^*\right)^2\left(\alpha_j^u - K_j^l\right)^2 - \left[r_j^u x_i^* + \left(r_j^u\right)^2 x_i^* x_j^*\right]\left(\alpha_j^u - K_j^l\right), \quad i,j = 1,2, i \neq j. \end{aligned} \tag{5.3}$$

Here, $x_i^* \overset{\text{def}}{=} (\alpha_i^u / r_i^u)\exp[\alpha_i^u(r_i^u - 1)]$, *and* $x_{i*} \overset{\text{def}}{=} K_i^l \exp[r_i^l(K_i^l - x_i^*)]$, $i = 1,2$.
Then, system (3.30) *admits a unique uniformly asymptotically stable almost periodic solution.*

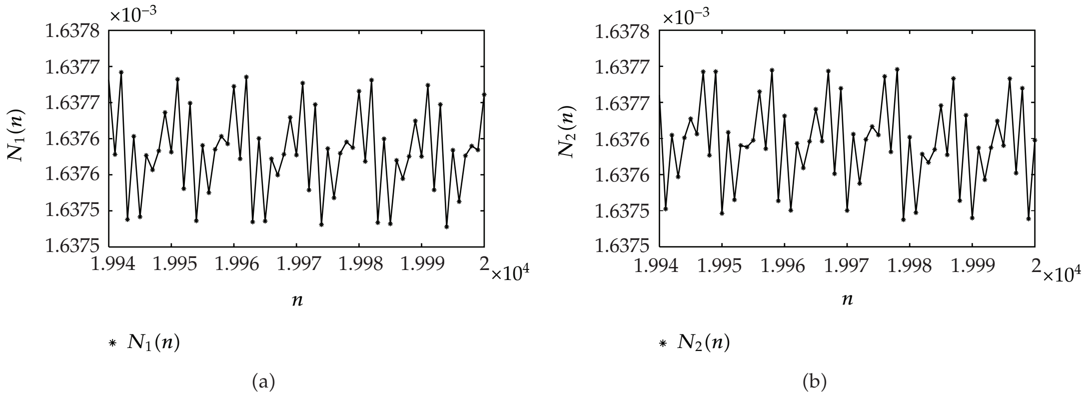

Figure 2: System (5.5) with $N_1(-1) = N_1(0) = N_2(-1) = N_2(0) = 0.002$.

Remark 5.3. In system (5.1), we can easily calculate

$$\min\{\Theta_{12}, \Theta_{21}\} < 0, \tag{5.4}$$

which implies that (H_6) of Theorem 5.2 is invalid. Therefore, it is impossible to obtain the existence of a unique globally stable almost periodic solution of system (5.1) by Theorem 5.2.

Example 5.4. Consider the following discrete mutualism model with constant delays:

$$\begin{aligned} N_1(n+1) &= N_1(n)\exp\left\{0.2\left[\frac{0.001 + 0.001\left|\cos\left(\sqrt{2}n\right)\right| + 0.002N_2(n)}{1 + N_2(n)} - N_1(n-1)\right]\right\}, \\ N_2(n+1) &= N_2(n)\exp\left\{0.2\left[\frac{0.001 + 0.001\left|\sin\left(\sqrt{3}n\right)\right| + 0.002N_1(n)}{1 + N_1(n)} - N_2(n-1)\right]\right\}. \end{aligned} \tag{5.5}$$

Then, system (5.5) admits a unique globally attractive almost periodic solution.

Proof. Corresponding to system (3.33), $r_i \equiv 0.2$, $K_i^u = 0.002$, $K_i^l = 0.001$, $\alpha_i \equiv 0.002$, $\mu_i = 0$, and $\nu_i = 1$, $i = 1, 2$. By calculation, we obtain

$$\widehat{M}_1 = \widehat{M}_2 \le 5 \times 10^{-3}, \qquad \widehat{m}_1 = \widehat{m}_2 \ge 2 \times 10^{-4}, \qquad \widehat{\sigma}_1 = \widehat{\sigma}_2 \le 1, \qquad \widehat{\upsilon}_1 = \widehat{\upsilon}_2 \le 5 \times 10^{-3}, \tag{5.6}$$

which implies that condition (H_4) of Corollary 3.3 is satisfied with $\lambda_1 = \lambda_2 = 1$. It is easy to verify that (H_1) holds, and the result follows from Theorem 4.5 (see Figure 2). □

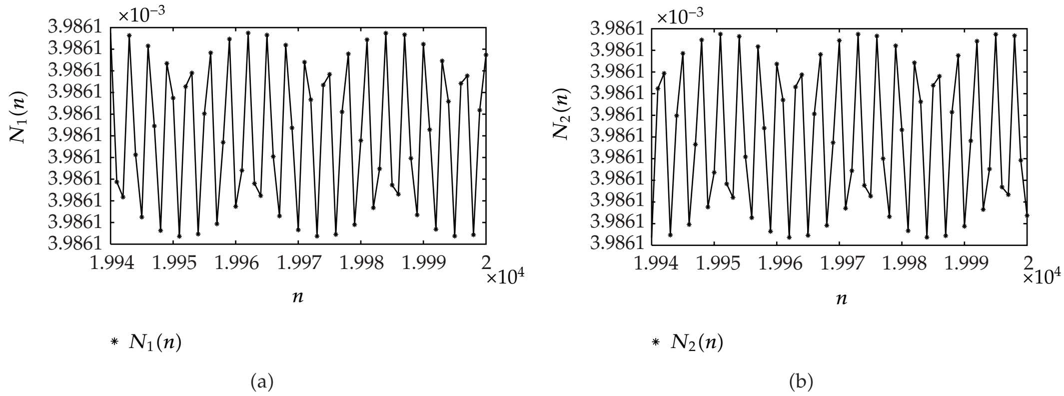

Figure 3: System (5.7) with $N_1(-1) = N_1(0) = N_2(-1) = N_2(0) = 0.002$.

Example 5.5. Consider the following discrete mutualism model with variable delays:

$$N_1(n+1) = N_1(n)\exp\left\{0.5\left[\frac{0.002 + \left(0.002 + 0.001\cos^2 n\right)N_2(n)}{1 + N_2(n)} - N_1\left(n - \frac{1 + (-1)^n}{2}\right)\right]\right\},$$
$$N_2(n+1) = N_2(n)\exp\left\{0.5\left[\frac{0.002 + \left(0.002 + 0.001\sin^2 n\right)N_1(n)}{1 + N_1(n)} - N_2(n)\right]\right\}. \tag{5.7}$$

Then, system (5.7) admits a unique globally attractive almost periodic solution.

Proof. Corresponding to system (1.2), $r_i \equiv 0.5$, $K_i \equiv 0.002$, $\alpha_i^l = 0.002$, $\alpha_i^u = 0.003$, $\nu_1^l = 0$, $\nu_1^u = 1$, $\nu_2 \equiv 0$, and $\mu_i \equiv 0$, $i = 1, 2$. By calculation, we obtain

$$M_2 < M_1 \le 5.03 \times 10^{-3}, \quad m_2 > m_1 \ge 1.974 \times 10^{-3}, \quad \sigma_1 = \sigma_2 \le 1, \quad \upsilon_1 = \upsilon_2 \le 5.03 \times 10^{-3}, \tag{5.8}$$

which implies that condition (H_2) of Theorem 4.3 is satisfied with $\lambda_1 = \lambda_2 = 1$. It is easy to verify that (H_1) holds, and the result follows from Theorem 4.3 (see Figure 3). □

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