Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 315158, 12 pages
http://dx.doi.org/10.1155/2013/315158

Research Article

Kamenev-Type Oscillation Criteria of Second-Order Nonlinear Dynamic Equations on Time Scales

Yang-Cong Qiu[1,2] and Qi-Ru Wang[2]

[1] *Department of Humanities & Education, Shunde Polytechnic, Foshan, Guangdong 528333, China*
[2] *School of Mathematics & Computational Science, Sun Yat-Sen University, Guangzhou, Guangdong 510275, China*

Correspondence should be addressed to Qi-Ru Wang; mcswqr@mail.sysu.edu.cn

Received 20 November 2012; Accepted 14 December 2012

Academic Editor: Elena Braverman

Using functions from some function classes and a generalized Riccati technique, we establish Kamenev-type oscillation criteria for second-order nonlinear dynamic equations on time scales of the form $(p(t)\psi(x(t))k \circ x^{\Delta}(t))^{\Delta} + f(t, x(\sigma(t))) = 0$. Two examples are included to show the significance of the results.

1. Introduction

In this paper, we study the second-order nonlinear dynamic equation

$$\left(p(t)\psi(x(t))k \circ x^{\Delta}(t)\right)^{\Delta} + f(t, x(\sigma(t))) = 0 \tag{1}$$

on a time scale $\mathbb{T}$.

Throughout this paper, we will assume that

(C1) $p \in C_{\rm rd}(\mathbb{T}, (0, \infty))$,

(C2) $\psi \in C(\mathbb{R}, (0, \eta])$, where η is a fixed positive constant,

(C3) $k \in C(\mathbb{R}, \mathbb{R})$, and there exist $\gamma_1 \geq \gamma_2 > 0$ such that $0 < \gamma_2 y k(y) \leq k^2(y) \leq \gamma_1 y k(y)$ for all $y \neq 0$,

(C4) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$.

Preliminaries about time scale calculus can be found in [1–3], and hence we omit them here. Note that for some typical time scales, we have the following properties, respectively:

(1) $\mathbb{T} = \mathbb{R}_+ := [0, \infty)$, we have

$$\sigma(t) = \rho(t) = t, \qquad f^{\Delta}(t) = f'(t),$$
$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt, \tag{2}$$

(2) $\mathbb{T} = \mathbb{N}_0$, we have

$$\sigma(t) = t + 1, \qquad \rho(t) = t - 1,$$
$$f^{\Delta}(t) = f(t+1) - f(t), \tag{3}$$
$$\int_a^b f(t)\,\Delta t = \sum_{k=a}^{b-1} f(k), \quad a \leq b,$$

(3) $\mathbb{T} = h\mathbb{N}_+$, $h \in \mathbb{R}_+ \setminus \{0\}$, we have

$$\sigma(t) = t + h, \qquad \rho(t) = t - h,$$
$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}, \tag{4}$$
$$\int_a^b f(t)\,\Delta t = \sum_{k=a/h}^{(b/h)-1} f(hk)\,h, \quad a \leq b;$$

(4) $\mathbb{T} = \{2^n, n \in \mathbb{N}_0\}$, we have

$$\sigma(t) = 2t, \qquad \rho(t) = \frac{t}{2}, \qquad f^{\Delta}(t) = \frac{f(2t) - f(t)}{t},$$
$$\int_a^b f(t)\,\Delta t = \sum_{k=\log_2 a}^{\log_2 b - 1} f\left(2^k\right) 2^k, \quad a \leq b. \tag{5}$$

Without loss of generality, we assume throughout that $\sup \mathbb{T} = \infty$ since we are interested in extending oscillation criteria for the typical time scales above.

Definition 1. A solution x of (1) is said to have a generalized zero at $t^* \in \mathbb{T}$ if $x(t^*)x(\sigma(t^*)) \leq 0$, and it is said to be nonoscillatory on $\mathbb{T}$ if there exists $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t > t_0$. Otherwise, it is oscillatory. Equation (1) is said to be oscillatory if all solutions of (1) are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis [4] in 1988 in order to unify continuous and discrete analysis; see also [5]. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales; for example, see [1–28] and the references therein. In Došlý and Hilger [10], the authors considered the second-order dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)x(\sigma(t)) = 0, \tag{6}$$

and gave necessary and sufficient conditions for the oscillation of all solutions on unbounded time scales. In Del Medico and Kong [8, 9], the authors employed the following Riccati transformation

$$u(t) = \frac{p(t)x^{\Delta}(t)}{x(t)} \tag{7}$$

and gave sufficient conditions for Kamenev-type oscillation criteria of (6) on a measure chain.

In Wang [25], the author considered second-order nonlinear damped differential equation

$$\begin{gathered}\left(a(t)\psi(x(t))k\left(x'(t)\right)\right)' + p(t)k\left(x'(t)\right) \\ + q(t)f(x(t)) = 0, \quad t \geq t_0,\end{gathered} \tag{8}$$

used the following generalized Riccati transformations

$$\begin{gathered}v(t) = \phi(t)a(t)\left[\frac{\psi(x(t))k\left(x'(t)\right)}{f(x(t))} + R(t)\right], \quad t \geq t_0, \\ v(t) = \phi(t)a(t)\left[\frac{\psi(x(t))k\left(x'(t)\right)}{x(t)} + R(t)\right], \quad t \geq t_0,\end{gathered} \tag{9}$$

where $\phi \in C^1([t_0,\infty),\mathbb{R}_+), R \in C([t_0,\infty),\mathbb{R})$, and gave a new oscillation criteria of (8). In Huang and Wang [16], the authors considered second-order nonlinear dynamic equation on time scales

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + f(t,x(\sigma(t))) = 0. \tag{10}$$

By using a similar generalized Riccati transformation which is more general than (7)

$$u(t) = \frac{A(t)p(t)x^{\Delta}(t)}{x(t)} + B(t), \tag{11}$$

where $A \in C^1_{rd}(\mathbb{T},\mathbb{R}_+ \setminus \{0\})$, $B \in C^1_{rd}(\mathbb{T},\mathbb{R})$, the authors extended the results in Del Medico and Kong [8, 9] and established some new Kamenev-type oscillation criteria.

In this paper, we will use functions in some function classes and a similar generalized Riccati transformation as (11) and was used in [25, 26] for nonlinear differential equations, and establish Kamenev-type oscillation criteria for (1) in Section 2. Finally, in Section 3, two examples are included to show the significance of the results.

For simplicity, throughout this paper, we denote $(a,b) \cap \mathbb{T} = (a,b)_{\mathbb{T}}$, where $a,b \in \mathbb{R}$, and $[a,b]_{\mathbb{T}}, [a,b)_{\mathbb{T}}, (a,b]_{\mathbb{T}}$ are denoted similarly.

2. Kamenev-Type Criteria

In this section we establish Kamenev-type criteria for oscillation of (1). Our approach to oscillation problems of (1) is based largely on the application of the Riccati transformation. Now, we give the first lemma.

Lemma 2. *Assume that (C1)–(C4) hold and that there exists a function $q \in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$. Also, suppose that $x(t)$ is a solution of (1) satisfies $x(t) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ with $t_0 \in \mathbb{T}$. For $t \in [t_0,\infty)_{\mathbb{T}}$, define*

$$u(t) = A(t)\frac{p(t)\psi(x(t))k \circ x^{\Delta}(t)}{x(t)} + B(t), \tag{12}$$

where $A \in C^1_{rd}(\mathbb{T},\mathbb{R}_+ \setminus \{0\})$, $B \in C^1_{rd}(\mathbb{T},\mathbb{R})$, and $\gamma_1 A - (\gamma_1 - \gamma_2)A^{\sigma} > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Then, $u(t)$ satisfies

$$\mu(t)u(t) - \mu(t)B(t) + \gamma_1\eta A(t)p(t) > 0, \tag{13}$$

$$\begin{aligned}&u^{\Delta}(t) + \Phi_1(t) \\ &\quad + \frac{\left[\gamma_1 A(t) - (\gamma_1-\gamma_2)A^{\sigma}(t)\right]u^2(t)}{\gamma_1 A(t)\left(\mu(t)u(t) - \mu(t)B(t) + \gamma_1\eta A(t)p(t)\right)} \\ &\quad + \frac{-\Phi_0(t)u(t) + \gamma_2 A^{\sigma}(t)B^2(t)}{\gamma_1 A(t)\left(\mu(t)u(t) - \mu(t)B(t) + \gamma_1\eta A(t)p(t)\right)} \leq 0,\end{aligned} \tag{14}$$

where $\Phi_0(t) = ((2\gamma_2 - \gamma_1)A^{\sigma}(t) + \gamma_1 A(t))B(t) + \gamma_1^2\eta A^{\Delta}(t) A(t)p(t)$, $\Phi_1(t) = A^{\sigma}(t)(q(t) - (B(t)/A(t))^{\Delta})$, $A^{\sigma}(t) = A(\sigma(t))$.

Proof. By (C3), we see that x^{Δ} and $k \circ x^{\Delta}$ are both positive, both negative, or both zero. When $x^{\Delta} > 0$, which implies that $k \circ x^{\Delta} > 0$, it follows that

$$\begin{aligned}&\mu u - \mu B + \gamma_1 Ap\psi(x) \\ &\quad \geq \mu\frac{Ap\psi(x)\left(k \circ x^{\Delta}\right)^2}{xk \circ x^{\Delta}} + \gamma_2 Ap\psi(x) \\ &\quad \geq \gamma_2\mu\frac{Ap\psi(x)x^{\Delta}k \circ x^{\Delta}}{xk \circ x^{\Delta}} \\ &\qquad + \gamma_2 Ap\psi(x) = \gamma_2 Ap\psi(x)\frac{x^{\sigma}}{x} > 0.\end{aligned} \tag{15}$$

When $x^\Delta < 0$, which implies that $k \circ x^\Delta < 0$, it follows that

$$\begin{aligned}\mu u - \mu B + \gamma_1 A p\psi(x) &\geq \mu \frac{Ap\psi(x)\left(k\circ x^\Delta\right)^2}{xk\circ x^\Delta} + \gamma_1 Ap\psi(x)\\ &\geq \gamma_1\mu\frac{Ap\psi(x)\,x^\Delta k\circ x^\Delta}{xk\circ x^\Delta} + \gamma_1 Ap\psi(x)\\ &= \gamma_1 Ap\psi(x)\frac{x^\sigma}{x} \geq \gamma_2 Ap\psi(x)\frac{x^\sigma}{x} > 0.\end{aligned} \tag{16}$$

When $x^\Delta = 0$, which implies that $k \circ x^\Delta = 0$ and $x = x^\sigma$, it follows that

$$\begin{aligned}\mu u - \mu B + \gamma_1 Ap\psi(x) &= \gamma_1 Ap\psi(x) \geq \gamma_2 Ap\psi(x)\\ &= \gamma_2 Ap\psi(x)\frac{x^\sigma}{x} > 0.\end{aligned} \tag{17}$$

Hence, we always have

$$\begin{aligned}&\mu u - \mu B + \gamma_1\eta Ap \geq \mu u - \mu B + \gamma_1 Ap\psi(x) > 0,\\ &\frac{x}{x^\sigma} \geq \frac{\gamma_2 Ap\psi(x)}{\mu u - \mu B + \gamma_1 Ap\psi(x)} \geq \frac{\gamma_2 Ap\psi(x)}{\mu u - \mu B + \gamma_1\eta Ap},\end{aligned} \tag{18}$$

that is, (13) holds. Then, differentiating (12) and using (1), it follows that

$$\begin{aligned}u^\Delta &= A^\Delta\left(\frac{p\psi(x)\,k\circ x^\Delta}{x}\right) + A^\sigma\left(\frac{p\psi(x)\,k\circ x^\Delta}{x}\right)^\Delta + B^\Delta\\ &= \frac{A^\Delta}{A}(u-B)\\ &\quad + A^\sigma\frac{\left(p\psi(x)\,k\circ x^\Delta\right)^\Delta x - p\psi(x)\,k\circ x^\Delta x^\Delta}{xx^\sigma} + B^\Delta\\ &= \frac{A^\Delta}{A}u + B^\Delta - \frac{A^\Delta}{A}B - A^\sigma\frac{f(t,x^\sigma)}{x^\sigma}\\ &\quad - A^\sigma p\psi(x)\frac{k\circ x^\Delta x^\Delta}{x^2}\frac{x}{x^\sigma}\\ &\leq \frac{A^\Delta}{A}u + A^\sigma\left(\frac{B}{A}\right)^\Delta - A^\sigma q - A^\sigma p\psi(x)\frac{\left(k\circ x^\Delta\right)^2}{\gamma_1 x^2}\frac{x}{x^\sigma}\\ &\leq \frac{A^\Delta}{A}u - \Phi_1 - \frac{1}{\gamma_1}A^\sigma p\psi(x)\\ &\quad\times\frac{(u-B)^2}{A^2p^2\psi^2(x)}\frac{\gamma_2 Ap\psi(x)}{\mu u - \mu B + \gamma_1\eta Ap}\\ &= \frac{A^\Delta}{A}u - \Phi_1 - \frac{\gamma_2}{\gamma_1}\frac{A^\sigma}{A}\frac{(u-B)^2}{\mu u - \mu B + \gamma_1\eta Ap}\\ &= \frac{-\left[\gamma_1 A - (\gamma_1-\gamma_2)A^\sigma\right]u^2 + \Phi_0 u - \gamma_2 A^\sigma B^2}{\gamma_1 A\left(\mu u - \mu B + \gamma_1\eta Ap\right)} - \Phi_1,\end{aligned} \tag{19}$$

that is, (14) holds. Lemma 2 is proved. □

Remark 3. In Lemma 2, the condition $\gamma_1 A - (\gamma_1 - \gamma_2)A^\sigma > 0$ ensures that the coefficient of u^2 in (14) is always negative. The condition is obvious and easy to be fulfilled. For example, when $A^\Delta(t) \leq 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, we have $A^\sigma = A + \mu A^\Delta \leq A$, by (C3), we see that $\gamma_1 A - (\gamma_1-\gamma_2)A^\sigma > 0$, and when $\gamma_1 = \gamma_2$, the condition $\gamma_1 A - (\gamma_1 - \gamma_2)A^\sigma > 0$ is also fulfilled.

Let $D_0 = \{s \in \mathbb{T} : s \geq 0\}$ and $D = \{(t,s) \in \mathbb{T}^2 : t \geq s \geq 0\}$. For any function $f(t,s)$: $\mathbb{T}^2 \to \mathbb{R}$, denote by f_1^Δ and f_2^Δ the partial derivatives of f with respect to t and s, respectively. For $E \subset \mathbb{R}$, denote by $L_{\text{loc}}(E)$ the space of functions which are integrable on any compact subset of E. Define

$$\begin{aligned}(\mathscr{A},\mathscr{B}) = &\Big\{(A,B) : A(s) \in C_{rd}^1\left(D_0, \mathbb{R}_+\setminus\{0\}\right),\\ &B(s) \in C_{rd}^1\left(D_0,\mathbb{R}\right), \gamma_1 A - \left(\gamma_1-\gamma_2\right)A^\sigma > 0,\\ &\gamma_1\eta A(s)\,p(s) \pm \mu(s)\,B(s) > 0, s \in D_0\Big\},\\ \mathscr{H}^* = &\Big\{H(t,s) \in C^1\left(D,\mathbb{R}_+\right) : H(t,t) = 0,\\ &H(t,s) > 0, H_2^\Delta(t,s) \leq 0, t > s \geq 0\Big\},\\ \mathscr{H}_* = &\Big\{H(t,s) \in C^1\left(D,\mathbb{R}_+\right) : H(t,t) = 0,\\ &H(t,s) > 0, H_1^\Delta(t,s) \geq 0, t > s \geq 0\Big\}.\end{aligned} \tag{20}$$

These function classes will be used throughout this paper. Now, we are in a position to give our first theorem.

Theorem 4. *Assume that (C1)–(C4) hold and that there exists a function $q \in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$. Also, suppose that there exist $(A,B) \in (\mathscr{A},\mathscr{B})$ and $H \in \mathscr{H}^*$ such that $M_1(t,\cdot) \in L([0,\rho(t)]_{\mathbb{T}})$ and for any $t_0 \in \mathbb{T}$,*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\,\Phi_1(s)\,\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\,\Delta s\right.$$

$$+H_2^{\Delta}(t,\rho(t))(\gamma_1\eta A(\rho(t))p(\rho(t)) -\mu(\rho(t))B(\rho(t)))\Big] = \infty, \quad (21)$$

where Φ_1 *is defined as before, and*

$$M_1(t,s) \triangleq \frac{\left\{\gamma_1 H(t,s)A(s)B(s) + (2\gamma_2-\gamma_1)H(t,\sigma(s))A^{\sigma}(s)B(s) + \gamma_1^2\eta A(s)p(s)(H(t,s)A(s))^{\Delta_s}\right\}^2}{\left\{\begin{array}{c}4\gamma_1 A(s)\min\{[\gamma_1 H(t,s)A(s) - (\gamma_1-\gamma_2)H(t,\sigma(s))A^{\sigma}(s)]\times[\gamma_1\eta A(s)p(s)-\mu(s)B(s)], \\ \gamma_2 H(t,\sigma(s))A^{\sigma}(s)[\gamma_1\eta A(s)p(s)+\mu(s)B(s)]\}\end{array}\right\}}. \quad (22)$$

Then, (1) *is oscillatory.*

Proof. Assume that (1) is not oscillatory. Without loss of generality, we may assume there exists $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Let $u(t)$ be defined by (12). Then, by Lemma 2, (13) and (14) hold.

For simplicity in the following, we let $H_\sigma = H(t,\sigma(s))$, $H = H(t,s)$, and $H_2^{\Delta} = H_2^{\Delta}(t,s)$ and omit the arguments in the integrals. For $s \in \mathbb{T}$,

$$H_\sigma - H = H_2^{\Delta}\mu. \quad (23)$$

Since $H_2^{\Delta} \le 0$ on D, we see that $H_\sigma \le H$. From $\gamma_1 A-(\gamma_1-\gamma_2)A^{\sigma} > 0$ and (C3), we have

$$\gamma_1 HA - (\gamma_1-\gamma_2)H_\sigma A^{\sigma} > \gamma_1 H_\sigma A - \gamma_1 H_\sigma A = 0. \quad (24)$$

Multiplying (14), where t is replaced by s, by H_σ and integrating it with respect to s from t_0 to t with $t \in \mathbb{T}$ and $t \ge \sigma(t_0)$, we obtain

$$\int_{t_0}^{t} H_\sigma\Phi_1\Delta s \le -\int_{t_0}^{t}\left(H_\sigma u^{\Delta} + H_\sigma\frac{[\gamma_1 A-(\gamma_1-\gamma_2)A^{\sigma}]u^2}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)} + \frac{-\Phi_0 u+\gamma_2 A^{\sigma}B^2}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)}\right)\Delta s. \quad (25)$$

Noting that $H(t,t) = 0$, by the integration by parts formula, we have

$$\begin{aligned}\int_{t_0}^{t} H_\sigma\Phi_1\Delta s &\le H(t,t_0)u(t_0) + \int_{t_0}^{t}\left(H_2^{\Delta}u - H_\sigma\frac{[\gamma_1 A-(\gamma_1-\gamma_2)A^{\sigma}]u^2-\Phi_0 u+\gamma_2 A^{\sigma}B^2}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)}\right)\Delta s \\ &\le H(t,t_0)u(t_0) + \int_{t_0}^{t}\left(H_2^{\Delta}u - H_\sigma\frac{[\gamma_1 A-(\gamma_1-\gamma_2)A^{\sigma}]u^2-\Phi_0 u}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)}\right)\Delta s \\ &= H(t,t_0)u(t_0) + \int_{\rho(t)}^{t} H_2^{\Delta}u\Delta s + \int_{t_0}^{\rho(t)}\left(H_2^{\Delta}u - H_\sigma\frac{[\gamma_1 A-(\gamma_1-\gamma_2)A^{\sigma}]u^2-\Phi_0 u}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)}\right)\Delta s.\end{aligned} \quad (26)$$

Since $H_2^{\Delta} \le 0$ on D, from (13) we see that for $t \ge \sigma(t_0)$,

$$\begin{aligned}\int_{\rho(t)}^{t} H_2^{\Delta}u\Delta s &= H_2^{\Delta}(t,\rho(t))u(\rho(t))\mu(\rho(t)) \\ &\le -H_2^{\Delta}(t,\rho(t))(\gamma_1\eta A(\rho(t))p(\rho(t)) -\mu(\rho(t))B(\rho(t))).\end{aligned} \quad (27)$$

For $t \ge \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, and $u(s) \le 0$, from (24), we have

$$\begin{aligned}H_2^{\Delta}u - H_\sigma\frac{[\gamma_1 A-(\gamma_1-\gamma_2)A^{\sigma}]u^2-\Phi_0 u}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)} &= -\frac{[\gamma_1 HA-(\gamma_1-\gamma_2)H_\sigma A^{\sigma}]u^2}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)} + \frac{[\gamma_1 HAB+(2\gamma_2-\gamma_1)H_\sigma A^{\sigma}B+\gamma_1^2\eta Ap(HA)^{\Delta}]u}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)} \\ &= -\frac{\gamma_1 HA-(\gamma_1-\gamma_2)H_\sigma A^{\sigma}}{\gamma_1 A(\mu u-\mu B+\gamma_1\eta Ap)}u^2 + \frac{\gamma_1 HAB+(2\gamma_2-\gamma_1)H_\sigma A^{\sigma}B+\gamma_1^2\eta Ap(HA)^{\Delta}}{\gamma_1 A(\gamma_1\eta Ap-\mu B)}u \\ &\quad - \frac{\gamma_1 HAB+(2\gamma_2-\gamma_1)H_\sigma A^{\sigma}B+\gamma_1^2\eta Ap(HA)^{\Delta}}{\gamma_1 A(\gamma_1\eta Ap-\mu B)} \times\frac{\mu u^2}{\mu u-\mu B+\gamma_1\eta Ap}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\gamma_2 H_\sigma A^\sigma (\gamma_1 \eta A p + \mu B)}{\gamma_1 A (\gamma_1 \eta A p - \mu B)(\mu u - \mu B + \gamma_1 \eta A p)} u^2 \\
&\quad + \frac{\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta}{\gamma_1 A (\gamma_1 \eta A p - \mu B)} u \\
&\leq -\frac{\gamma_2 H_\sigma A^\sigma (\gamma_1 \eta A p + \mu B)}{\gamma_1 A (\gamma_1 \eta A p - \mu B)^2} u^2 \\
&\quad + \frac{\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta}{\gamma_1 A (\gamma_1 \eta A p - \mu B)} u \\
&= -\frac{\gamma_2 H_\sigma A^\sigma (\gamma_1 \eta A p + \mu B)}{\gamma_1 A (\gamma_1 \eta A p - \mu B)^2} \\
&\quad \times \left[u - \frac{(\gamma_1 \eta A p - \mu B)}{2\gamma_2 H_\sigma A^\sigma (\gamma_1 \eta A p + \mu B)} \right. \\
&\quad \left. \times \left(\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta \right) \right]^2 \\
&\quad + \frac{\left(\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta \right)^2}{4\gamma_1 \gamma_2 H_\sigma A^\sigma A (\gamma_1 \eta A p + \mu B)} \\
&\leq \left(\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta \right)^2 \\
&\quad \times (4\gamma_1 A \min \{ (\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma) \\
&\qquad \times (\gamma_1 \eta A p - \mu B), \gamma_2 H_\sigma A^\sigma \\
&\qquad \times (\gamma_1 \eta A p + \mu B) \})^{-1} = M_1.
\end{aligned} \tag{28}
$$

For $t \geq \sigma(t_0)$, $s \in [t_0, \rho(t))_{\mathbb{T}}$, and $u(s) > 0$, from (24), we have

$$
\begin{aligned}
&H_2^\Delta u - H_\sigma \frac{[\gamma_1 A - (\gamma_1 - \gamma_2) A^\sigma] u^2 - \Phi_0 u}{\gamma_1 A (\mu u - \mu B + \gamma_1 \eta A p)} \\
&\quad = \left(- [\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma] u^2 \right. \\
&\qquad \left. + \left[\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta \right] u \right) \\
&\qquad \times (\gamma_1 A (\mu u - \mu B + \gamma_1 \eta A p))^{-1} \\
&\quad = -\frac{\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma}{\gamma_1 A (\mu u - \mu B + \gamma_1 \eta A p)} \\
&\qquad \times \left[u - \frac{\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta}{2 (\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma)} \right]^2 \\
&\qquad + \frac{\left(\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta \right)^2}{4\gamma_1 A (\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma)(\mu u - \mu B + \gamma_1 \eta A p)} \\
&\quad \leq \left(\gamma_1 HAB + (2\gamma_2 - \gamma_1) H_\sigma A^\sigma B + \gamma_1^2 \eta A p (HA)^\Delta \right)^2 \\
&\qquad \times (4\gamma_1 A \min \{ (\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma) \\
&\qquad\quad \times (\gamma_1 \eta A p - \mu B), \gamma_2 H_\sigma A^\sigma \\
&\qquad\quad \times (\gamma_1 \eta A p + \mu B) \})^{-1} = M_1.
\end{aligned} \tag{29}
$$

Therefore, for all $t \geq \sigma(t_0)$, $s \in [t_0, \rho(t))_{\mathbb{T}}$, we have

$$
H_2^\Delta u - H_\sigma \frac{[\gamma_1 A - (\gamma_1 - \gamma_2) A^\sigma] u^2 - \Phi_0 u}{\gamma_1 A (\mu u - \mu B + \gamma_1 \eta A p)} \leq M_1. \tag{30}
$$

Then, from (26), (27), and (30), we obtain that for $t \in \mathbb{T}$ and $t > \sigma(t_0)$,

$$
\begin{aligned}
\int_{t_0}^{t} H_\sigma \Phi_1 \Delta s &\leq H(t, t_0) u(t_0) \\
&\quad + \int_{t_0}^{\rho(t)} M_1 \Delta s - H_2^\Delta (t, \rho(t)) \\
&\quad \times (\gamma_1 \eta A(\rho(t)) p(\rho(t)) - \mu(\rho(t)) B(\rho(t))).
\end{aligned} \tag{31}
$$

Hence,

$$
\begin{aligned}
\frac{1}{H(t, t_0)} &\left[\int_{t_0}^{t} H(t, \sigma(s)) \Phi_1(s) \Delta s - \int_{t_0}^{\rho(t)} M_1(t, s) \Delta s \right. \\
&\quad + H_2^\Delta (t, \rho(t)) (\gamma_1 \eta A(\rho(t)) p(\rho(t)) \\
&\quad \left. - \mu(\rho(t)) B(\rho(t))) \right] \leq u(t_0) < \infty,
\end{aligned} \tag{32}
$$

which contradicts (21) and completes the proof. □

Remark 5. If we change the condition $\gamma_1 A - (\gamma_1 - \gamma_2) A^\sigma > 0$ in the definition of $(\mathscr{A}, \mathscr{B})$ with a stronger one $A^\Delta(t) \leq 0$, (24) in the proof of Theorem 4 will be changed with

$$
\begin{aligned}
&\gamma_1 HA - (\gamma_1 - \gamma_2) H_\sigma A^\sigma \\
&\quad \geq \gamma_1 H_\sigma A - (\gamma_1 - \gamma_2) H_\sigma A = \gamma_2 H_\sigma A > 0,
\end{aligned} \tag{33}
$$

Then, the definition of M_1 can be simplified as

$$M_1(t,s) \triangleq \frac{\left\{\gamma_1 H(t,s)A(s)B(s) + (2\gamma_2-\gamma_1)H(t,\sigma(s))A^\sigma(s)B(s) + \gamma_1^2\eta A(s)p(s)(H(t,s)A(s))^{\Delta_s}\right\}^2}{4\gamma_1\gamma_2 H(t,\sigma(s))A(s)\min\left\{A(s)\left[\gamma_1\eta A(s)p(s)-\mu(s)B(s)\right], A^\sigma(s)\left[\gamma_1\eta A(s)p(s)+\mu(s)B(s)\right]\right\}}. \quad (34)$$

In the sequel, we define

$$\mathbb{T}_1 = \{s \in \mathbb{T} : s \text{ is right-dense}\}, \quad (35)$$

$$\mathbb{T}_2 = \{s \in \mathbb{T} : s \text{ is right-scattered}\}. \quad (36)$$

When $\gamma_1 = \gamma_2 = 1$, by (C3), we see that $k(y) = y$ and (1) is simplified as

$$\left(p(t)\psi(x(t))x^\Delta(t)\right)^\Delta + f(t,x(\sigma(t))) = 0. \quad (37)$$

Now, we have the following theorem, but we should note that this result does not apply to the case where all points in $\mathbb{T}$ are right dense.

Theorem 6. *Assume that (C1)–(C4) with $\gamma_1 = \gamma_2 = 1$ hold and that there exists a function $q \in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u) \ge q(t)u^2$. Let $(A,B) \in (\mathscr{A},\mathscr{B}), H \in \mathscr{H}_*, M_2(\cdot,t) \in L_{\text{loc}}([\sigma(t),\infty)_\mathbb{T})$, and $\mathbb{T}_1, \mathbb{T}_2$ be defined by (35) and (36). Then, (37) is oscillatory provided there exists $\{t_n\}_{n=1}^\infty \subset \mathbb{T}_2$, $t_n \to \infty$, such that for any $t_0 \in \mathbb{T}$, one of the following holds*

(i) $\lim_{n\to\infty} N(t_n,t_0) = \infty$ *and*

$$\limsup_{n\to\infty}\frac{1}{N(t_n,t_0)} \times \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] = \infty, \quad (38)$$

(ii) $\limsup_{n\to\infty} N(t_n,t_0) = \infty$ *and*

$$\lim_{n\to\infty}\frac{1}{N(t_n,t_0)} \times \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] = \infty, \quad (39)$$

(iii) $\limsup_{n\to\infty} N(t_n,t_0) < \infty$ *and*

$$\limsup_{n\to\infty}\left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] = \infty, \quad (40)$$

where $N(t,s) = H(t,s)(\eta A(t)p(t) - \mu(t)B(t))/\mu(t)$, Φ_1 is defined as before, and

$$M_2(s,t) \triangleq \frac{\left\{H(s,t)A(s)B(s) + H(\sigma(s),t)A^\sigma(s)B(s) + \eta A(s)p(s)(H(s,t)A(s))^{\Delta_s}\right\}^2}{4H(s,t)A(s)\min\left\{A(s)\left[\eta A(s)p(s)-\mu(s)B(s)\right], A^\sigma(s)\left[\eta A(s)p(s)+\mu(s)B(s)\right]\right\}}. \quad (41)$$

Proof. Assume that (37) is not oscillatory. Without loss of generality, we may assume there exists $t_0 \in [0,\infty)_\mathbb{T}$ such that $x(t) > 0$ for $t \in [t_0,\infty)_\mathbb{T}$. Let $u(t)$ be defined by (12) with $k(y) = y$. Then, by Lemma 2, (13) and (14) hold for $\gamma_1 = \gamma_2 = 1$. So, we have

$$\mu(t)u(t) - \mu(t)B(t) + \eta A(t)p(t) > 0, \quad (13)'$$

$$u^\Delta(t) + \Phi_1(t) + \frac{A(t)u^2(t)}{A(t)\left(\mu(t)u(t) - \mu(t)B(t) + \eta A(t)p(t)\right)} - \frac{\left[\left(A^\sigma(t)+A(t)\right)B(t) + \eta A^\Delta(t)A(t)p(t)\right]u(t)}{A(t)\left(\mu(t)u(t) - \mu(t)B(t) + \eta A(t)p(t)\right)} + \frac{A^\sigma(t)B^2(t)}{A(t)\left(\mu(t)u(t) - \mu(t)B(t) + \eta A(t)p(t)\right)} \le 0, \quad (14)'$$

where $\Phi_1(t)$ and $A^\sigma(t)$ are defined as in Lemma 2.

For simplicity in the following, we let $H'_\sigma = H(\sigma(s),t_0)$, $H' = H(s,t_0)$, and $H_1^\Delta = H_1^\Delta(s,t_0)$ and omit the arguments in the integrals. Multiplying $(14)'$, where t is replaced by s, by H'_σ and integrating it with respect to s

from t_0 to t and then using the integration by parts formula, we have that

$$\begin{aligned}
&\int_{t_0}^{t} H'_\sigma \Phi_1 \Delta s \\
&\leq -\int_{t_0}^{t}\Bigg(H'_\sigma u^\Delta + H'_\sigma \\
&\qquad \times \frac{Au^2 - \left[(A^\sigma + A)B + \eta A^\Delta Ap\right]u + A^\sigma B^2}{A(\mu u - \mu B + \eta Ap)}\Bigg)\Delta s \\
&= -H(t,t_0)u(t) \\
&\quad + \int_{t_0}^{t}\Bigg(H_1^\Delta u - H'_\sigma \\
&\qquad \times \frac{Au^2 - \left[(A^\sigma + A)B + \eta A^\Delta Ap\right]u + A^\sigma B^2}{A(\mu u - \mu B + \eta Ap)}\Bigg)\Delta s \\
&\leq -H(t,t_0)u(t) \\
&\quad + \left(\int_{t_0}^{\sigma(t_0)} + \int_{\sigma(t_0)}^{t}\right) \\
&\quad \left(H_1^\Delta u - H'_\sigma \frac{Au^2 - \left[(A^\sigma + A)B + \eta A^\Delta Ap\right]u}{A(\mu u - \mu B + \eta Ap)}\right)\Delta s.
\end{aligned} \tag{42}$$

For $s \in [t_0, t)_{\mathbb{T}}$,

$$H'_\sigma - H_1^\Delta \mu = H'. \tag{43}$$

Hence,

$$\begin{aligned}
&\int_{t_0}^{\sigma(t_0)}\left(H_1^\Delta u - H'_\sigma \frac{Au^2 - \left[(A^\sigma + A)B + \eta A^\Delta Ap\right]u}{A(\mu u - \mu B + \eta Ap)}\right)\Delta s \\
&= \mu(t_0)\left(H_1^\Delta u - H'_\sigma \frac{Au^2 - \left[(A^\sigma + A)B + \eta A^\Delta Ap\right]u}{A(\mu u - \mu B + \eta Ap)}\right)\Bigg|_{s=t_0} \\
&= \frac{\left[-AH'u^2 + \left(H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta\right)u\right]\mu}{A(\mu u - \mu B + \eta Ap)}\Bigg|_{s=t_0} \\
&= \frac{\left(H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta\right)u\mu}{A(\mu u - \mu B + \eta Ap)}\Bigg|_{s=t_0} \\
&\leq \left(\eta p\left(H'A\right)^\Delta + \frac{H'_\sigma A^\sigma B}{A}\right)\Bigg|_{s=t_0} \\
&= \eta p(t_0) H_1^\Delta(t_0,t_0) A^\sigma(t_0) + \frac{H(\sigma(t_0),t_0)A^\sigma(t_0)B(t_0)}{A(t_0)}.
\end{aligned} \tag{44}$$

Furthermore, for $t \geq t_0$, $s \in [\sigma(t_0), t)_{\mathbb{T}}$, and $u(s) \leq 0$,

$$\begin{aligned}
&H_1^\Delta u - H'_\sigma \frac{Au^2 - \left[(A^\sigma + A)B + \eta A^\Delta Ap\right]u}{A(\mu u - \mu B + \eta Ap)} \\
&= \frac{-H'Au^2 + \left[H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta\right]u}{A(\mu u - \mu B + \eta Ap)} \\
&= -\frac{H'}{\mu u - \mu B + \eta Ap}u^2 \\
&\quad + \frac{H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta}{A(\eta Ap - \mu B)}u \\
&\quad - \frac{H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta}{A(\eta Ap - \mu B)}\frac{\mu u^2}{\mu u - \mu B + \eta Ap} \\
&= -\frac{H'_\sigma A^\sigma(\eta Ap + \mu B)}{A(\eta Ap - \mu B)(\mu u - \mu B + \eta Ap)}u^2 \\
&\quad + \frac{H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta}{A(\eta Ap - \mu B)}u \\
&\leq -\frac{H'_\sigma A^\sigma(\eta Ap + \mu B)}{A(\eta Ap - \mu B)^2}u^2 \\
&\quad + \frac{H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta}{A(\eta Ap - \mu B)}u \\
&= -\frac{H'_\sigma A^\sigma(\eta Ap + \mu B)}{A(\eta Ap - \mu B)^2} \\
&\quad \times \left[u - \frac{(\eta Ap - \mu B)\left(H'AB + H'_\sigma A^\sigma B + \eta Ap\left(H'A\right)^\Delta\right)}{2H'_\sigma A^\sigma(\eta Ap + \mu B)}\right]^2
\end{aligned}$$

$$+\frac{\left(H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta\right)^2}{4H'_\sigma A^\sigma A\left(\eta Ap+\mu B\right)}$$
$$\leq\frac{\left(H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta\right)^2}{4H'A\min\left\{A\left(\eta Ap-\mu B\right),A^\sigma\left(\eta Ap+\mu B\right)\right\}}=M_2. \tag{45}$$

For $t\geq t_0$, $s\in[\sigma(t_0),t)_{\mathbb{T}}$, and $u(s)>0$,

$$\begin{aligned}
&H_1^\Delta u-H'_\sigma\frac{Au^2-\left[\left(A^\sigma+A\right)B+\eta A^\Delta Ap\right]u}{A\left(\mu u-\mu B+\eta Ap\right)}\\
&=-\frac{H'Au^2+\left[H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta\right]u}{A\left(\mu u-\mu B+\eta Ap\right)}\\
&=-\frac{H'}{\mu u-\mu B+\eta Ap}\left[u-\frac{H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta}{2H'A}\right]^2\\
&\quad+\frac{\left(H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta\right)^2}{4H'A^2\left(\mu u-\mu B+\eta Ap\right)}\\
&\leq\frac{\left(H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta\right)^2}{4H'A^2\left(\eta Ap-\mu B\right)}\\
&\leq\frac{\left(H'AB+H'_\sigma A^\sigma B+\eta Ap\left(H'A\right)^\Delta\right)^2}{4H'A\min\left\{A\left(\eta Ap-\mu B\right),A^\sigma\left(\eta Ap+\mu B\right)\right\}}=M_2.
\end{aligned} \tag{46}$$

Hence, for all $t\geq t_0$, $s\in[\sigma(t_0),t)_{\mathbb{T}}$, we have

$$H_1^\Delta u-H'_\sigma\frac{Au^2-\left[\left(A^\sigma+A\right)B+\eta A^\Delta Ap\right]u}{A\left(\mu u-\mu B+\eta Ap\right)}\leq M_2. \tag{47}$$

From (42), (44), and (47), we have

$$\begin{aligned}
&\int_{t_0}^t H'_\sigma\Phi_1\Delta s\leq -H\left(t,t_0\right)u\left(t\right)+\int_{\sigma(t_0)}^t M_2\left(s,t_0\right)\Delta s\\
&+\left[\eta p\left(t_0\right)H_1^\Delta\left(t_0,t_0\right)A^\sigma\left(t_0\right)+\frac{H\left(\sigma\left(t_0\right),t_0\right)A^\sigma\left(t_0\right)B\left(t_0\right)}{A\left(t_0\right)}\right].
\end{aligned} \tag{48}$$

For $t\in\mathbb{T}_2$, $(13)'$ implies that

$$-H\left(t,t_0\right)u\left(t\right)\leq H\left(t,t_0\right)\frac{\eta A\left(t\right)p\left(t\right)-\mu\left(t\right)B\left(t\right)}{\mu\left(t\right)}=N\left(t,t_0\right). \tag{49}$$

Hence,

$$\begin{aligned}
&\int_{t_0}^t H\left(\sigma\left(s\right),t_0\right)\Phi_1\left(s\right)\Delta s\\
&\leq N\left(t,t_0\right)+\int_{\sigma(t_0)}^t M_2\left(s,t_0\right)\Delta s\\
&\quad+\left[\eta p\left(t_0\right)H_1^\Delta\left(t_0,t_0\right)A^\sigma\left(t_0\right)\right.\\
&\quad\left.+\frac{H\left(\sigma\left(t_0\right),t_0\right)A^\sigma\left(t_0\right)B\left(t_0\right)}{A\left(t_0\right)}\right].
\end{aligned} \tag{50}$$

Assume that condition (i) holds. Let $t=t_n$ in (50). Then, we obtain

$$\begin{aligned}
&\frac{1}{N\left(t_n,t_0\right)}\left[\int_{t_0}^{t_n}H\left(\sigma\left(s\right),t_0\right)\Phi_1\left(s\right)\Delta s-\int_{\sigma(t_0)}^{t_n}M_2\left(s,t_0\right)\Delta s\right]\\
&\leq 1+\frac{1}{N\left(t_n,t_0\right)}\left[\eta p\left(t_0\right)H_1^\Delta\left(t_0,t_0\right)A^\sigma\left(t_0\right)\right.\\
&\quad\left.+\frac{H\left(\sigma\left(t_0\right),t_0\right)A^\sigma\left(t_0\right)B\left(t_0\right)}{A\left(t_0\right)}\right].
\end{aligned} \tag{51}$$

Taking the lim sup as $n\rightarrow\infty$ on both sides, we have

$$\begin{aligned}
&\limsup_{n\rightarrow\infty}\frac{1}{N\left(t_n,t_0\right)}\left[\int_{t_0}^{t_n}H\left(\sigma\left(s\right),t_0\right)\Phi_1\left(s\right)\Delta s\right.\\
&\quad\left.-\int_{\sigma(t_0)}^{t_n}M_2\left(s,t_0\right)\Delta s\right]<\infty,
\end{aligned} \tag{52}$$

which contradicts (38).

The conclusions with conditions (ii) and (iii) can be proved similarly. We omit the details. The proof is complete. □

When $(A,B)=(1,0)$, Theorems 4 and 6 can be simplified as the following corollaries, respectively.

Corollary 7. *Assume that (C1)–(C4) hold and that there exists a function $q\in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u)\geq q(t)u^2$. Also, suppose that there exists $H\in\mathscr{H}^*$ such that for any $t_0\in\mathbb{T}$,*

$$\begin{aligned}
&\limsup_{t\rightarrow\infty}\frac{1}{H\left(t,t_0\right)}\left[\int_{t_0}^t H\left(t,\sigma\left(s\right)\right)q\left(s\right)\Delta s\right.\\
&\quad-\frac{\gamma_1^2\eta}{4\gamma_2}\int_{t_0}^{\rho(t)}\frac{\left(H_2^\Delta\left(t,s\right)\right)^2}{H\left(t,\sigma\left(s\right)\right)}p\left(s\right)\Delta s\\
&\quad\left.+\gamma_1\eta H_2^\Delta\left(t,\rho\left(t\right)\right)p\left(\rho\left(t\right)\right)\right]=\infty.
\end{aligned} \tag{53}$$

Then, (1) is oscillatory.

Corollary 8. *Assume that (C1)–(C4) with $\gamma_1=\gamma_2=1$ hold and that there exists a function $q\in C_{rd}(\mathbb{T},\mathbb{R})$ such that*

$uf(t,u) \geq q(t)u^2$. *Let* $H \in \mathscr{H}_*$, $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *be defined by* (35) *and* (36). *Then,* (37) *is oscillatory provided that there exists* $\{t_n\}_{n=1}^{\infty} \subset \mathbb{T}_2$, $t_n \to \infty$, *such that for any* $t_0 \in \mathbb{T}$, *one of the following holds*

(i) $\lim_{n\to\infty}(H(t_n,t_0)p(t_n))/\mu(t_n) = \infty$ *and*

$$\limsup_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)p(t_n)} \times \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)q(s)\Delta s - \frac{\eta}{4}\int_{\sigma(t_0)}^{t_n} \frac{\left(H_1^{\Delta}(s,t_0)\right)^2}{H(s,t_0)} p(s)\Delta s\right] = \infty, \tag{54}$$

(ii) $\limsup_{n\to\infty}(H(t_n,t_0)p(t_n))/\mu(t_n) = \infty$ *and*

$$\lim_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)p(t_n)} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)q(s)\Delta s - \frac{\eta}{4}\int_{\sigma(t_0)}^{t_n} \frac{\left(H_1^{\Delta}(s,t_0)\right)^2}{H(s,t_0)} p(s)\Delta s\right] = \infty, \tag{55}$$

(iii) $\limsup_{n\to\infty}(H(t_n,t_0)p(t_n))/\mu(t_n) < \infty$ *and*

$$\limsup_{n\to\infty} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)q(s)\Delta s - \frac{\eta}{4}\int_{\sigma(t_0)}^{t_n} \frac{\left(H_1^{\Delta}(s,t_0)\right)^2}{H(s,t_0)} p(s)\Delta s\right] = \infty. \tag{56}$$

Remark 9. When $\psi(x) \equiv 1$ and $k(y) = y$, Theorems 4 and 6 reduce to [16, Theorems 2.1 and 2.2], respectively. When $\psi(x) \equiv 1$, $k(y) = y$, $f(t,u) = q(t)u$, and $(A,B) = (1,0)$, Theorems 4 and 6 reduce to [8, Theorems 2.1 and 2.2], respectively.

3. Examples

In this section, we will show the application of our oscillation criteria in two examples. We first give an example to demonstrate Theorem 4 (or Corollary 7).

Example 10. Consider the equation

$$\left[t(2+\sin x(t))\frac{x^{\Delta}(t)\left(1+\left(x^{\Delta}(t)\right)^2\right)}{2+\left(x^{\Delta}(t)\right)^2}\right]^{\Delta} + t^2\left(t^2+1\right)x(\sigma(t)) = 0, \tag{57}$$

where $p(t) = t$, $\psi(x(t)) = 2+\sin x(t)$, $k \circ x^{\Delta}(t) = (x^{\Delta}(t)(1+(x^{\Delta}(t))^2))/(2+(x^{\Delta}(t))^2)$, and $q(t) = t^2$, so we have $\gamma_1 = 1$, $\gamma_2 = 1/2$, and $\eta = 3$. Let $(A,B) = (1,0)$ and $H(t,s) = (t-s)^2$, we have

(1) $\mathbb{T} = \mathbb{R}_+$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{H(t,t_0)} &\left[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s - \frac{\gamma_1^2\eta}{4\gamma_2}\int_{t_0}^{\rho(t)} \frac{\left(H_2^{\Delta}(t,s)\right)^2}{H(t,\sigma(s))}p(s)\Delta s + \gamma_1\eta H_2^{\Delta}(t,\rho(t))p(\rho(t))\right] \\ &= \limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\left[\int_{t_0}^{t}(t-s)^2s^2\,ds - \frac{3}{2}\int_{t_0}^{t} s\,ds\right] = \infty,\end{aligned} \tag{58}$$

That is, (53) holds. By Corollary 7, we see that (57) is oscillatory;

(2) $\mathbb{T} = \mathbb{N}_0$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{H(t,t_0)} &\left[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s - \frac{\gamma_1^2\eta}{4\gamma_2}\int_{t_0}^{\rho(t)} \frac{\left(H_2^{\Delta}(t,s)\right)^2}{H(t,\sigma(s))}p(s)\Delta s + \gamma_1\eta H_2^{\Delta}(t,\rho(t))p(\rho(t))\right] \\ &= \limsup_{n\to\infty}\frac{1}{(n-l)^2} \times \left[\sum_{k=l}^{n-1}(n-k-1)^2k^2 - \frac{3}{2}\sum_{k=l}^{n-2}\frac{(2n-2k-1)^2k}{(n-k-1)^2} - 3(n-1)\right] \\ &\geq \limsup_{n\to\infty}\left[\sum_{k=l}^{n-2}\frac{k^2}{(n-l)^2} - \sum_{k=1}^{n-2}\frac{27k}{2(n-l)^2} - \frac{3(n-1)}{(n-l)^2}\right] = \infty,\end{aligned} \tag{59}$$

that is, (53) holds. By Corollary 7, we see that (57) is oscillatory;

(3) $\mathbb{T} = h\mathbb{N}_0, h \in \mathbb{R}_+ \setminus \{0\}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \left[\int_{t_0}^{t} H(t,\sigma(s))\,q(s)\,\Delta s \right. \\
&\qquad - \frac{\gamma_1^2\eta}{4\gamma_2} \int_{t_0}^{\rho(t)} \frac{\left(H_2^{\Delta}(t,s)\right)^2}{H(t,\sigma(s))} p(s)\,\Delta s \\
&\qquad \left. + \gamma_1\eta H_2^{\Delta}(t,\rho(t))\,p(\rho(t)) \right] \\
&= \limsup_{n\to\infty} \frac{1}{(hn-hl)^2} \left[\sum_{k=l}^{n-1} (hn-hk-h)^2 (hk)^2 h \right. \\
&\qquad - \frac{3}{2} \sum_{k=l}^{n-2} \frac{(2hn-2hk-h)^2 hk\cdot h}{(hn-hk-h)^2} \\
&\qquad \left. - 3h(hn-h) \right] \\
&\geq \limsup_{n\to\infty} \left[h^3 \sum_{k=l}^{n-2} \frac{k^2}{(n-l)^2} - \sum_{k=1}^{n-2} \frac{27k}{2(n-l)^2} - \frac{3(n-1)}{(n-l)^2} \right] = \infty,
\end{aligned} \tag{60}$$

that is, (53) holds. By Corollary 7, we see that (57) is oscillatory;

(4) $\mathbb{T} = \{2^n, n \in \mathbb{N}_0\}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \left[\int_{t_0}^{t} H(t,\sigma(s))\,q(s)\,\Delta s \right. \\
&\qquad - \frac{\gamma_1^2\eta}{4\gamma_2} \int_{t_0}^{\rho(t)} \frac{\left(H_2^{\Delta}(t,s)\right)^2}{H(t,\sigma(s))} p(s)\,\Delta s \\
&\qquad \left. + \gamma_1\eta H_2^{\Delta}(t,\rho(t))\,p(\rho(t)) \right] \\
&= \limsup_{t\to\infty} \frac{1}{(t-t_0)^2} \left[\int_{t_0}^{t} (t-2s)^2 s^2\,\Delta s \right. \\
&\qquad \left. - \frac{3}{2} \int_{t_0}^{t/2} \frac{(2t-3s)^2 s}{(t-2s)^2}\,\Delta s - 3\cdot\frac{t}{2}\cdot\frac{t}{2} \right] \\
&\geq \limsup_{n\to\infty} \frac{1}{(2^n-2^l)^2} \left[\sum_{k=l}^{n-1} \left(\left(2^n - 2^{k+1}\right)^2 2^{2k}\cdot 2^k \right) \right. \\
&\qquad - \sum_{k=l}^{n-2} \frac{3\cdot 2^k\cdot 2^k\cdot \left(2^{n+1} - 3\cdot 2^k\right)^2}{2\left(2^n - 2^{k+1}\right)^2} \\
&\qquad \left. - \frac{3\cdot 2^{2n}}{4} \right] = \infty,
\end{aligned} \tag{61}$$

that is, (53) holds. By Corollary 7, we see that (57) is oscillatory.

The second example illustrates Theorem 6.

Example 11. Consider the equation

$$\left[\frac{1}{t} \frac{2 + x^2(t)}{1 + x^2(t)} x^{\Delta}(t) \right]^{\Delta} + t(2+\cos t)\,x(\sigma(t)) = 0, \tag{62}$$

where $p(t) = 1/t$, $\psi(x(t)) = (2 + x^2(t))/(1 + x^2(t))$, $k \circ x^{\Delta}(t) = x^{\Delta}(t)$, and $q(t) = t$, so we have $\gamma_1 = \gamma_2 = 1$, $\eta = 2$. Let $H(t,s) = (t-s)^2$, we have

(1) $\mathbb{T} = \mathbb{N}$, let $(A, B) = (s, 1/s)$. When $t_0 = l$ is sufficiently large, there exists $t_n = n + l$ such that

$$\begin{aligned}
&\lim_{n\to\infty} H(t_n,t_0) \frac{\eta A(t_n)\,p(t_n) - \mu(t_n)\,B(t_n)}{\mu(t_n)} \\
&= \lim_{n\to\infty} \frac{(t_n-t_0)^2(2t_n-1)}{t_n} = \infty, \\
&\limsup_{n\to\infty} \frac{\mu(t_n)}{H(t,t_0)\left(\eta A(t_n)\,p(t_n) - \mu(t_n)\,B(t_n)\right)} \\
&\quad \times \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\,\Phi_1(s)\,\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\,\Delta s \right] \\
&= \limsup_{n\to\infty} \frac{t_n}{(t_n-t_0)^2(2t_n-1)} \\
&\quad \times \left[\int_{t_0}^{t_n} (s+1-t_0)^2 \frac{s^3(s+1)^2 + 2s + 1}{s^2(s+1)}\,\Delta s \right. \\
&\qquad - \int_{t_0+1}^{t_n} \left((s-t_0)^2 + (s-t_0+1)^2((s+1)/s) + 6s^2 + 6s + 2 \right)^2 \\
&\qquad \left. \times \left(4s(2s-1)(s-t_0)^2 \right)^{-1} \Delta s \right] \\
&\geq \limsup_{n\to\infty} \frac{t_n}{(t_n-t_0)^2(2t_n-1)} \\
&\quad \times \left[\int_{t_0}^{t_n} \frac{s^5}{2s^3}\,\Delta s - \int_{t_0+1}^{t_n} \frac{\left(s^2 + 2s^2 + 6s^2 + s^2\right)^2}{4s^2(s-t_0)^2}\,\Delta s \right] \\
&\geq \limsup_{n\to\infty} \frac{n}{(n-l)^2(2n-1)} \left[\sum_{k=l}^{n-1} \frac{k^2}{2} - 25 \sum_{k=l+1}^{n-1} \frac{k^2}{(k-l)^2} \right] = \infty,
\end{aligned} \tag{63}$$

that is, in Theorem 6, (i) and (38) hold. Then, (62) is oscillatory;

(2) $\mathbb{T} = h\mathbb{N}_0$, $h \in \mathbb{R}_+ \setminus \{0\}$, let $(A,B) = (s, 1/h)$. When $t_0 = hl$ is sufficiently large, there exists $t_n = h(n+l)$ such that

$$\limsup_{n\to\infty} H(t_n,t_0)\frac{\eta A(t_n)p(t_n)-\mu(t_n)B(t_n)}{\mu(t_n)}$$
$$=\limsup_{n\to\infty}\frac{(t_n-t_0)^2}{h}=\infty,$$
$$\lim_{n\to\infty}\frac{\mu(t_n)}{H(t,t_0)(\eta A(t_n)p(t_n)-\mu(t_n)B(t_n))}\times\left[\int_{t_0}^{t_n}H(\sigma(s),t_0)\Phi_1(s)\Delta s-\int_{\sigma(t_0)}^{t_n}M_2(s,t_0)\Delta s\right]$$
$$=\lim_{n\to\infty}\frac{h}{(t_n-t_0)^2}\times\left[\int_{t_0}^{t_n}(s+h-t_0)^2\frac{s^3+hs^2+1}{s}\Delta s-\int_{t_0+h}^{t_n}\Big(\big(\big(s(s-t_0)^2\big)/h\big)+\big(\big((s+h)(s+h-t_0)^2\big)/h\big)+6s^2+6hs+2h^2\Big)^2\big(4s^2(s-t_0)^2\big)^{-1}\Delta s\right]$$
$$\geq\lim_{n\to\infty}\frac{h}{(t_n-t_0)^2}\times\left[\int_{t_0}^{t_n}(s+h-t_0)^2s^2\Delta s-\int_{t_0+h}^{t_n}\frac{\big((s^3/h)+(2s^3/h)+(s^3/h)\big)^2}{4s^2(s-t_0)^2}\Delta s\right]$$
$$=\lim_{n\to\infty}\frac{h}{(hn-hl)^2}\left[\sum_{k=l}^{n-1}(hk-hl+h)^2\left(hk^2\right)h-\frac{4}{h}\sum_{k=l+1}^{n-1}\frac{(hk)^4}{(hk-hl)^2}\right]=\infty, \tag{64}$$

that is, in Theorem 6, (ii) and (39) hold. Then, (62) is oscillatory;

(3) $\mathbb{T} = \{2^n, n \in \mathbb{N}_0\}$, let $(A,B) = (1, 1/s^2)$. When $t_0 = 2^l$ is sufficiently large, there exists $t_n = 2^{n+l}$ such that

$$\limsup_{n\to\infty} H(t_n,t_0)\frac{\eta A(t_n)p(t_n)-\mu(t_n)B(t_n)}{\mu(t_n)}$$
$$=\limsup_{n\to\infty}\frac{(t_n-t_0)^2}{t_n^2}=1<\infty,$$

$$\limsup_{n\to\infty}\left[\int_{t_0}^{t_n}H(\sigma(s),t_0)\Phi_1(s)\Delta s-\int_{\sigma(t_0)}^{t_n}M_2(s,t_0)\Delta s\right]$$
$$=\limsup_{n\to\infty}\left[\int_{t_0}^{t_n}(2s-t_0)^2\left(s+\frac{2s+1}{s^2(s+1)^2}\right)\Delta s-\int_{2t_0}^{t_n}s\Big(\big((s-t_0)^2/s^2\big)+\big((2s-t_0)^2/s^2\big)+(2/s)(3s-2t_0)\Big)^2\times\big(4(s-t_0)^2\big)^{-1}\Delta s\right]$$
$$\geq\limsup_{n\to\infty}\left[\int_{t_0}^{t_n}(2s-t_0)^2s\Delta s-\int_{2t_0}^{t_n}\frac{(1+4+6)^2s}{4(s-t_0)^2}\Delta s\right]$$
$$=\limsup_{n\to\infty}\left[\sum_{k=l}^{n-1}\left(2^{k+1}-2^l\right)^22^k\cdot2^k-\frac{121}{4}\sum_{k=l+1}^{n-1}\frac{2^k\cdot2^k}{(2^k-2^l)^2}\right]=\infty, \tag{65}$$

that is, in Theorem 6, (iii) and (40) hold. Then, (62) is oscillatory.

Acknowledgments

The authors sincerely thank the referees for their valuable comments and suggestions. This Project was supported by the NNSF of China (nos. 10971231, 11071238, and 11271379).

References

[1] R. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, "Dynamic equations on time scales: a survey," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 1–26, 2002.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[4] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten [Ph.D. thesis]*, Universität Würzburg, 1988.

[5] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[6] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results in Mathematics*, vol. 35, no. 1-2, pp. 3–22, 1999.

[7] R. P. Agarwal, D. O'Regan, and S. H. Saker, "Philos-type oscillation criteria for second order half-linear dynamic equations on time scales," *The Rocky Mountain Journal of Mathematics*, vol. 37, no. 4, pp. 1085–1104, 2007.

[8] A. Del Medico and Q. Kong, "Kamenev-type and interval oscillation criteria for second-order linear differential equations

on a measure chain," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 2, pp. 621–643, 2004.

[9] A. Del Medico and Q. Kong, "New Kamenev-type oscillation criteria for second-order differential equations on a measure chain," *Computers & Mathematics with Applications*, vol. 50, no. 8-9, pp. 1211–1230, 2005.

[10] O. Došlý and S. Hilger, "A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equation on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 147–158, 2002, Dynamic equations on time scales.

[11] L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, "Interval oscillation criteria for forced second-order nonlinear delay dynamic equations with oscillatory potential," *Dynamics of Continuous, Discrete & Impulsive Systems Series A*, vol. 17, no. 4, pp. 533–542, 2010.

[12] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear dynamic equations on time scales," *Journal of the London Mathematical Society*, vol. 67, no. 3, pp. 701–714, 2003.

[13] L. Erbe, A. Peterson, and S. H. Saker, "Kamenev-type oscillation criteria for second-order linear delay dynamic equations," *Dynamic Systems and Applications*, vol. 15, no. 1, pp. 65–78, 2006.

[14] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 505–522, 2007.

[15] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for a forced second-order nonlinear dynamic equation," *Journal of Difference Equations and Applications*, vol. 14, no. 10-11, pp. 997–1009, 2008.

[16] H. Huang and Q.-R. Wang, "Oscillation of second-order nonlinear dynamic equations on time scales," *Dynamic Systems and Applications*, vol. 17, no. 3-4, pp. 551–570, 2008.

[17] R. M. Mathsen, Q.-R. Wang, and H.-W. Wu, "Oscillation for neutral dynamic functional equations on time scales," *Journal of Difference Equations and Applications*, vol. 10, no. 7, pp. 651–659, 2004.

[18] Y. C. Qiu and Q. R. Wang, "Interval oscillation criteria of second-order nonlinear dynamic equations on time scales," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 952932, 16 pages, 2012.

[19] S. H. Saker, "Oscillation of nonlinear dynamic equation on time scales," *Applied Mathematics and Computation*, vol. 148, pp. 81–91, 2004.

[20] S. H. Saker, "Oscillation of second-order delay and neutral delay dynamic equations on time scales," *Dynamic Systems and Applications*, vol. 16, no. 2, pp. 345–359, 2007.

[21] S. H. Saker, R. P. Agarwal, and D. O'Regan, "Oscillation of second-order damped dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 1317–1337, 2007.

[22] S. H. Saker and D. O'Regan, "New oscillation criteria for second-order neutral functional dynamic equations via the generalized Riccati substitution," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 423–434, 2011.

[23] S. H. Saker, D. O'Regan, and R. P. Agarwal, "Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales," *Acta Mathematica Sinica*, vol. 24, no. 9, pp. 1409–1432, 2008.

[24] A. Tiryaki and A. Zafer, "Interval oscillation of a general class of second-order nonlinear differential equations with nonlinear damping," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 60, no. 1, pp. 49–63, 2005.

[25] Q.-R. Wang, "Oscillation criteria for nonlinear second order damped differential equations," *Acta Mathematica Hungarica*, vol. 102, no. 1-2, pp. 117–139, 2004.

[26] Q.-R. Wang, "Interval criteria for oscillation of certain second order nonlinear differential equations," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 12, no. 6, pp. 769–781, 2005.

[27] H.-W. Wu, Q.-R. Wang, and Y.-T. Xu, "Oscillation and asymptotics for nonlinear second-order differential equations," *Computers & Mathematics with Applications*, vol. 48, no. 1-2, pp. 61–72, 2004.

[28] Q. Yang, "Interval oscillation criteria for a forced second order nonlinear ordinary differential equations with oscillatory potential," *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 49–64, 2003.