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Research Article

Several Types of Convergence Rates of the M/G/1 Queueing System

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We study the workload process of the M/G/1 queueing system. Firstly, we give the explicit criteria for the geometric rate of convergence and the geometric decay of stationary tail. And the parameters ε_0 and s_0 for the geometric rate of convergence and the geometric decay of the stationary tail are obtained, respectively. Then, we give the explicit criteria for the rate of convergence and decay of stationary tail for three specific types of subgeometric cases. And we give the parameters ε_1 and s_1 of the rate of convergence and the decay of the stationary tail, respectively, for the subgeometric rate $r(n) = \exp(sn^{1/(1+\alpha)})$, $s > 0$, $\alpha > 0$.

1. Introduction

We consider several types of convergence rates of the M/G/1 queueing system by using drift conditions. The M/G/1 queueing system discussed here is that the arrivals form a Poisson process with parameter λ. The service times $v_1, v_2, \ldots$ for the customers are independently identically distributed random variables with a common distribution function $B(x)$. Let

$$\frac{1}{\mu} \equiv \int_0^\infty x dB(x), \qquad \rho \equiv \frac{\lambda}{\mu}, \tag{1}$$

where $\mu > 0$ is a constant, and ρ is called the service intensity. Denote the workload process of the M/G/1 queueing system by $W(t)$; then, $\{W(t), t \geq 0\}$ is a Markov process.

Ergodicity, specially ordinary ergodicity, has been well studied for Markov processes. There are a large volume of references devoted to the geometric case (or exponential case) and the subgeometric case (e.g., see [1–3]). Hou and Liu [4, 5] discussed ergodicity of embedded M/G/1 and GI/M/n queues, polynomial and geometric ergodicity for M/G/1-type Markov chain, and processes by generating function of the first return probability. Hou and Li [6, 7] obtained the explicit necessary and sufficient conditions for polynomial ergodicity and geometric ergodicity for the class of quasi-birth-and-death processes by using matrix geometric solutions.

There is much work on decay of the tail in the stationary distribution. Li and Zhao [8, 9] studied heavy-tailed asymptotic and light-tailed asymptotic of stationary probability vectors of Markov chains of GI/G/1 type. Jarner and Roberts [10] discussed Foster-Lyapounov-type drift conditions for Markov chains which imply polynomial rate convergence to stationarity in appropriate V-norms. Jarner and Tweedie [11] proved that the geometric decay of the tail in the stationary distribution is a necessary condition for the geometric-ergodicity for random walk-type Markov chains. We will discuss several types of ergodicity and the tail asymptotic behavior of the stationary distribution by Foster-Lyapounov-drift conditions. We give the relationship of ergodicity and the decay of the tail in the stationary distribution for h-skeleton chain in M/G/1 queueing system, which is different from the former; ergodicity and the decay of the tail are discussed, respectively. We shall give the bounded interval in which geometric and subexponential parameter s lies and prove that it is determined by the tail of the service distribution. The parameters ε_0 and s_0 for geometric rate of convergence and the geometric decay of the stationary tail are obtained, respectively. We shall also give explicit criteria for the rate of convergence and decay of stationary tail for three specific types of subgeometric cases (Case 1: the rate function $r(n) = \exp(sn^{1/(1+\alpha)})$, $\alpha > 0$, $s > 0$; Case 2: polynomial rate function

$r(n) = n^{\alpha}, \alpha > 0$; Case 3: logarithmic rate function $r(n) = \log^{\alpha} n, \alpha > 0$). And we give the parameters ε_1 and s_1 of the rate of convergence and the decay of the stationary tail, respectively, for the subgeometric rate in Case 1.

We organize the paper as follows. In Section 2, we shall introduce basic definitions and theorems, including the main result, Theorem 6. In Section 3, we shall prove the geometric rates of convergence in Theorem 6. In Section 4, we shall prove the rates of convergence for the subgeometric Cases 1–3 in Theorem 6.

2. Basic Definitions and the Main Results

Let $\{X_n, n \geq 0\}$ be a discrete time Markov chain on the state space $(E, \mathscr{E})$ with transition kernel P. Assume that it is ψ-irreducible, aperiodic, and positive recurrent. Now, we discuss the convergence in f-norm of the iterates P^n of the kernel to the stationary distribution π at rate $r := (r(n), n \geq 0)$; that is, for all $A \in \mathscr{E}$,

$$\lim_{n \to \infty} r(n) \left\| P^{(n)}(x, A) - \pi(A) \right\|_f = 0, \quad \pi\text{-a.e.}, \tag{2}$$

where $f : E \to [1, +\infty)$ satisfies $\pi(f) < +\infty$, and for all signed measures σ, the f-norm $\|\sigma\|_f$ is defined as $\sup_{|g| \leq f} |\sigma(g)|$.

Geometric Rate Function. That is, the function r satisfies

$$0 < \liminf \frac{\log r(n)}{n} \leq \limsup \frac{\log r(n)}{n} < +\infty. \tag{3}$$

Subgeometric Rate Function. That is, the function r satisfies

$$\lim_{n \to \infty} \frac{\log r(n)}{n} = 0. \tag{4}$$

The class of subgeometric rates function includes polynomial rates functions; that is, $r(n) = n^{\alpha}, \alpha > 0$, and rate functions which increase faster than the polynomial ones; $r(n) = \exp(s n^{1/(1+\alpha)}), \alpha > 0, s > 0$.

We shall discuss geometric rates of convergence $r(n) = \exp(sn), s > 0$, subgeometric rate of convergence $r(n) = \exp(s n^{1/(1+\alpha)}), \alpha > 0, s > 0$, polynomial rate of convergence $r(n) = n^{\alpha}, \alpha > 0$, and logarithmic rate of convergence $r(n) = \log^{\alpha} n, \alpha > 0$.

Condition $D(\phi, V, C)$. There exist a function $V : E \to [1, \infty)$, a concave monotone nondecreasing differentiable function $\phi : [1, \infty] \to (0, \infty]$, a measurable set C, and a finite constant b such that

$$\Delta V(x) = PV(x) - V(x) \leq \phi \circ V + b I_C, \tag{5}$$

where I_C is the indicator function of the set C.

Now we shall give Theorems 1 and 2 which we will use in this paper.

Theorem 1 (Theorem 14.0.1 in [1]). *If $D(\phi, V, C)$ holds for some petite set C and there exists $x_0 \in E$ such that $V(x_0) < \infty$, then there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$ and*

$$\lim_{n \to \infty} \left\| P^{(n)}(x, A) - \pi(A) \right\|_{\phi \circ V} = 0, \quad \pi\text{-a.e.}, \tag{6}$$

where $\phi \circ V(x) \geq 1, x \in E$.

Theorem 2 (Proposition 2.5 in Douc et al. [12]). *Let P be a ψ-irreducible and aperiodic kernel. Assume that $D(\phi, V, C)$ holds for function ϕ with $\lim_{t \to +\infty} \phi'(t) = 0$, a petite set C, and a function V with $\{V < +\infty\} \neq \emptyset$. Then, there exists an invariant probability measure π, and for all x in the full and absorbing set $\{V < \infty\}$,*

$$\lim_{n \to \infty} r_{\phi}(n) \left\| P^{(n)}(x, A) - \pi(A) \right\| = 0, \tag{7}$$

where $(r_{\phi}(n)) = \phi \circ H_{\phi}^{-1}(n)$, $H_{\phi}(v) := \int_1^v (1/\phi(x)) dx$.

Since ϕ is a concave monotone nondecreasing differentiable function, ϕ' is nonincreasing. Then, there exists $c \in [0, 1)$, such that $\lim_{t \to +\infty} \phi'(t) = c$. In Theorem 2, for the case $c \in (0, 1)$, condition $D(\phi, V, C)$ implies that the chain is geometric ergodic, but the rate in the geometric convergence property cannot be achieved under the condition that $\lim_{t \to +\infty} \phi'(t) = c > 0$.

The workload process $\{W(t), t \geq 0\}$ of the M/G/1 queueing system is a Markov process on the state space $\{R_+, \mathscr{B}(R_+)\}$. $\{W(nh)\}$ is an h-skeleton of $\{W(t), t \geq 0\}$. We choose $h = 1$, and denote $\{W(nh)\}_{h=1}$ by $\{X_n\}$. Suppose that the workload can be decreased by $\min\{1, X_n\}$ during the time interval $[n, n+1]$. And suppose that the transition kernel of $\{X_n\}$ is $P(x, \cdot)$. For convenience, let $\sigma_k = \nu_1 + \nu_2 + \cdots + \nu_k$. Then,

$$X_{n+1} = \sum_{k=1}^{+\infty} I_{\{\xi = k\}} \sigma_k + \min\{X_n - 1, 0\}, \quad n = 1, 2, \ldots, \tag{8}$$

where ξ is the number of arrivals in a time interval of unit length.

Lemma 3. *$\{X_n\}$ is irreducible and aperiodic.*

Proof. Let φ be a measure on R_+ with $\varphi(\{0\}) = 1, \varphi(\{0\}^c) = 0$. For all $x \in R_+$, there exists a k satisfying $k - 1 < t \leq k$, such that

$$P^k(x, \{0\}) \geq \exp(-\lambda k) > 0. \tag{9}$$

Hence, $\{X_n\}$ is irreducible. From

$$P^n(x, \{0\}) > 0, \quad n \geq 1, \tag{10}$$

we know that $\{X_n\}$ is also aperiodic. □

Lemma 4. *$C = [0, c]$ is petite set, where c ($c \geq 0$) is a real number.*

Proof. Let $[c]$ be the maximum integer no more than c. Since

$$P^{[c]+1}(x, \{0\}) \geq \exp\{-\lambda([c] + 1)\} > 0, \quad \forall x \in C, \tag{11}$$

and C is a closed set, we know that $\min_{x\in C}P^{[c]+1}(x,\{0\})>0$. Let ν_2 be a measure on R_+ satisfying, for all $B\in\mathscr{B}(R_+)$,

$$\nu_2(B)=\begin{cases}0, & \{0\}\notin B,\\ \min_{x\in C}P^{[c]+1}(x,\{0\}), & \{0\}\in B.\end{cases} \quad (12)$$

Obviously, for all $x\in C$,

$$P^{[c]+1}(x,B)\geq\nu_2(B),\quad \forall\mathscr{B}(R_+). \quad (13)$$

Thus, we get that C is a petite set. □

Lemma 5. *The Markov chain $\{X_n\}$ is stochastically monotonic.*

Proof. For every fixed y, from

$$P\{X_{n+1}\leq y\mid X_n=x\}=P\left\{\sum_{k=1}^{+\infty}I_{\{\xi=k\}}\sigma_k\leq y\right\},\quad \forall x\in[0,1],$$

$$\begin{aligned}P\{X_{n+1}\leq y\mid X_n=x\}&=P\left\{\sum_{k=1}^{+\infty}I_{\{\xi=k\}}\sigma_k+x-1\leq y\right\},\\&=P\left\{\sum_{k=1}^{+\infty}I_{\{\xi=k\}}\sigma_k-y-1\leq -x\right\},\quad \forall x\in[1,+\infty),\end{aligned} \quad (14)$$

we obtain that $P\{X_{n+1}\leq y\mid X_n=x\}$ is nonincreasing in x. That is, $\{X_n\}$ is stochastically monotonic. □

For two sequences u_n and v_n, we write $u_n\asymp v_n$, if there exist positive constants c_1 and c_2 such that, for large n, $c_1u_n\leq v_n\leq c_2u_n$.

Let us say that the distribution function G of a random variable ξ is in $\mathscr{G}^+(r)$ if

$$Ee^{s\xi}=\int_0^{+\infty}e^{sx}G(dx)<+\infty,\quad 0<s\leq r; \quad (15)$$

the distribution function G of a random variable ξ is in $\mathscr{S}^+(r,\alpha)$ if

$$Ee^{s\xi^{1/(1+\alpha)}}=\int_0^{+\infty}e^{sx^{1/(1+\alpha)}}G(dx)<+\infty,\quad 0<s\leq r, \quad (16)$$

where $r>0$, and $\alpha>0$.

Now, we give the main result.

Theorem 6. *Suppose that $\rho<1$ and π is the stationary distribution of $\{X_n\}$.*

(1) If $B\in\mathscr{G}^+(r)$, then one has

$$\int_1^{+\infty}\exp(sx)\,\pi(dx)<+\infty,\quad \forall 0<s<s_0, \quad (17)$$

where s_0 is the minimum positive root of the equation $Ee^{s\nu_1}=1+(s/\lambda)$. Moreover, $\{X_n\}$ is geometrically ergodic,

$$\lim_{n\to\infty}e^{\varepsilon_0 n}\left\|P^{(n)}(x,A)-\pi(A)\right\|=0, \quad (18)$$

where $\varepsilon_0=\lambda+\bar{s}-\lambda Ee^{\bar{s}\nu_1}$, and $\bar{s}\in(0,s_0)$ is a root of the equation $E\nu_1e^{s\nu_1}=1/\lambda$.

(2) If $B\in\mathscr{S}^+(r,\alpha)$, then one has

$$\int_1^{+\infty}x^{-\alpha/(1+\alpha)}\exp\left(sx^{1/(1+\alpha)}\right)\pi(dx)<\infty,\quad 0<s<s_1, \quad (19)$$

where s_1 is the minimal positive solution of $\beta(s)=0$ ($\beta(s)=x^{\alpha/(1+\alpha)}\{1-\sum_{k=0}^{\infty}(\lambda^ke^{-\lambda}/k!)E\exp\{s(\sigma_k-1+x)^{1/(1+\alpha)}-sx^{1/(1+\alpha)}\}\}$). And

$$\lim_{n\to\infty}n^{(-\alpha/(1+\alpha))}\exp\left(\varepsilon_1n^{1/(1+\alpha)}\right)\left\|P^{(n)}(x,\cdot)-\pi(\cdot)\right\|=0, \quad (20)$$

where $\varepsilon_1=\max_{s\in(0,s_1)}[(\alpha+1)\beta(s)]^{1/(\alpha+1)}$.

(3) If there exists a constant $\alpha>1$, such that $E\nu_1^{\alpha}=\int_0^{+\infty}x^{\alpha}B(dx)<+\infty$, then

$$\lim_{n\to\infty}n^{\alpha-1}\left\|P^{(n)}(x,\cdot)-\pi(\cdot)\right\|=0,\qquad \int_1^{+\infty}x^{\alpha-1}\pi(dx)<+\infty. \quad (21)$$

(4) If there exists an integer number $\alpha>0$, such that $E\nu_1\log^{\alpha}(\nu_1+1)=\int_0^{+\infty}x\log^{\alpha}(x+1)B(dx)<+\infty$, then

$$\lim_{n\to\infty}\log^{\alpha}n\left\|P^{(n)}(x,\cdot)-\pi(\cdot)\right\|=0,\qquad \int_1^{+\infty}\log^{\alpha}x\,\pi(dx)<+\infty. \quad (22)$$

We shall prove Theorem 6 in Sections 3 and 4.

3. Geometric Rate of Convergence

The Markov chain $\{X_n\}$ is geometrically ergodic if (2) holds with $r(n)=e^{sn}$ for some $s>0$. By Theorem 15.0.1 in [1], an equivalent condition of geometric ergodicity is that there exist a petite set C, constants $\beta>0$ and $b<\infty$, and a function $V\geq 1$ finite for at least one $x_0\in E$ satisfying

$$\Delta V(x)<-\beta V(x)+bI_C,\quad x\in E. \quad (23)$$

By using the drift previous condition, we usually obtain the geometric ergodicity, but we could not get the parameters for the geometric rate of convergence. Now, we will study the geometric decay of the stationary tail and geometric rate of convergence to the stationary distribution.

Let $V_s(x)=\exp(sx)$, $0<s\leq r$, $x\in R_+$. Taking the petite set $C=[0,1]$, for all $x\in[0,1]$,

$$\begin{aligned}\Delta V_s(x)&=PV_s(x)-V_s(x)\\&<\sum_{k=0}^{\infty}\frac{\lambda^ke^{-\lambda}}{k!}E\exp\{s(\nu_1+\cdots+\nu_k+x)\}\\&\leq\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(Ee^{s\nu_1}\right)^ke^x=\exp\left(\lambda Ee^{s\nu_1}\right)e^x.\end{aligned} \quad (24)$$

Since $B \in \mathcal{G}^+(r)$(i.e., $Ee^{s\nu_1} < \infty, 0 < s \le r$), we know that

$$\Delta V_s(x) < \exp\left(\lambda E e^{s\nu_1}\right) e^x < \infty, \quad \forall x \in [0,1]. \tag{25}$$

For all $x > 1$ (i.e., $x \in C^C$),

$$\begin{aligned}\Delta V_s(x) &= PV_s(x) - V_s(x)\\ &= -\exp(sx) + \exp(sx)\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!} E\exp\{s(\sigma_k - 1)\}\\ &= -\exp(sx)\left\{1 - \exp(-\lambda - s)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(Ee^{s\nu_1}\right)^k\right\}\\ &= -\exp(sx)\left\{1 - \exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)\right\}.\end{aligned} \tag{26}$$

Let $\beta(s) = 1 - \exp(-\lambda - s + \lambda E e^{s\nu_1})$. Now, we prove that there exists an $s' > 0$ such that $\beta(s') > 0$. By the stated condition $B \in \mathcal{G}^+(r)$, we know that $\beta(s)$ is a finite differentiable function for $s \in [0, r)$. Furthermore,

$$\begin{aligned}\beta(0) &= \left\{1 - \exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)\right\}_{s=0} = 0,\\ \beta'(s)\big|_{s=0} &= \left\{-\left(-1 + \lambda E\nu_1 e^{s\nu_1}\right)\exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)\right\}_{s=0}\\ &= 1 - \rho > 0.\end{aligned} \tag{27}$$

Proposition 7. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$. If $B \in \mathcal{G}^+(r)$; then,*

$$\int_1^{+\infty} \exp(sx)\,\pi(dx) < +\infty, \quad \forall 0 < s < s_0, \tag{28}$$

where s_0 is the minimum positive root of the equation $Ee^{s\nu_1} = 1 + (s/\lambda)$.

Proof. By (27), we know that $\beta(0) = 0$ and $\beta'(0) > 0$. So, there exists an $s' \in (0, r)$ such that $\beta(s') > 0$. The function $\beta(s)$ is continuous in the interval $[s', r]$, and it is easy to see that $\beta(r) < 0$, $\beta(s') > 0$. By the zero theorem, we know that there exists at least one root of the equation $\beta(s) = 0$ (i.e., $Ee^{s\nu_1} = 1 + (s/\lambda)$). Let s_0 be the minimum positive root; then, $\beta(s) > 0$, for all $0 < s < s_0$.

Let $b = \sup_{s\in(0,s_0)} \exp(\lambda E e^{s\nu_1}) < \infty$; then, we have

$$\begin{aligned}\Delta V_s(x) \le& -\beta(s)V_s(x) + bI_{[0,1]}\\ =& -\phi\circ V_s(x) + bI_{[0,1]}, \quad 0 < s < s_0,\ x \in R_+,\end{aligned} \tag{29}$$

where $\phi(x) = \beta(s)x$ (i.e., condition $D(\phi, V_s, C)$ holds). By Theorem 1, we know that $\pi(\phi\circ V_s) < \infty$; that is,

$$\int_1^{+\infty} \exp(sx)\,\pi(dx) < \infty, \quad 0 < s < s_0. \tag{30}$$

□

Proposition 8. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$. If $B \in \mathcal{G}^+(r)$; then,*

$$\lim_{n\to\infty} e^{\varepsilon_0 n}\left\|P^{(n)}(x, A) - \pi(A)\right\| = 0, \tag{31}$$

where $\varepsilon_0 = \alpha(\bar{s})$, $\alpha(s) = \lambda + s - \lambda E e^{s\nu_1}$, and $\bar{s} \in (0, s_0)$ is the root of the equation $1 - \lambda E\nu_1 e^{s\nu_1} = 0$.

Proof. From (29),

$$\Delta V_s(x) \le -\beta(s)V_s(x) + bI_{[0,1]}, \quad 0 < s < s_0,\ x \in R, \tag{32}$$

where $\beta(s) = 1 - \exp(-\lambda - s + \lambda E e^{s\nu_1})$, and s_0 is the minimum positive root of the equation $Ee^{s\nu_1} = 1 + (s/\lambda)$. We have

$$\begin{aligned}PV_s(x) \le \exp\left(-\lambda - s + \lambda E e^{s\nu_1}\right)V_s(x) + bI_{[0,1]},\\ 0 < s < s_0,\ x \in R_+.\end{aligned} \tag{33}$$

From Lemma 5, we know that $\{X_n\}$ is a stochastically monotonic Markov chain. By using Theorem 1.1 in [13], we have

$$\begin{aligned}\lim_{n\to\infty} \exp\left\{\left(\lambda + s - \lambda E e^{s\nu_1}\right)n\right\}\left\|P^{(n)}(x,\cdot) - \pi(\cdot)\right\| = 0,\\ 0 < s < s_0.\end{aligned} \tag{34}$$

Let $\alpha(s) = \lambda + s - \lambda E e^{s\nu_1}$. From $\alpha''(s) = -\lambda E\nu_1^2 e^{s\nu_1} < 0$, we know that $\alpha(x)$ is a concave function. Together with $\alpha(0) = 0$, $\alpha(s_0) = 0$, there exists a unique point $\bar{s} \in (0, s_0)$, such that $\alpha'(\bar{s}) = 1 - \lambda E\nu_1 e^{\bar{s}\nu_1} = 0$, and $\alpha(s)$ has a maximum $\alpha(\bar{s})$ at the point $\bar{s}$ in the interval $(0, s_0)$. So,

$$\lim_{n\to\infty} e^{\varepsilon_0 n}\left\|P^{(n)}(x, A) - \pi(A)\right\| = 0, \tag{35}$$

where $\varepsilon_0 = \alpha(\bar{s})$. The proof is completed. □

4. Subgeometric Rates of Convergence for Cases 1–3

Case 1 (The Rate Function $r(n) = \exp(sn^{1/(1+\alpha)})$). The rate function $r(n) = \exp(sn^{1/(1+\alpha)})$, which increases to infinity faster than the polynomial one, and slower than the geometrical one, has been discussed only recently in the literature.

Proposition 9. *Suppose that $\rho < 1$ and π is the stationary distribution of $\{X_n\}$. If $B \in \mathcal{S}^+(r, \alpha)$, then one has*

$$\int_1^{+\infty} x^{-\alpha/(1+\alpha)}\exp\left(sx^{1/(1+\alpha)}\right)\pi(dx) < \infty, \quad 0 < s < s_1, \tag{36}$$

where s_1 is the minimal positive solution of $\beta(s) = 0$ ($\beta(s) = x^{\alpha/(1+\alpha)}\{1 - \sum_{k=0}^{\infty}(\lambda^k e^{-\lambda}/k!)E\exp\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\}\}$). And

$$\lim_{n\to\infty} n^{-\alpha/(1+\alpha)}\exp\left(\varepsilon_1 n^{1/(1+\alpha)}\right)\left\|P^{(n)}(x,\cdot) - \pi(\cdot)\right\| = 0, \tag{37}$$

where $\varepsilon_1 = \max_{s\in(0,s_1)}[(\alpha+1)\beta(s)]^{(1/(1+\alpha))}$.

Proof. Let $V_s(x) = \exp(sx^{1/(1+\alpha)})$, $0 < s \le r, x \in R_+$. For all $x \in C$,

$$\begin{aligned}
\Delta V_s(x) &= PV_s(x) - V_s(x) \le -\exp\left(sx^{1/(1+\alpha)}\right) + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\left\{s(\sigma_k + x)^{1/(1+\alpha)}\right\} \\
&\le -\exp\left(sx^{1/(1+\alpha)}\right) + \exp\left(sx^{1/(1+\alpha)}\right) \cdot \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\left\{s\left(v_1^{1/(1+\alpha)} + v_2^{1/(1+\alpha)} + \cdots + v_k^{1/(1+\alpha)}\right)\right\} \\
&= -\exp\left(sx^{1/(1+\alpha)}\right) + \exp\left(sx^{1/(1+\alpha)}\right) \cdot \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\left[E\exp\left(sv_1^{1/(1+\alpha)}\right)\right]^k = -\exp\left(sx^{1/(1+\alpha)}\right) \\
&\quad + \exp\left(sx^{1/(1+\alpha)}\right)\exp\left\{\lambda E\exp\left(sv_1^{1/(1+\alpha)}\right)\right\},
\end{aligned} \tag{38}$$

where the second inequality holds by using the condition that $f(x) = x^{1/(1+\alpha)}$ is concave. Let $\alpha(s) = -\exp(sx^{1/(1+\alpha)}) + \exp(sx^{1/(1+\alpha)})\exp\{\lambda E\exp(sv_1^{1/(1+\alpha)})\}$. Since $E\exp(sv_1^{1/(1+\alpha)}) < \infty$, we know that

$$\Delta V_s(x) \le \alpha(s) < \infty, \quad x \in C. \tag{39}$$

For all $x \in C^C$,

$$\begin{aligned}
\Delta V_s(x) = PV_s(x) - V_s(x) &= -\exp\left(sx^{1/(1+\alpha)}\right) + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)}\right\} \\
&= -\exp\left(sx^{1/(1+\alpha)}\right) \cdot \left\{1 - \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\right\}\right\} \\
&= -\frac{\exp\left(sx^{1/(1+\alpha)}\right)}{x^{1/(1+\alpha)}}x^{\alpha/(1+\alpha)} \cdot \left\{1 - \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\right\}\right\}.
\end{aligned} \tag{40}$$

Let

$$\beta(s) = x^{\alpha/(1+\alpha)} \cdot \left\{1 - \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\exp\left\{s(\sigma_k - 1 + x)^{1/(1+\alpha)} - sx^{1/(1+\alpha)}\right\}\right\}. \tag{41}$$

Now, we prove that there exists an $s_1 > 0$ such that $\beta(s) > 0$ for all $s \in (0, s_1)$. Similar to the proof of the case $x \in C$, we know that $\beta(s)$ is a finite function for $s \in [0, r)$. Furthermore,

$$\begin{aligned}
\beta(0) &= x^{\alpha/(1+\alpha)}\left(1 - \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\right) = 0, \\
\beta'(s)\big|_{s=0} &= -x^{\alpha/(1+\alpha)}\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{(\sigma_k - 1 + x)^{1/(1+\alpha)} - x^{1/(1+\alpha)}\right\} \\
&= x\left[1 - \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left(1 + \frac{\sigma_k - 1}{x}\right)^{1/(1+\alpha)}\right] \\
&\ge x\left[1 - \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\left(1 + \frac{(k/\mu) - 1}{x}\right)^{1/(1+\alpha)}\right] \\
&\ge x\left[1 - \left(1 + \frac{\rho - 1}{x}\right)^{1/(1+\alpha)}\right] > 0.
\end{aligned} \tag{42}$$

Let s_1 be the minimum positive root of the equation $\beta(s) = 0$; then, we have $\beta(s) > 0$ for all $0 < s < s_1$.

Let $V_s(x) = \exp(sx^{1/(1+\alpha)})$, for all $s \in (0, s_1)$, and let $b = \max_{s\in(0,s_1)}\{\alpha(s)\} < \infty$; then, we have

$$\begin{aligned}
\Delta V_s(x) &\le -\beta(s)\frac{\exp\left(sx^{1/(1+\alpha)}\right)}{x^{\alpha/(1+\alpha)}} + bI_C, \\
&= -\phi \circ V_s(x) + bI_C, \quad x \in R_+,
\end{aligned} \tag{43}$$

where $\phi(x) = \beta(s)(x/\log^{\alpha}(x))$, (i.e., Condition $D(\phi, V_s, C)$ holds). By Theorem 1, we know that there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$, that is

$$\int_1^{+\infty} x^{-\alpha/(1+\alpha)}\exp\left(sx^{1/(1+\alpha)}\right)\pi(dx) < \infty. \tag{44}$$

From

$$H_\phi(x) = \int_1^x \frac{dx}{\phi(x)} = \int_1^x \frac{\log^{\alpha}(x)\,dx}{\beta(s)\,x} = \frac{\log^{1+\alpha}(x)}{(\alpha+1)\beta(s)}, \tag{45}$$

we have $H_{\phi}^{-1}(x) = \exp[((\alpha+1)\beta(s)x)^{1/(1+\alpha)}]$. So,

$$\begin{aligned} r_{\phi}(n) &= \phi \circ H_{\phi}^{-1}(n) \\ &= \beta(s)\frac{\exp\left[((\alpha+1)\beta(s)n)^{1/(1+\alpha)}\right]}{((\alpha+1)\beta(s)n)^{\alpha/(1+\alpha)}} \\ &= \beta(s)^{1/(1+\alpha)}[(\alpha+1)n]^{-\alpha/(1+\alpha)} \\ &\quad \cdot \exp\left[((\alpha+1)\beta(s)n)^{1/(1+\alpha)}\right] \\ &\asymp n^{-\alpha/(1+\alpha)}\exp\left[((\alpha+1)\beta(s)n)^{1/(1+\alpha)}\right]. \end{aligned} \tag{46}$$

Let $\varepsilon_1 = \max_{s\in(0,s_1)}\{[(\alpha+1)\beta(s)]^{1/(1+\alpha)}\}$; then we have,

$$\lim_{n\to\infty} n^{-\alpha/(1+\alpha)}\exp\left(\varepsilon_1 n^{1/(1+\alpha)}\right)\left\|P^{(n)}(x,\cdot)-\pi(\cdot)\right\| = 0. \tag{47}$$

The proof is completed. □

Case 2 (Polynomial Rate of Convergence). Consider the following.

Proposition 10. *If $\rho < 1$ and there exists a constant $\alpha > 1$ such that*

$$E\nu_1^{\alpha} = \int_0^{+\infty} x^{\alpha}B(dx) < +\infty, \tag{48}$$

then

$$\begin{gathered} \lim_{n\to\infty} n^{\alpha-1}\left\|P^{(n)}(x,\cdot)-\pi(\cdot)\right\| = 0, \\ \int_1^{+\infty} x^{\alpha-1}\pi(dx) < +\infty. \end{gathered} \tag{49}$$

Proof. Let $V(x) = (x+1)^{\alpha} \geq 1,\ x \in R_+$, and let m_{α} be the αth moment of the poisson distribution with parameter λ. From $(a+b)^{\alpha} \leq 2^{\alpha}(a^{\alpha}+b^{\alpha})$, where $a > 0, b > 0$, we have, for all $x \in C$ (where C is the petite $[0,c]$),

$$\begin{aligned} \Delta V(x) &= PV(x) - V(x) \leq -(x+1)^{\alpha} \\ &\quad + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E(\nu_1+\nu_2+\cdots+\nu_k+x+1)^{\alpha} \\ &\leq -(x+1)^{\alpha} + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}2^{\alpha}E\left((\sigma_k)^{\alpha}+(x+1)^{\alpha}\right) \\ &\leq -(x+1)^{\alpha} + 2^{\alpha}(x+1)^{\alpha} + 2^{\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}k^{\alpha}E\nu_1^{\alpha} \\ &= -(x+1)^{\alpha} + 2^{\alpha}(x+1)^{\alpha} + 2^{\alpha}m_{\alpha}E\nu_1^{\alpha}. \end{aligned} \tag{50}$$

Let $b = -(c+1)^{\alpha} + 2^{\alpha}(c+1)^{\alpha} + 2^{\alpha}m_{\alpha}E\nu_1^{\alpha}$. Since $E\nu_1^{\alpha} < \infty$, we know that

$$\Delta V(x) \leq b < \infty, \quad x \in C. \tag{51}$$

Let C_n^i denote the binomial coefficient, and let $n = \lfloor\alpha\rfloor$, $\theta = \alpha - n$; then; $\alpha = n + \theta$. For all $x \in C^C$,

$$\begin{aligned} \Delta V(x) &= PV(x) - V(x) = -(x+1)^{\alpha} \\ &\quad + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E(\sigma_k - 1 + x + 1)^{\alpha} = -(x+1)^{\alpha} \\ &\quad + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E(\sigma_k - 1 + x + 1)^{n+\theta} = -(x+1)^{\alpha} \\ &\quad + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!} \\ &\qquad \cdot E\left\{\left[(x+1)^n + C_n^1(x+1)^{n-1}(\sigma_k - 1)\right.\right. \\ &\qquad \left.\left. + \sum_{i=2}^{n}C_n^i(x+1)^{n-i}(\sigma_k - 1)^i\right](\sigma_k + x)^{\theta}\right\} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{(x+1)^n(\sigma_k + x)^{\theta} - (x+1)^{\alpha}\right\} \\ &\quad + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{C_n^1(x+1)^{n-1}(\sigma_k - 1)(\sigma_k + x)^{\theta}\right\} \\ &\quad + \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!} \\ &\qquad \cdot E\left\{\sum_{i=2}^{n}\left[C_n^i(x+1)^{n-i}(\sigma_k - 1)^i\right](\sigma_k + x)^{\theta}\right\}. \end{aligned} \tag{52}$$

Since $f_1(\xi) = \xi^{\theta}$ is a concave function, we know that

$$E(\xi + x)^{\theta} \leq (E\xi + x)^{\theta}. \tag{53}$$

Thus, the first part of (52) is

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{(x+1)^n(\sigma_k + x)^{\theta} - (x+1)^{\alpha}\right\} \\ &\quad \leq (x+1)^n(\rho + x)^{\theta} - (x+1)^{\alpha} < 0. \end{aligned} \tag{54}$$

If α is integer (i.e., $\theta = 0$), then the second part of (52) is

$$\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{C_n^1(x+1)^{n-1}(\sigma_k - 1)\right\} = C_n^1(\rho - 1)(x+1)^{\alpha-1}. \tag{55}$$

If α is not integer (i.e., $\theta \neq 0$), then the second part of (52) is

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{C_n^1(x+1)^{n-1}(\sigma_k-1)(\sigma_k+x)^{\theta}\right\} \\
&\quad= C_n^1(x+1)^{n-1}\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left[(\sigma_k-1)(x+1)^{\theta}\right] \\
&\qquad+ C_n^1(x+1)^{n-1}\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!} \\
&\qquad\cdot E\left\{(\sigma_k-1)\left[(\sigma_k+x)^{\theta}-(x+1)^{\theta}\right]\right\} \\
&\quad\le C_n^1(\rho-1)(x+1)^{\alpha-1} \\
&\qquad+ C_n^1(x+1)^{n-1}\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{1_{\{\sigma_k>1\}}\sigma_k^{\theta+1}\right\} \\
&\qquad+ E\left\{1_{\{\sigma_k<1\}}\left((x+1)^{\theta}-x^{\theta}\right)\right\} \\
&\quad\le C_n^1(\rho-1)(x+1)^{\alpha-1} \\
&\qquad+ C_n^1(x+1)^{n-1}m_{\theta+1}\left(E\nu_1^{\theta+1}+1\right) \\
&\quad= C_n^1(\rho-1)(x+1)^{\alpha-1}+a_1(x+1)^{n-1},
\end{aligned} \tag{56}$$

where $a_1 = C_n^1 m_{\theta+1}(E\nu_1^{\theta+1}+1)$. From (55) and (56), we obtain that the second part of (52) is

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}E\left\{C_n^1(x+1)^{n-1}(\sigma_k-1)(\sigma_k+x)^{\theta}\right\} \\
&\quad= C_n^1(\rho-1)(x+1)^{\alpha-1}+a_1(x+1)^{n-1},
\end{aligned} \tag{57}$$

where $a_1 = 0$ if α is integer. The third part of (52) is

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\sum_{i=2}^{n}C_n^i(x+1)^{n-i}E\left[(\sigma_k-1)^i(\sigma_k+x)^{\theta}\right] \\
&\le \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\sum_{i=2}^{n}C_n^i(x+1)^{n-i} \\
&\qquad\cdot E\left\{\left(\left((\sigma_k)^i+1\right)(\sigma_k)^{\theta}+(x+1)^{\theta}\right)\right\} \\
&\le \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\sum_{i=2}^{n}C_n^i(x+1)^{n-i} \\
&\qquad\cdot E\left\{(\sigma_k)^{i+\theta}+(\sigma_k)^i(x+1)^{\theta}+(\sigma_k)^{\theta}+(x+1)^{\theta}\right\} \\
&= \sum_{i=2}^{n}C_n^i(x+1)^{n-i} \\
&\qquad\cdot\left\{m_{i+\theta}E\nu_1^{i+\theta}+m_iE\nu_1^i(x+1)^{\theta}+m_{\theta}E\nu_1^{\theta}+(x+1)^{\theta}\right\} \\
&\le \sum_{i=2}^{n}C_n^i\left(m_{i+\theta}E\nu_1^{i+\theta}+m_iE\nu_1^i+m_{\theta}E\nu_1^{\theta}+1\right)(x+1)^{n-i+\theta} \\
&= \sum_{i=2}^{n}a_i(x+1)^{\alpha-i},
\end{aligned} \tag{58}$$

where $a_i = C_n^i(m_{i+\theta}E\nu_1^{i+\theta}+m_iE\nu_1^i+m_{\theta}E\nu_1^{\theta}+1) < \infty$, $1 \le i \le n$ (by $E\nu_1^{\alpha} < \infty$). Combining (52), (54), (57), and (58), we have

$$\begin{aligned}
\Delta V(x) \le{}& C_n^1(\rho-1)(x+1)^{\alpha-1} \\
&+ a_1(x+1)^{n-1}+\sum_{i=2}^{n}a_i(x+1)^{\alpha-i},
\end{aligned} \tag{59}$$

where $a_1 = 0$ if α is integer. Choose c large enough such that, for all $x > c$ (i.e., $x \in C^c, C = [0,c]$),

$$a_1(x+1)^{n-1}+\sum_{i=2}^{n}a_i(x+1)^{\alpha-i} < -\frac{1}{2}C_n^1(\rho-1)(x+1)^{\alpha-1}. \tag{60}$$

Thus

$$\Delta V(x) \le \frac{1}{2}C_n^1(\rho-1)(x+1)^{\alpha-1}, \quad \forall x \in C^c. \tag{61}$$

Together with (51), we have

$$\begin{aligned}
\Delta V(x) &\le -\beta(x+1)^{\alpha-1}+bI_C, \\
&= -\phi\circ V(x+1)+bI_C, \quad \forall x \in R_+,
\end{aligned} \tag{62}$$

where $\beta = (1/2)C_n^1(1-\rho) > 0$, $\phi(z) = \beta z^{(\alpha-1)/\alpha}$ (i.e., condition $D(\phi, V, C)$ holds). By Theorem 1, we know that there exists a unique invariant distribution π, $\pi(\phi\circ V) < \infty$ (i.e., $\int_1^{+\infty}x^{\alpha-1}\pi(dx) < \infty$) and

$$\lim_{n\to\infty}\left\|P^{(n)}(x,A)-\pi(A)\right\|_{\phi\circ V} = 0, \quad \pi\text{-}a.e.. \tag{63}$$

From

$$H_{\phi}(x) = \int_1^x\frac{dz}{\phi(z)} = \frac{1}{\beta}\int_1^x z^{(1-\alpha)/\alpha}dz = \frac{\alpha}{\beta}\left(x^{1/\alpha}-1\right), \tag{64}$$

we have $H_{\phi}^{-1}(x) = ((\beta/\alpha)x+1)^{\alpha}$. So,

$$\begin{aligned}
r_{\phi}(n) &= \phi\circ H_{\phi}^{-1}(n) \\
&= \beta\left[\left(\frac{\beta}{\alpha}n+1\right)^{\alpha}\right]^{(\alpha-1)/\alpha} \\
&= \beta\left(\frac{\beta}{\alpha}n+1\right)^{\alpha-1} \asymp n^{\alpha-1}.
\end{aligned} \tag{65}$$

That is,

$$\lim_{n\to\infty}n^{\alpha-1}\left\|P^{(n)}(x,\cdot)-\pi(\cdot)\right\| = 0. \tag{66}$$

□

Case 3 (Logarithmic Rate of Convergence). Now, we consider the logarithmic case which is slower than that for any polynomial.

Proposition 11. *If $\rho < 1$ and there exists a positive integer α such that*

$$E\left(\nu_1 \log^{\alpha}(\nu_1+1)\right) = \int_0^{+\infty} x\log^{\alpha}(x+1)\,B(dx) < +\infty, \tag{67}$$

then

$$\lim_{n\to\infty} \log^{\alpha} n \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0, \tag{68}$$

$$\int_1^{+\infty} \log^{\alpha} x \pi(dx) < \infty. \tag{69}$$

Proof. For all $x \in R_+$, let $V(x) = (x+e)\log^{\alpha}(x+e)$; then, we have $V(x) > e, x \in R_+$. Let $c > e^{\alpha}$, and choose $C = [0, c]$. For all $x \in E$,

$$\begin{aligned}
&\Delta V(x) \\
&= PV(x) - V(x) \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(\sigma_k - 1 + x + e\right)\log^{\alpha}\left(\sigma_k - 1 + x + e\right) \\
&\quad - (x+e)\log^{\alpha}(x+e) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(\sigma_k - 1\right)\log^{\alpha}(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left(\sigma_k - 1 + x + e\right) \\
&\qquad \cdot \left[\log^{\alpha}\left(\sigma_k - 1 + x + e\right) - \log^{\alpha}(x+e)\right] = (\rho - 1)\log^{\alpha}(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&\qquad \cdot E\left\{\left(\sigma_k - 1 + x + e\right)\right. \\
&\qquad\qquad \left.\cdot \left[\log^{\alpha}\left((x+e)\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right) - \log^{\alpha}(x+e)\right]\right\} \\
&\le (\rho - 1)\log^{\alpha}(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > 1\}}\left(\sigma_k - 1 + x + e\right)\right. \\
&\qquad\qquad \left.\cdot \sum_{i=1}^{\alpha} C_{\alpha}^{i} \log^{\alpha - i}(x+e)\log^{i}\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right\} \\
&\le (\rho - 1)\log^{\alpha}(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{1<\sigma_k \le x+e\}} 2(x+e)\right. \\
&\qquad\qquad \left.\cdot \sum_{i=1}^{\alpha} C_{\alpha}^{i} \log^{\alpha - i}(x+e)\log^{i}\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right\} \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}} 2\sigma_k \sum_{i=1}^{\alpha} C_{\alpha}^{i} \log^{\alpha - i}\right. \\
&\qquad\qquad \left.\cdot (x+e)\log^{i}\left(1 + \frac{\sigma_k - 1}{x+e}\right)\right\} \\
&\le (\rho - 1)\log^{\alpha}(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{1\le\sigma_k \le x+e\}} 2\left(\sigma_k - 1\right) \sum_{i=1}^{\alpha} C_{\alpha}^{i} \log^{\alpha - i}(x+e)\right\} \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}} 2\sigma_k\right. \\
&\qquad\qquad \left.\cdot \sum_{i=1}^{\alpha} C_{\alpha}^{i} \log^{\alpha - i}(x+e)\log^{i}\sigma_k\right\} \\
&\le (\rho - 1)\log^{\alpha}(x+e) \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{1\le\sigma_k \le x+e\}} 2^{\alpha+1}\sigma_k \log^{\alpha-1}(x+e)\right\} \\
&\quad + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}} 2^{\alpha+1}\sigma_k\right. \\
&\qquad\qquad \left.\cdot \log^{\alpha-1}(x+e)\log^{\alpha}\sigma_k\right\} \\
&\le (\rho - 1)\log^{\alpha}(x+e) \\
&\quad + 2^{\alpha+1}\log^{\alpha-1}(x+e) \\
&\quad \cdot \left[\rho + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}}\sigma_k \log^{\alpha}\sigma_k\right\}\right] \\
&\le (\rho - 1)\log^{\alpha}(x+e) + 2^{\alpha+1}\log^{\alpha-1}(x+e) \\
&\quad \cdot \left[\rho + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} E\left\{1_{\{\sigma_k > x+e\}}\sigma_k 2^{\alpha}\right.\right. \\
&\qquad\qquad \left.\left.\cdot \left(\log^{\alpha} k + \log^{\alpha}\left(\nu_1 + 1\right)\right)\right\}\right] \\
&\le (\rho - 1)\log^{\alpha}(x+e) + 2^{\alpha+1}\log^{\alpha-1}(x+e) \\
&\quad \cdot \left[\rho + 2^{\alpha}\left(m_2 E\nu_1 + m_1 E\left(\nu_1 \log^{\alpha}\left(\nu_1 + 1\right)\right)\right)\right] \\
&= (\rho - 1)\log^{\alpha}(x+e) + a_0 \log^{\alpha-1}(x+e),
\end{aligned} \tag{70}$$

where $a_0 = 2^{\alpha+1}[\rho + 2^{\alpha}(m_2 E\nu_1 + m_1 E(\nu_1 \log^{\alpha}(\nu_1 + 1)))]$. Since $E(\nu_1 \log^{\alpha}(\nu_1 + 1)) < \infty$, we get that $a_0 < +\infty$. Let $b = a_0 \log^{\alpha-1}(c+e)$; then, we have

$$\Delta V(x) \le b < \infty, \quad x \in C. \tag{71}$$

Choose c large enough such that, if $x > c$ (i.e., $x \in [0,c]^c$),

$$a_0 \log^{\alpha-1}(x+e) < \frac{1}{2}(1-\rho)\log^{\alpha}(x+e). \tag{72}$$

Thus,

$$\Delta V(x) \le -\frac{1}{2}(1-\rho)\log^{\alpha}(x+e), \quad \forall x \in C^c. \tag{73}$$

Together with (71), we have

$$\begin{aligned} \Delta V(x) &\le -\beta \log^{\alpha}(x+e) + bI_C, \\ &= -\phi \circ V(x) + bI_C, \quad \forall x \in R_+, \end{aligned} \tag{74}$$

where $\beta = (1/2)(1-\rho) > 0$, $\phi(z) = \beta \log^{\alpha} z$ (i.e., condition $D(\phi, V, C)$ holds). A straightforward calculation shows that

$$r_{\phi}(n) \asymp \log^{\alpha} n. \tag{75}$$

That is,

$$\lim_{n\to\infty} \log^{\alpha} n \left\| P^{(n)}(x,\cdot) - \pi(\cdot) \right\| = 0. \tag{76}$$

By Theorem 1, we know that there exists a unique invariant distribution π, $\pi(\phi \circ V) < \infty$ (i.e., $\int_1^{+\infty} \log^{\alpha} x \pi(dx) < \infty$) and

$$\lim_{n\to\infty} \left\| P^{(n)}(x,A) - \pi(A) \right\|_{\phi\circ V} = 0, \quad \pi\text{-}a.e.. \tag{77}$$

□

4.1. Conclusion and Future Research. We studied the M/G/1 queueing system, and the waiting time process of the queueing system is a Markov process. For the workload process of the M/G/1 queueing system, we got an h-skeleton process and discussed its properties of the irreducible and aperiodic and the property of stochastic monotone. Then, we got the parameters ε_0 and s_0 for geometric rate of convergence and the geometric decay of the stationary tail, respectively. For three specific types of subgeometric cases: Case 1: the rate function $r(n) = \exp(sn^{1/(1+\alpha)})$, $\alpha > 0$, $s > 0$; Case 2: polynomial rate function $r(n) = n^{\alpha}$, $\alpha > 0$; Case 3: logarithmic rate function $r(n) = \log^{\alpha} n$, $\alpha > 0$, we gave explicit criteria for the rate of convergence and decay of stationary tail. We gave the parameters ε_1 and s_1 of the rate of convergence and the decay of the stationary tail, respectively, for the subgeometric rate $r(n) = \exp(sn^{1/(1+\alpha)})$, $\alpha > 0$, $s > 0$. These results are important in the study of the stability of M/G/1 queueing system.

For future research, much could be done. Our work could be used to the convergence analysis of Markov chain Monte Carlo (MCMC) theory. It could also be used to further discuss queue length, congestion, and so forth. Using similar techniques, these results may be extended to storage models, nonlinear autoregressive model, stochastic unit root models, multidimensional random walk, and other queueing systems.

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References

[1] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springe, London, UK; Beijing World Publishing Corporation, Beijing, China, 1999.

[2] P. Tuominen and R. L. Tweedie, "Subgeometric rates of convergence of f-ergodic Markov chains," *Advances in Applied Probability*, vol. 26, no. 3, pp. 775–798, 1994.

[3] D. Bakry, P. Cattiaux, and A. Guillin, "Rate of convergence for ergodic continuous Markov processes: lyapunov versus Poincaré," *Journal of Functional Analysis*, vol. 254, no. 3, pp. 727–759, 2008.

[4] Z. Hou and Y. Liu, "Explicit criteria for several types of ergodicity of the embedded $M/G/1$ and $GI/M/n$ queues," *Journal of Applied Probability*, vol. 41, no. 3, pp. 778–790, 2004.

[5] Y. Y. Liu and Z. T. Hou, "Several types of ergodicity for M/G/1-type Markov chains and Markov processes," *Journal of Applied Probability*, vol. 43, no. 1, pp. 141–158, 2006.

[6] Z. T. Hou and X. H. Li, "Ergodicity of quasi-birth and death processes (I)," *Acta Mathematica Sinica*, vol. 23, no. 2, pp. 201–208, 2007.

[7] Z. T. Hou and X. H. Li, "Ergodicity of quasi-birth and death processes (II)," *Chinese Annals of Mathematics A*, vol. 26, no. 2, pp. 181–192, 2005.

[8] Q.-L. Li and Y. Q. Zhao, "Heavy-tailed asymptotics of stationary probability vectors of Markov chains of $GI/G/1$ type," *Advances in Applied Probability*, vol. 37, no. 2, pp. 482–509, 2005.

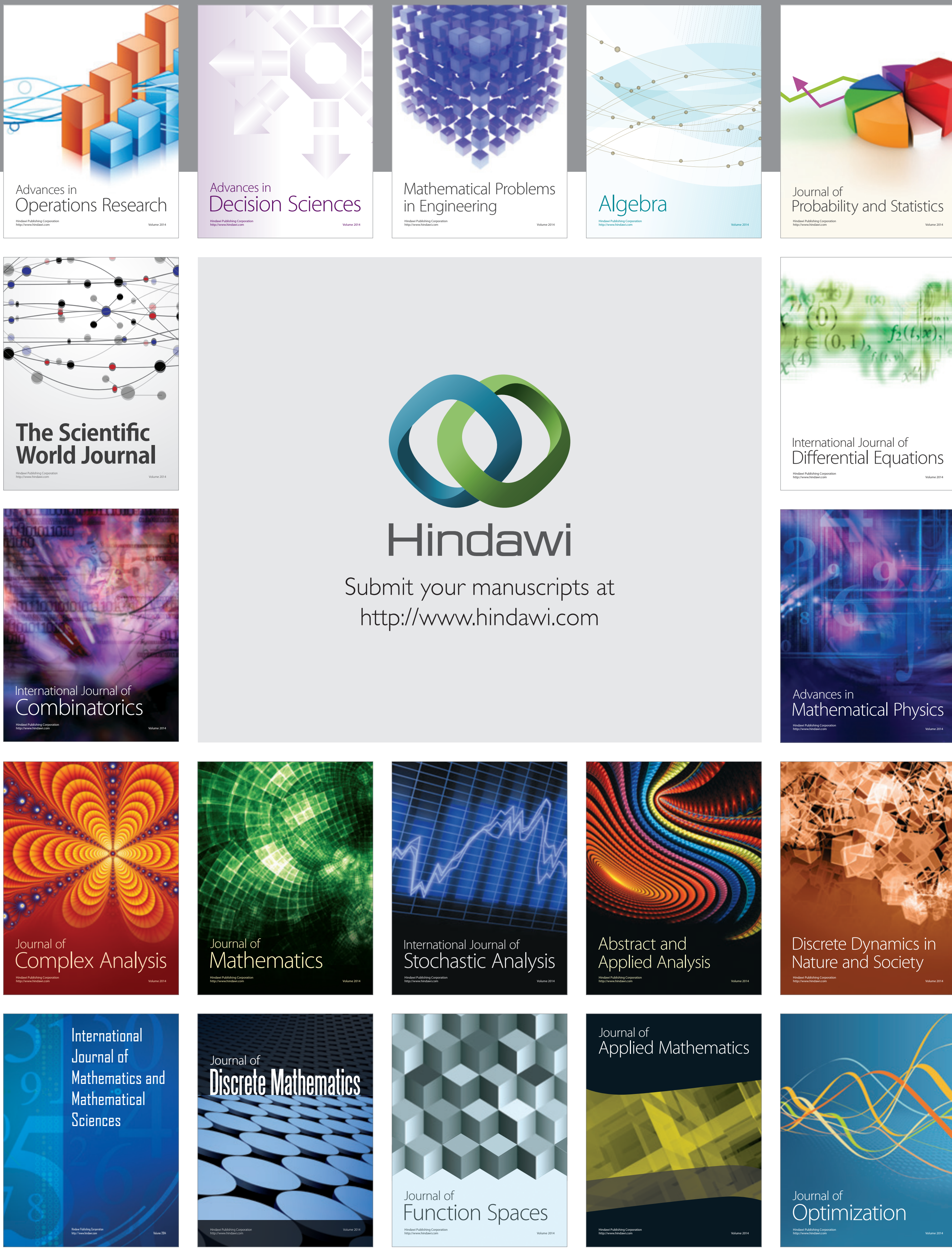
[9] Q.-L. Li and Y. Q. Zhao, "Light-tailed asymptotics of stationary probability vectors of Markov chains of $GI/G/1$ type," *Advances in Applied Probability*, vol. 37, no. 4, pp. 1075–1093, 2005.

[10] S. F. Jarner and G. O. Roberts, "Polynomial convergence rates of Markov chains," *The Annals of Applied Probability*, vol. 12, no. 1, pp. 224–247, 2002.

[11] S. F. Jarner and R. L. Tweedie, "Necessary conditions for geometric and polynomial ergodicity of random-walk-type Markov chains," *Bernoulli*, vol. 9, no. 4, pp. 559–578, 2003.

[12] R. Douc, G. Fort, E. Moulines, and P. Soulier, "Practical drift conditions for subgeometric rates of convergence," *The Annals of Applied Probability*, vol. 14, no. 3, pp. 1353–1377, 2004.

[13] R. B. Lund and R. L. Tweedie, "Geometric convergence rates for stochastically ordered Markov chains," *Mathematics of Operations Research*, vol. 21, no. 1, pp. 182–194, 1996.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