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Research Article

Conditions for Oscillation of a Neutral Differential Equation

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For a neutral differential equation with positive and changeable sign coefficients $[x(t) - a(t)x(\delta(t))]' + p(t)F(x(\tau(t))) - q(t)G(x(\sigma(t))) = 0$, oscillation criteria are established, where $q(t)$ is not required as nonnegative. Several new results are obtained.

1. Introduction

Consider the delay neutral differential equation with positive and changeable sign coefficients

$$[x(t) - a(t)x(\delta(t))]' + p(t)F(x(\tau(t))) - q(t)G(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where the following conditions are assumed to hold throughout this paper:

(A_1) $p, q, a \in C([t_0, \infty), R)$, $p \geq 0$, $a \geq 0$;

(A_1) $\delta \in C([t_0, \infty), R), \tau, \sigma \in C^1([t_0, \infty), R)$, and $\tau'(t) \geq 0$, $\sigma'(t) \geq 0$, $\tau(t) \leq \sigma(t) \leq t$, $\delta(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, and $\lim_{t\to\infty}\delta(t) = \infty$;

(A_3) $F, G \in C(R, R)$ and $xF(x) > 0$, $xG(x) > 0$, $|F(x)| \geq |x|$, $|G(x)| \leq |x|$ for all $x \neq 0$.

Recently, oscillation of first-order differential equations and difference equations with positive and negative coefficients has been investigated by many authors. Several interesting results have been obtained. We refer to [1–12] and the references cited therein. However, to the best of our knowledge, up to now, there are not works on oscillation of solutions of (1.1) with $q(t)$ able to change sign. The purpose of this paper is to study oscillation properties of

(1.1) by some new technique. Our results improve and extend several known results in the literature. In particular, our results can be applied to linear neutral differential equation

$$[x(t) - a(t)x(\delta(t))]' + p(t)x(\tau(t)) - q(t)x(\sigma(t)) = 0. \tag{1.1$'$}$$

By a solution of (1.1) we mean a function $x(t) \in C([\bar{t}_0, \infty), R)$ for some $\bar{t}_0 \geq t_0$ such that $x(t) - a(t)x(\delta(t))$ is continuously differentiable on $[\bar{t}_0, \infty)$ and satisfies (1.1) for $t \geq \bar{t}$, where $\bar{t} = \inf_{t \geq \bar{t}_0}\{\delta(t), \tau(t)\}$. As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

In the sequel, unless otherwise specified, when we write a functional inequality on t it will hold for all sufficiently large t.

First, we establish the following lemma. It extends and improves in [3, Lemma 3.7.1], [4, Lemma 2.6.1], [7, Lemma 2.1], and [9, Lemma 1].

Lemma 1.1. *Assume that*

$$p(t) - \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))} \geq 0 \text{ is not identically zero,} \tag{1.2}$$

$$a(t) + \int_{\tau(t)}^{\sigma(t)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))}\mathrm{d}s \leq 1, \qquad a(t) \neq 1, \tag{1.3}$$

where $q_+(t) = \max\{q(t), 0\}$, *and* $\sigma^{-1}(t)$ *is the inverse function of* $\sigma(t)$. *Let* $x(t)$ *be a nonoscillatory solution of* (1.1) *and*

$$y(t) = x(t) - a(t)x(\delta(t)) - \int_{\tau(t)}^{\sigma(t)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))}x(s)\mathrm{d}s. \tag{1.4}$$

Then

$$x(t)y(t) > 0, \qquad y(t)y'(t) \leq 0. \tag{1.5}$$

Proof. Let $x(t)$ be an eventually positive solution. The case when $x(t)$ is an eventually negative solution is similar and its proof is omitted. Thus we have

$$x(\tau(t)) > 0, \qquad x(\sigma(t)) > 0, \qquad x(\delta(t)) > 0. \tag{1.6}$$

By (A_3), (1.4), and (1.2), we obtain

$$\begin{aligned} y'(t) &= -p(t)F(x(\tau(t))) + q(t)G(x(\sigma(t))) - q_+(t)x(\sigma(t)) + \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))}x(\tau(t)) \\ &\leq -\left[p(t) - \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))}\right]x(\tau(t)), \end{aligned} \tag{1.7}$$

which implies that $y(t)$ is decreasing. Hence, if (1.5) does not hold, then eventually $y(t) < 0$, and there exist $T \geq t_0$ and positive constant α such that $y(t) \leq -\alpha < 0$ for all $t \geq T$, that is,

$$x(t) \leq -\alpha + a(t)x(\delta(t)) + \int_{\tau(t)}^{\sigma(t)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} x(s)\,\mathrm{d}s, \quad t \geq T. \tag{1.8}$$

We consider the following two possible cases.

The first case. $x(t)$ is unbounded, that is, $\limsup_{t\to\infty} x(t) = \infty$. Thus there exists a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x(s_n) = \infty$ and $x(s_n) = \max\{x(t) : T \leq t \leq s_n, n = 1, 2, \ldots\}$. From (1.8) we have

$$\begin{aligned} x(s_n) &\leq -\alpha + a(s_n)x(\delta(s_n)) + \int_{\tau(s_n)}^{\sigma(s_n)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))} x(s)\,\mathrm{d}s \\ &\leq -\alpha + x(s_n)\left[a(s_n) + \int_{\tau(s_n)}^{\sigma(s_n)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))}\,\mathrm{d}s\right] \\ &\leq -\alpha + x(s_n). \end{aligned} \tag{1.9}$$

This is a contradiction.

The second case. $x(t)$ is bounded, that is, $\limsup_{t\to\infty} x(t) < \infty$. Choose a sequence of points $\{\bar{s}_n\}_{n=1}^{\infty}$ such that $\bar{s}_n \to \infty$ and $x(\bar{s}_n) \to l$ as $n \to \infty$. Let $\xi(t) = \min\{\delta(t), \tau(t)\}$, $\eta(t) = \max\{\delta(t), \sigma(t)\}$, and $x(t_n) = \max\{x(s) : \xi(\bar{s}_n) \leq s \leq \eta(\bar{s}_n),\ t_n \in [\xi(\bar{s}_n), \eta(\bar{s}_n)], n = 1, 2, \ldots, \}$. Then $\limsup_{n\to\infty} x(t_n) \leq l$. Thus, in view of (1.8) we obtain

$$x(\bar{s}_n) \leq -\alpha + x(t_n)\left[a(\bar{s}_n) + \int_{\tau(\bar{s}_n)}^{\sigma(\bar{s}_n)} \frac{q_+(\sigma^{-1}(s))}{\sigma'(\sigma^{-1}(s))}\,\mathrm{d}s\right] \leq -\alpha + x(t_n). \tag{1.10}$$

Therefore $l \leq -\alpha + l$, which is also a contradiction. Hence (1.5) holds. The proof of Lemma 1.1 is complete. □

2. Main Results

In this section, we will prove a comparison theorem on oscillation for (1.1). For convenience of discussions, in the rest of this paper we will use the following notations:

$$\begin{gathered} P(t) := p(t) - \frac{q_+(\sigma^{-1}(\tau(t)))\tau'(t)}{\sigma'(\sigma^{-1}(\tau(t)))}, \qquad Q(t) := \frac{q_+(\sigma^{-1}(t))}{\sigma'(\sigma^{-1}(t))}, \qquad \delta(t) \text{ is increasing for } t \geq t_0, \\ \delta^0(t) = t, \qquad \delta^{k+1}(t) = \delta\left(\delta^k(t)\right), \quad t \geq \delta^{-(k+1)}(t_0),\ k = 1, 2, \ldots, \end{gathered} \tag{2.1}$$

where $\delta^{-k}(t)$ is the inverse function of $\delta^k(t)$.

The following comparison theorem is the main result of this paper.

Theorem 2.1. *Assume that* (1.2) *and* (1.3) *hold and there exists a nonnegative integer* m *such that all solutions of the following delay differential equation:*

$$\begin{aligned} y'(t) + P(t)\Bigg\{ y(\tau(t)) + a(\tau(t))\Bigg[y(\delta(\tau(t))) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i+1}(\tau(t))\Big) y\Big(\delta^{k+2}(\tau(t))\Big)\Bigg] \\ + \int_{\tau^2(t)}^{\sigma(\tau(t))} Q(s)\Bigg[y(s) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i}(s)\Big) y\Big(\delta^{k+1}(s)\Big)\Bigg] \mathrm{d}s \Bigg\} = 0 \end{aligned} \tag{2.2}$$

are oscillatory. Then all solutions of (1.1) *are also oscillatory.*

Proof. Suppose that $x(t)$ is an eventually positive solution of (1.1). The proof of the case where $x(t)$ is eventually negative is similar and will be omitted. By Lemma 1.1, we have

$$y(t) > 0, \qquad y'(t) \le 0, \tag{2.3}$$

where $y(t)$ is given by (1.4). Thus

$$x(t) \ge y(t) + a(t)x(\delta(t)) \ge y(t) + a(t)\Big[y(\delta(t)) + a(\delta(t))x\Big(\delta^2(t)\Big)\Big]. \tag{2.4}$$

By induction, we see that

$$x(t) \ge y(t) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i}(t)\Big) y\Big(\delta^{k+1}(t)\Big). \tag{2.5}$$

From (1.4), (2.3), and (2.5), we obtain

$$\begin{aligned} x(t) &= y(t) + a(t)x(\delta(t)) + \int_{\tau(t)}^{\sigma(t)} Q(s)x(s)\mathrm{d}s \\ &\ge y(t) + a(t)\Bigg[y(\delta(t)) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i+1}(t)\Big) y\Big(\delta^{k+2}(t)\Big)\Bigg] \\ &\quad + \int_{\tau(t)}^{\sigma(t)} Q(s)\Bigg[y(s) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i}(s)\Big) y\Big(\delta^{k+1}(s)\Big)\Bigg] \mathrm{d}s. \end{aligned} \tag{2.6}$$

It follows from (1.7) that

$$\begin{aligned} y'(t) \le - P(t)\Bigg\{ y(\tau(t)) + a(\tau(t))\Bigg[y(\delta(\tau(t))) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i+1}(\tau(t))\Big) y\Big(\delta^{k+2}(\tau(t))\Big)\Bigg] \\ + \int_{\tau^2(t)}^{\sigma(\tau(t))} Q(s)\Bigg[y(s) + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i}(s)\Big) y\Big(\delta^{k+1}(s)\Big)\Bigg] \mathrm{d}s \Bigg\}. \end{aligned} \tag{2.7}$$

By a well-known result (see, e.g., [4, Corollary 3.2.2]) we can conclude that (2.2) has also an eventually positive solution. This is a contradiction. The proof of Theorem 2.1 is complete. □

Following the proof of Theorem 2.1 and taking into account in (2.7) the positivity of the functions P and Q and the properties of the delay functions δ and τ, we state as corollary the following claim.

Corollary 2.2. *Assume that* (1.2) *and* (1.3) *hold and there exists a nonnegative integer* m *such that all solutions of the delay differential equation*

$$\begin{aligned} y'(t) + P(t)\Bigg\{1 + a(\tau(t))\left[1 + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i+1}(\tau(t))\Big)\right] \\ + \int_{\tau^2(t)}^{\sigma(\tau(t))} Q(s)\left[1 + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i}(s)\Big)\right] \mathrm{d}s\Bigg\} y(\tau(t)) = 0. \end{aligned} \tag{2.8}$$

are oscillatory. Then all solutions of (1.1) *are also oscillatory.*

Corollary 2.3. *Consider* (1.1) *with* $q(t) \equiv 0$*. Assume that there exists a nonnegative integer* m *such that all solutions of the delay differential equation*

$$y'(t) + P(t)\left\{1 + a(\tau(t))\left[1 + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i+1}(\tau(t))\Big)\right]\right\} y(\tau(t)) = 0 \tag{2.9}$$

are oscillatory. Then all solutions of (1.1) *are also oscillatory.*

Remark 2.4. Corollaries 2.2 and 2.3 extend and improve [3, Theorems 3.7.1 and 3.2.1].

3. Explicit Oscillation Conditions

In this section, we will give several explicit oscillation conditions for (1.1). Let

$$\begin{aligned} R_m(t) &:= P(t)\left\{1 + a(\tau(t))\left[1 + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i+1}(\tau(t))\Big)\right] + \int_{\tau(t)}^{\sigma(t)} Q(s)\left[1 + \sum_{k=0}^{m}\prod_{i=0}^{k} a\Big(\delta^{i}(s)\Big)\right] \mathrm{d}s\right\}, \\ I_n(R_m, \tau(t)) &:= \int_{\tau(t)}^{t} R_m(s_1) \int_{\tau(s_1)}^{s_1} R_m(s_2) \int_{\tau(s_{n-1})}^{s_{n-1}} R_m(s_n)\mathrm{d}s_n \mathrm{d}s_{n-1} \cdots \mathrm{d}s_2 \mathrm{d}s_1, \end{aligned} \tag{3.1}$$

where $n \geq 1$ and $s_0 := t$.

By [11, Corollary 2.1 and Theorem 1] with a well-known oscillation criterion for first-order linear delay differential equations, we have the following result.

Theorem 3.1. *Assume that* (1.2) *and* (1.3) *hold and there exists a nonnegative integer* m *and* $n \geq 1$ *such that*

$$\liminf_{t\to\infty} I_n(R_m,\tau(t)) > \frac{1}{e^n}, \tag{3.2}$$

or there exists a nonnegative integer m *such that*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} R_m(s)\mathrm{d}s > 1. \tag{3.3}$$

Then all solutions of (1.1) *are oscillatory.*

Remark 3.2. Theorem 3.1 extends and improves [4, Theorem 2.6.1], [2, Theorem 3], and the relative results in [12].

Consider the autonomous neutral differential equation

$$(x(t) - ax(t-\delta))' + px(t-\tau) - qx(t-\sigma) = 0, \quad t \geq 0, \tag{3.4}$$

where p, δ, τ, and σ are positive constants, a and q are real number, and

$$\tau > \sigma, \qquad 0 \leq a + q_+(\tau-\sigma) \leq 1, \qquad p - q_+ > 0, \qquad q_+ = \max\{q, 0\}. \tag{3.5}$$

Theorem 3.3. *Assume that* (3.5) *holds and there exists a nonnegative integer* m *satisfying*

$$(p-q_+)\left[1 + a + q_+(\tau-\sigma)\right]\left(1 + \sum_{k=0}^{m} a^{k+1}\right)\tau > \frac{1}{e}. \tag{3.6}$$

Then all solutions of (3.4) *are oscillatory.*

Proof. By (3.2), we find that for (3.4)

$$I_n(R_m,\tau) = (p-q_+)^n\left[1 + a\left(1 + \sum_{k=0}^{m} a^{k+1}\right) + q_+(\tau-\sigma)\left(1 + \sum_{k=0}^{m} a^{k+1}\right)\right]^n \tau^n > \frac{1}{e^n}, \tag{3.7}$$

which implies that (3.6) holds. By Theorem 3.1, (3.6) leads to that all solutions of (3.4) are oscillatory. The proof of Theorem 3.3 is complete. □

Corollary 3.4. *Assume that* (3.5) *with* $0 \leq a < 1$ *holds and*

$$(p-q_+)\left[1 + a + q_+(\tau-\sigma)\right]\frac{\tau}{1-a} > \frac{1}{e}. \tag{3.8}$$

Then all solutions of (3.4) *are oscillatory.*

Remark 3.5. When $a = 0$, (3.8) reduces to

$$\left(p - q_+\right)\left[1 + q_+(\tau - \sigma)\right]\tau > \frac{1}{e}. \tag{3.9}$$

Equation (3.9) improves conditions (2) and (4) in [1, Corollary 2.3] where the following two oscillation criteria for the solutions of (3.4) with $a = 0$ and $p > q > 0$ are obtained:

$$\begin{aligned} (p - q)\left[\tau + \sigma q(\tau - \sigma)\right] &> \frac{1}{e}, \\ (p - q)\left[1 + q(\tau - \sigma)\right]\sigma &> \frac{1}{e}. \end{aligned} \tag{3.10}$$

Obviously, since $\tau > \sigma$, it follows that (3.9) is respectively better than (3.10).

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