

Review Article

Variational Iteration Method for Initial and Boundary Value Problems Using He's Polynomials

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This paper outlines a detailed study of the coupling of He's polynomials with correction functional of variational iteration method (VIM) for solving various initial and boundary value problems. The elegant coupling gives rise to the modified versions of VIM which is very efficient in solving nonlinear problems of diversified nature. It is observed that the variational iteration method using He's polynomials (VIMHP) is very efficient, easier to implement, and more user friendly. Several examples are given to reconfirm the efficiency of the proposed VIMHP.

1. Introduction

With the rapid development of nonlinear sciences, many analytical and numerical techniques have been developed by various scientists. Most of the developed techniques have their limitations like limited convergence, divergent results, linearization, discretization unrealistic assumptions, and noncompatibility with the physical problems [1–67]. He [20–35] developed the variational iteration (VIM) and homotopy perturbation methods (HPM) which proved to be fully synchronized with the versatile nature of the physical problems showing the intellect of the author; see [1–8, 12–35, 41, 46–58, 63–65] and the references therein. In order to improve the efficiency of these algorithms several modifications have been introduced by different researchers for time and again. Abbasbandy [1, 2] made the elegant coupling of Adomian's polynomials and correction functional of He's VIM and solved quadratic Riccati differential and Klein-Gordon equations. This concept was subsequently exploited by Mohyud-Din and Noor [49, 54] for solving various singular and nonsingular initial and boundary value

problems. Recently, Ghorbani and Saberi-Nadjafi [17] and Ghorbani [18] introduced He's polynomials by splitting the nonlinear term and also proved that He's polynomials are fully compatible with Adomian's polynomials but are easier to calculate. More recently, Noor and Mohyud-Din coupled He's polynomials and correction functional of the VIM and applied this reliable version (VIMHPS) to a number of physical problems; see [51–53]. The VIMHPS [51–53] has a very simple solution procedure and absorbs all of the positive features of He's variational iteration (VIM) and homotopy perturbation (HPM) methods and is highly compatible with the diversity of the physical problems. Moreover, the inclusion of He's polynomials p in the correction functional enhances its capability to deal with the physical nature of the problems and makes this version closer to the versatility of the physical nature of the problems. The basic motivation of the present study is the implementation of VIMHPS for solving various initial and boundary value problems of diversified physical nature. Several examples are given for the comparison and to measure the efficiency of these couplings. It is to be highlighted that the variational iteration method using He's polynomials (VIMHPS) [51–53] has certain advantages as compared to the decomposition method. Firstly, the use of Lagrange multiplier reduces the successive applications of the integral operator and hence minimizes the computational work to a tangible level while still maintaining a very high level of accuracy. Moreover, He's polynomials are easier to calculate as compared to Adomian's polynomials and this gives it a clear edge over the traditional decomposition method. The VIMHPS is also independent of the small parameter assumption (which is either not there in the physical problems or difficult to locate) and hence is more convenient to apply as compare to the traditional perturbation method. It is worth mentioning that the VIMHPS is applied without any discretization, restrictive assumption or transformation and is free from round off errors. We apply the proposed VIMHPS for all the nonlinear terms in the problem without discretizing either by finite difference or spline techniques at the nodes, involving laborious calculations coupled with a strong possibility of the ill-conditioned resultant equations which is a complicated problem to solve. Moreover, unlike the method of separation of variables that requires initial and boundary conditions, the VIMHPS provides the solution by using the initial conditions only; see [51–53]. It is to be highlighted that new developments of variational iteration and homotopy perturbation methods are available in [8, 12, 40]. Moreover, the convergence of variational iteration method has been discussed in [58]. It is worth mentioning that the examples which have been discussed in this paper are of utmost importance in applied, engineering, and nonlinear sciences.

2. Variational Iteration Method (VIM)

To illustrate the basic concept of He's VIM, we consider the following general differential equation:

$$Lu + Nu = g(x), \quad (2.1)$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the inhomogeneous term. According to variational iteration method [1–8, 13, 14, 21–24, 26, 27, 41, 46, 51–54], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s)) ds, \quad (2.2)$$

where λ is a Lagrange multiplier [21–24], which can be identified optimally via variational iteration method. The subscripts n denote the n th approximation, and \tilde{u}_n is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$; (2.2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [21–24]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1} , $n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0 ; consequently, the solution is given by $u = \lim_{n \rightarrow \infty} u_n$. The convergence of variational iteration method has been discussed in [58].

3. Homotopy Perturbation Method (HPM) and He's Polynomials

To explain He's homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0, \quad (3.1)$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u), \quad (3.2)$$

where $F(u)$ is a functional operator with known solutions v_0 , which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \quad (3.3)$$

we have

$$H(u, 0) = F(u), \quad H(u, 1) = L(u). \quad (3.4)$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [17, 18, 29–35]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [29–35] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots. \quad (3.5)$$

If $p \rightarrow 1$, then (3.5) corresponds to (3.2) and becomes the approximate solution of the form

$$f = \lim_{p \rightarrow 1} u = \sum_{i=0}^{\infty} u_i. \quad (3.6)$$

It is well known that series (3.6) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see [29–35]. We assume that (3.6) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [17, 18], He's HPM considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \dots, \quad (3.7)$$

where H_n 's are the so-called He's polynomials [17, 18], which can be calculated by using the formula

$$H_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N \left(\sum_{i=0}^n p^i u_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \dots \quad (3.8)$$

4. Adomian's Decomposition Method (ADM)

Consider the differential equation [59–61]

$$Lu + Ru + Nu = g, \quad (4.1)$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser than L , Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (4.1) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \quad (4.2)$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [59–61] defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (4.3)$$

where the components $u_n(x)$ are usually determined recurrently by using the relation

$$u_0 = f, \quad u_{k+1} = L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0. \quad (4.4)$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials $F(u) = \sum_{n=0}^{\infty} A_n$, where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [59–61] which yields

$$A_n = \left(\frac{1}{n!} \right) \left(\frac{d^n}{d\lambda^n} \right) N \left(\sum_{i=0}^n (\lambda^i u_i) \right)_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (4.5)$$

5. Variational Iteration Method Using He's Polynomials (VIMHPs)

This modified version of variational iteration method [49–53] is obtained by the elegant coupling of correction functional (3.1) of variational iteration method (VIM) with He's polynomials [17, 18] and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p \int_0^x \lambda(s) \left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n) \right) ds - \int_0^x \lambda(s) g(s) ds. \quad (5.1)$$

Comparisons of like powers of p give solutions of various orders.

6. Numerical Applications

In this section, we apply the VIMHPs for solving various initial and boundary value problems.

Example 1. Consider the following one-dimensional Burger's equation:

$$u_t + uu_{xx} - \nu u_{xx} = 0, \quad (6.1)$$

with initial condition

$$u(x, 0) = \frac{\alpha + \beta + (\beta - \alpha) \exp(\gamma)}{1 + \exp(\gamma)}, \quad t \geq 0, \quad (6.2)$$

where $\gamma = (\alpha/\nu)(x - \lambda)$ and the parameters α , β , λ are the arbitrary constants. The correction functional is given by

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(s) \left(\frac{\partial u_n}{\partial t} + \tilde{u}_n \frac{\partial^2 \tilde{u}_n}{\partial x^2} - \nu \frac{\partial^2 \tilde{u}_n}{\partial x^2} \right) ds. \quad (6.3)$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda(s) = -1$, and we get the following iterative scheme:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left(\frac{\partial u_n}{\partial t} + u_n \frac{\partial^2 u_n}{\partial x^2} - \nu \frac{\partial^2 u_n}{\partial x^2} \right) ds. \quad (6.4)$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$\begin{aligned}
 & u_0 + pu_1 + p^2u_2 + \dots \\
 &= \left(\frac{\alpha + \beta + (\beta - \alpha) \exp(\gamma)}{1 + \exp(\gamma)} \right) - p \int_0^t \left(\frac{\partial u_0}{\partial t} + p \frac{\partial u_1}{\partial t} + p^2 \frac{\partial u_2}{\partial t} + \dots \right) ds \\
 &\quad - p \int_0^t \left((u_0 + pu_1 + p^2u_2 + \dots) \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + \dots \right) - v \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + \dots \right) \right) ds.
 \end{aligned} \tag{6.5}$$

Comparing the coefficient of like powers of p ,

$$\begin{aligned}
 p^{(0)} : u_0(x, t) &= \frac{\alpha + \beta + (\beta - \alpha) \exp(\gamma)}{1 + \exp(\gamma)}, \\
 p^{(1)} : u_1(x, t) &= \frac{\alpha + \beta + (\beta - \alpha) \exp(\gamma)}{1 + \exp(\gamma)} + \frac{2\alpha\beta^2 \exp(\gamma)}{v(1 + \exp(\gamma))} t, \\
 p^{(2)} : u_2(x, t) &= \left(\frac{\alpha + \beta + (\beta - \alpha) \exp(\gamma)}{1 + \exp(\gamma)} \right) + \left(\frac{2\alpha\beta^2 \exp(\gamma)}{v(1 + \exp(\gamma))} \right) t \\
 &\quad + \left(\frac{\alpha^3 \beta^2 \exp(\gamma) (-1 + \exp(\gamma))}{v^2 (1 + \exp(\gamma))^3} \right) t^3, \\
 &\quad \vdots
 \end{aligned} \tag{6.6}$$

The series solution is given by

$$\begin{aligned}
 u(x, t) &= \left(\frac{\alpha + \beta + (\beta - \alpha) \exp(\gamma)}{1 + \exp(\gamma)} \right) + \left(\frac{2\alpha\beta^2 \exp(\gamma)}{v(1 + \exp(\gamma))} \right) t \\
 &\quad + \left(\frac{\alpha^3 \beta^2 \exp(\gamma) (-1 + \exp(\gamma))}{v^2 (1 + \exp(\gamma))^3} \right) t^3 \\
 &\quad + \left(\frac{\alpha^4 \beta^3 \exp(\gamma) (1 - 4 \exp(\gamma) + \exp(\gamma)^2)}{3v^3 (1 + \exp(\gamma))^4} \right) t^4 + \dots,
 \end{aligned} \tag{6.7}$$

and in a closed-form by

$$u(x, t) = \left(\frac{\alpha + \beta + (\beta - \alpha) \exp(x/v)(x - \beta t - \lambda)}{1 + \exp(x/v)(x - \beta t - \lambda)} \right). \tag{6.8}$$

Figure 1 depicts series solution at $\alpha = \beta = v = \lambda = 1$.

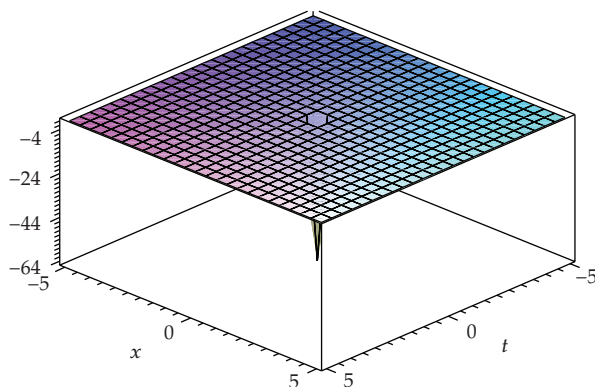


Figure 1: ($\alpha = \beta = \nu = \lambda = 1$).

Example 2. Consider the following homogeneous coupled Burger's equation:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \quad v_t - v_{xx} - 2vv_x + (uv)_x = 0, \quad (6.9)$$

with initial conditions

$$u(x, 0) = \sin x, \quad v(x, 0) = \sin x. \quad (6.10)$$

The correction functional for the above coupled system is given by

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) + \int_0^t \lambda(s) \left(\frac{\partial u_n}{\partial s} - \frac{\partial^2 \tilde{u}_n}{\partial x^2} - 2\tilde{u}_n(\tilde{u}_n)_x + (\tilde{u}_n \tilde{v}_n)_x \right) ds, \\ v_{n+1}(x, t) &= v_n(x, t) + \int_0^t \lambda(s) \left(\frac{\partial v_n}{\partial s} - \frac{\partial^2 \tilde{v}_n}{\partial x^2} - 2\tilde{v}_n(\tilde{v}_n)_x + (\tilde{u}_n \tilde{v}_n)_x \right) ds. \end{aligned} \quad (6.11)$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda(s) = -1$, and we get

$$\begin{aligned} u_{n+1}(x, t) &= u_n(x, t) - \int_0^t \left(\frac{\partial u_n}{\partial s} - \frac{\partial^2 u_n}{\partial x^2} - 2u_n(u_n)_x + (u_n v_n)_x \right) ds, \\ v_{n+1}(x, t) &= v_n(x, t) - \int_0^t \left(\frac{\partial v_n}{\partial s} - \frac{\partial^2 v_n}{\partial x^2} - 2v_n(v_n)_x + (u_n v_n)_x \right) ds. \end{aligned} \quad (6.12)$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$\begin{aligned}
 & u_0 + pu_1 + p^2u_2 + \dots \\
 &= u_0(x, t) + p \int_0^t \lambda(s) \left(\left(\frac{\partial u_0}{\partial t} + p \frac{\partial u_1}{\partial t} + \dots \right) - \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + \dots \right) \right) ds \\
 &\quad - p \int_0^t \left(2(u_0 + pu_1 + \dots) \left(\frac{\partial u_0}{\partial x} + p \frac{\partial u_1}{\partial x} + \dots \right) - ((u_0 + pu_1 + \dots)(v_0 + pv_1 + \dots))_x \right) ds, \\
 & v_0 + pv_1 + p^2v_2 + \dots \\
 &= v_0(x, t) + p \int_0^t \lambda(s) \left(\left(\frac{\partial v_0}{\partial t} + p \frac{\partial v_1}{\partial t} + \dots \right) - \left(\frac{\partial^2 v_0}{\partial x^2} + p \frac{\partial^2 v_1}{\partial x^2} + p^2 \frac{\partial^2 v_2}{\partial x^2} + \dots \right) \right) ds \\
 &\quad - p \int_0^t \left(2(v_0 + pv_1 + \dots) \left(\frac{\partial v_0}{\partial x} + p \frac{\partial v_1}{\partial x} + \dots \right) - ((u_0 + pu_1 + \dots)(v_0 + pv_1 + \dots))_x \right) ds.
 \end{aligned} \tag{6.13}$$

Comparing the coefficient of like powers of p ,

$$\begin{aligned}
 p^{(0)} : & \begin{cases} u_0(x, t) = \sin x, \\ v_0(x, t) = \sin x, \end{cases} \\
 p^{(1)} : & \begin{cases} u_1(x, t) = \sin x - t \sin x, \\ v_1(x, t) = \sin x - t \sin x, \end{cases} \\
 p^{(2)} : & \begin{cases} u_2(x, t) = \sin x - t \sin x + \frac{t^2}{2!} \sin x, \\ v_2(x, t) = \sin x - t \sin x + \frac{t^2}{2!} \sin x, \end{cases} \\
 & \vdots
 \end{aligned} \tag{6.14}$$

The series solutions are given by

$$\begin{aligned}
 u(x, t) &= \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right), \\
 v(x, t) &= \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right),
 \end{aligned} \tag{6.15}$$

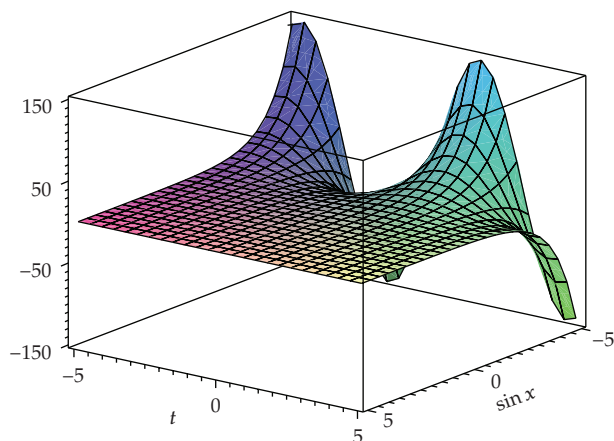


Figure 2

and the closed-form solutions are given as

$$\begin{aligned} u(x, t) &= \exp(-t) \sin x, \\ v(x, t) &= \exp(-t) \sin x. \end{aligned} \quad (6.16)$$

Figure 2 depicts the series solutions $u(x, t)$ or $v(x, t)$.

Example 3. Consider the Thomas-Fermi equation

$$y''(x) = \frac{y^{3/2}}{x^{1/2}}, \quad (6.17)$$

with boundary conditions

$$y(0) = 1, \quad \lim_{x \rightarrow \infty} y(x) = 0. \quad (6.18)$$

The correction functional is given as

$$y_{n+1}(x) = y_0(x) + \int_0^x \lambda(s) \left(\frac{d^2 y_n(x, s)}{ds^2} - x^{-1/2} \tilde{y}_n^{3/2} \right) ds. \quad (6.19)$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s - x)$, and the following iterative scheme is obtained:

$$y_{n+1}(x) = y_0(x) + \int_0^x (s - x) \left(\frac{d^2 y_n(x, s)}{ds^2} - x^{-1/2} y_n^{3/2} \right) ds. \quad (6.20)$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$y_0 + y_1 + y_2 + \dots = y_0(x) + p \int_0^x (s - x) \left(\left(\frac{d^2 y_0}{ds^2} + p \frac{d^2 y_1}{ds^2} + p^2 \frac{d^2 y_2}{ds^2} + \dots \right) - \left(x^{-1/2} (y_0 + p y_1 + p^2 y_2 + \dots)^{3/2} \right) \right) ds. \quad (6.21)$$

Now, we apply a slight modification in the conventional initial value and take $y_0(x) = 1$, instead of $y_0(x) = 1 + Bx$, where $B = y'(0)$. Comparing the coefficient of like powers of p ,

$$\begin{aligned} p^{(0)} : y_0(x) &= 1, \\ p^{(1)} : y_1(x) &= 1 + Bx + \frac{4}{3}x^{3/2}, \\ p^{(2)} : y_2(x) &= 1 + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3, \\ p^{(3)} : y_3(x) &= 1 + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{1}{3}x^3, \\ &\vdots \end{aligned} \quad (6.22)$$

The series solution is given as

$$\begin{aligned} y(x) &= 1 + Bx + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} \\ &\quad - \frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} + \frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 \\ &\quad + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2} - \frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 - \frac{512}{351000}B^3x^{15/2} \\ &\quad - \frac{46}{45045}B^2x^8 - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9 \dots, \\ &\quad \vdots \end{aligned} \quad (6.23)$$

Table 1: Padé approximants and initial slopes $y'(0)$ [56].

Padé approximants	Initial slope $y'(0)$	Error (%)
[2/2]	-1.211413729	23.71
[4/4]	-1.550525919	2.36
[7/7]	-1.586021037	12.9×10^{-2}
[8/8]	-1.588076820	3.66×10^{-4}
[10/10]	-1.588076779	3.64×10^{-4}

Setting $x^{1/2} = t$, the series solution is obtained as

$$\begin{aligned}
 y(t) = & 1 + Bt^2 + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^6 + \frac{3}{70}B^2t^7 + \frac{2}{15}Bt^8 + \left(-\frac{1}{252}B^3 + \frac{2}{27}\right)t^9 \\
 & + \frac{1}{175}B^2t^{10} + \left(\frac{1}{1056}B^4 + \frac{31}{1485}B\right)t^{11} + \left(\frac{4}{1575}B^3 + \frac{4}{405}\right)t^{12} \\
 & + \left(-\frac{3}{9152}B^5 + \frac{557}{100100}B^2\right)t^{13} + \left(-\frac{29}{24255}B^4 + \frac{4}{693}B\right)t^{14} + \left(\frac{7}{499}B^6 - \frac{623}{351000}B^3 + \frac{101}{52650}\right)t^{15} \\
 & + \left(\frac{68}{105105}B^4 - \frac{46}{45045}B^2\right)t^{16} + \left(-\frac{3}{43520}B^7 + \frac{153173}{116424000}B^4 - \frac{113}{1178100}B\right)t^{17} + \dots
 \end{aligned}
 \tag{6.24}$$

The diagonal Padé approximants can be applied [53] in order to study the mathematical behavior of the potential $y(x)$ and to determine the initial slope of the potential $y'(0)$.

Example 4. Consider the following nonlinear third-order boundary layer problem which appears mostly in the mathematical modeling of physical phenomena in fluid mechanics:

$$f'''(x) + (n - 1)f(x)f''(x) - 2n(f'(x))^2 = 0, \quad n > 0,
 \tag{6.25}$$

with boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad n > 0.
 \tag{6.26}$$

The correction functional is given as

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda(s) \left(f_n'''(x) + (n - 1)\tilde{f}_n(x)\tilde{f}_n''(x) - 2(\tilde{f}_n'(x))^2 \right) ds = 0, \quad n > 0.
 \tag{6.27}$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -(1/2!)(s - x)^2$; consequently

$$f_{n+1}(x) = f_n(x) - \int_0^x \frac{1}{2!}(s - x)^2 \left(f_n'''(x) + (n - 1)f_n(x)f_n''(x) - 2n(f_n'(x))^2 \right) ds = 0, \quad n > 0.
 \tag{6.28}$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$\begin{aligned}
 & f_0 + pf_1 + p^2f_2 + \dots \\
 &= x + \frac{1}{2}\alpha x^2 \\
 &\quad - p \int_0^x \frac{1}{3!} (s-x)^2 ((f_0''' + pf_1''' + \dots) + (n-1)(f_0 + pf_1 + \dots)(f_0'' + pf_1'' + \dots)) dx \\
 &\quad + 2np \int_0^x \frac{1}{3!} (s-x)^2 (f_0' + pf_1 + p^2f_2 + \dots)^2 dx,
 \end{aligned} \tag{6.29}$$

where $f''(0) = \alpha < 0$. Comparing the coefficient of like powers of p ,

$$\begin{aligned}
 p^{(0)} : f_0(x) &= x, \\
 p^{(1)} : f_1(x) &= x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3, \\
 p^{(2)} : f_2(x) &= x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5, \\
 p^{(3)} : f_3(x) &= x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5 + \frac{1}{120}\alpha^2(3n+1)x^5 \\
 &\quad + \frac{1}{720}\alpha(19n^2 + 18n + 3)x^6 + \frac{1}{315}n(2n^2 + 2n + 1)x^7, \\
 &\quad \vdots
 \end{aligned} \tag{6.30}$$

The series solution is given as

$$\begin{aligned}
 f(x) &= x + \frac{\alpha x^2}{2} + \frac{nx^3}{3} + \left(\frac{1}{8}n\alpha + \frac{1}{24}\alpha\right)x^4 + \left(\frac{1}{30}n^2 + \frac{1}{40}n\alpha^2 + \frac{1}{120}\alpha^2 + \frac{1}{30}n\right)x^5 \\
 &\quad + \left(\frac{19}{720}n^2\alpha + \frac{1}{240}\alpha + \frac{1}{40}n\alpha\right)x^6 + \left(\frac{1}{120}n\alpha^2 + \frac{1}{315}n + \frac{2}{315}n^3 + \frac{11}{5040}\alpha^2\right. \\
 &\quad \quad \quad \left. + \frac{3}{560}n^2\alpha^2 + \frac{2}{315}n^2\right)x^7 \\
 &\quad + \left(\frac{11}{40320}\alpha^3 + \frac{33}{4480}n^2\alpha + \frac{3}{4480}\alpha^3n^2 + \frac{23}{5760}n\alpha + \frac{1}{2688}\alpha + \frac{167}{40320}n^3\alpha + \frac{1}{960}\alpha^3n\right)x^8 \\
 &\quad + \left(\frac{1}{3780}n + \frac{527}{362880}n^3\alpha^2 + \frac{19}{11340}n^3 + \frac{709}{362880}n\alpha^2 + \frac{23}{8064}n^2\alpha^2 + \frac{23}{22680}n^2\right. \\
 &\quad \quad \quad \left. + \frac{13}{22680}n^4 + \frac{43}{120960}\alpha^2\right)x^9 + \dots
 \end{aligned} \tag{6.31}$$

Example 5. Consider the following nonlinear inhomogeneous Goursat problem:

$$\begin{aligned} u_{xt} &= -u^3 + x^3 + 3x^2t + 3xt^2 + t^3, \\ u(x, 0) &= x, \quad u(0, t) = t, \quad u(0, 0) = 0. \end{aligned} \quad (6.32)$$

The correction functional for the above problem is given by

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial x \partial s} + \tilde{u}_n^3 - x^3 - 3x^2s - 3xs^2 - s^3 \right) ds, \quad n \geq 0. \quad (6.33)$$

Making the above correctional functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$; consequently

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left(\frac{\partial^2 u_n}{\partial x \partial s} + u_n^3 - x^3 - 3x^2s - 3xs^2 - s^3 \right) ds, \quad n \geq 0. \quad (6.34)$$

Applying the variational iteration method using He's polynomials (5.1) (VIMHPS), we get

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \dots &= u_0(x, t) - p \int_0^t \left(\left(\frac{\partial^2 u_0}{\partial x \partial s} + p \frac{\partial^2 u_1}{\partial x \partial s} + \dots \right) + (u_0 + pu_1 + p^2u_2 + \dots)^3 \right. \\ &\quad \left. - x^3 - 3x^2s - 3xs^2 - s^3 \right) ds. \end{aligned} \quad (6.35)$$

The initial approximation can be selected (by utilizing the boundary conditions) as $u_0(x, t) = Ax + Bt$, where A, B are constants and would be determined by using the boundary conditions; consequently

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \dots &= Ax + Bt - p \int_0^t \left(\left(\frac{\partial^2 u_0}{\partial x \partial s} + p \frac{\partial^2 u_1}{\partial x \partial s} + \dots \right) + (u_0 + pu_1 + p^2u_2 + \dots)^3 \right. \\ &\quad \left. - x^3 - 3x^2s - 3xs^2 - s^3 \right) ds. \end{aligned} \quad (6.36)$$

Table 2: Numerical values for $\alpha = f''(0)$ for $0 < n < 1$ by using diagonal Padé approximants [51].

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	-0.3872983347	-0.3821533832	-0.3819153845	-0.3819148088	-0.3819121854
1/3	-0.5773502692	-0.5615999244	-0.5614066588	-0.5614481405	-0.561441934
0.4	-0.6451506398	-0.6397000575	-0.6389732578	-0.6389892681	-0.6389734794
0.6	-0.8407967591	-0.8393603021	-0.8396060478	-0.8395875381	-0.8396056769
0.8	-1.007983207	-1.007796981	-1.007646828	-1.007646828	-1.007792100

Table 3: Numerical values for $\alpha = f''(0)$ for $n > 1$ by using diagonal Padé approximants [51].

n	α
4	-2.483954032
10	-4.026385103
100	-12.84334315
1000	-40.65538218
5000	-104.8420672

Comparing the coefficient of like powers of p ,

$$\begin{aligned}
 p^{(0)} : u_0(x, t) &= Ax + Bt, \\
 p^{(1)} : u_1(x, t) &= Ax + Bt - \frac{1}{4}(B^3 - 1)t^4 - (AB^2 - 1)xt^3 - \frac{3}{2}(A^2B - 1)x^2t^2 - A^3x^3t + x^3t, \\
 &\vdots
 \end{aligned}
 \tag{6.37}$$

Imposing the boundary conditions $u(x, 0) = x$, $u(0, t) = t$, to find the constants A and B will yield $A = 1$, and $B = 1$, consequently

$$u(x, t) = x + t. \tag{6.38}$$

Figure 3 depicts the solution $u(x, t)$.

Example 6. Consider the following nonlinear differential equation which governs the unsteady flow of gas through a porous medium:

$$y''(x) + \frac{2x}{\sqrt{1 - \alpha y}} y'(x) = 0, \quad 0 < \alpha < 1, \tag{6.39}$$

with the following typical boundary conditions imposed by the physical properties:

$$y(0) = 1, \quad \lim_{x \rightarrow \infty} y(x) = 0. \tag{6.40}$$

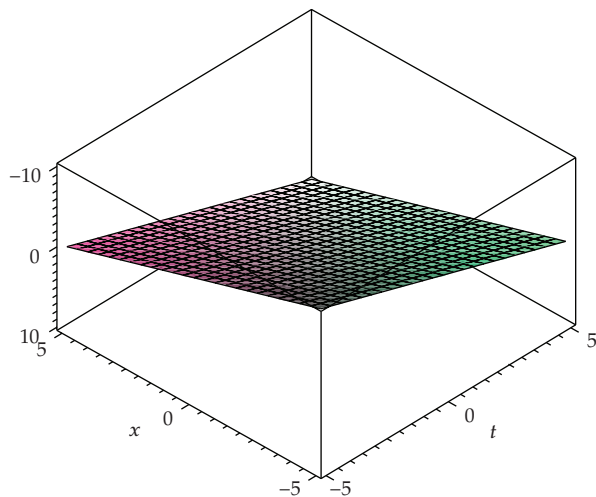


Figure 3

The correction functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(y''(x) + \frac{2s}{\sqrt{1-\alpha y}} y'(x) \right) ds, \quad 0 < \alpha < 1. \quad (6.41)$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda = s - x$, and thus we get

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x) \left(y''_n(s) + \frac{2s}{\sqrt{1-\alpha y_n(s)}} y'_n(s) \right) ds, \quad 0 < \alpha < 1. \quad (6.42)$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$\begin{aligned} & y_0 + p y_1 + p^2 y_2 + \dots \\ &= y_0 + \int_0^x (s - x) (y''_0 + p y''_1 + p^2 y''_2 + \dots) ds \\ &+ 2 \int_0^x (s - x) x (1 - \alpha (y'_0 + p y'_1 + p^2 y'_2 + \dots))^{-1/2} (y'_0 + p y'_1 + p^2 y'_2 + \dots) ds, \end{aligned} \quad (6.43)$$

Table 4

α	$B_{[2/2]} = y'(0)$	$B_{[3/3]} = y'(0)$
0.1	-3.556558821	-1.957208953
0.2	-2.441894334	-1.786475516
0.3	-1.928338405	-1.478270843
0.4	-1.606856838	-1.231801809
0.5	-1.373178096	-1.025529704
0.6	-1.185519607	-0.8400346085
0.7	-1.021411309	-0.6612047893
0.8	-0.8633400217	-0.4776697286
0.9	-0.6844600642	-0.2772628386

where $A = y'(0)$. Comparing the coefficient of like powers of p , consequently,

$$\begin{aligned}
 p^{(0)} : y_0(x) &= 1, \\
 p^{(1)} : y_1(x) &= 1 + Ax, \\
 p^{(2)} : y_2(x) &= 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3, \\
 p^{(3)} : y_3(x) &= 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5, \\
 &\vdots
 \end{aligned}
 \tag{6.44}$$

The series solution is given as

$$\begin{aligned}
 y(x) &= 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \left(\frac{A}{10(1-\alpha)} - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}} \right)x^5 \\
 &+ \left(\frac{\alpha A^2}{15(1-\alpha)^2} - \frac{\alpha^3 A^4}{48(1-\alpha)^{7/2}} \right)x^6 + \dots
 \end{aligned}
 \tag{6.45}$$

Table 4 exhibits the initial slopes $A = y'(0)$ for various values of α .

Table 5 exhibits the values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to 1.0 .

Example 7. Consider the following singularly perturbed sixth-order Boussinesq equation

$$u_{tt} = u_{xx} + (p(u))_{xx} + \alpha u_{xxxx} + \beta u_{xxxxx}.
 \tag{6.46}$$

Taking $\alpha = 1$, $\beta = 0$, and $p(u) = 3u^2$, the model equation is given as

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx},
 \tag{6.47}$$

Table 5

x	Y_{kidder}	$Y_{[2/2]}$	$Y_{[3/3]}$
0.1	0.8816588283	0.8633060641	0.8979167028
0.2	0.7663076781	0.7301262261	0.7985228199
0.3	0.6565379995	0.6033054140	0.7041129703
0.4	0.5544024032	0.4848898717	0.6165037901
0.5	0.4613650295	0.3761603869	0.5370533796
0.6	0.3783109315	0.2777311628	0.4665625669
0.7	0.3055976546	0.1896843371	0.4062426033
0.8	0.2431325473	0.1117105165	0.3560801699
0.9	0.1904623681	0.04323673236	0.3179966614
1.0	0.1587689826	0.01646750847	0.2900255005

with initial conditions

$$u(x, 0) = \frac{2ak^2e^{kx}}{(1 + ae^{kx})^2}, \quad u_t(x, 0) = \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3}, \tag{6.48}$$

where a and k are arbitrary constants. The exact solution $u(x, t)$ of the problem is given as

$$u(x, t) = 2 \frac{ak^2 \exp(kx + k\sqrt{1+k^2}t)}{\left(\left(1 + a \exp(kx + k\sqrt{1+k^2}t)\right)\right)^2}. \tag{6.49}$$

The correction functional is given by

$$u_{n+1}(x, t) = \frac{2ak^2e^{kx}}{(1 + ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3}t + \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial t^2} - \left((\tilde{u}_n)_{xx} + 3(\tilde{u}_n^2)_{xx} + (\tilde{u}_n)_{xxxx} \right) \right) ds. \tag{6.50}$$

Making the correction functional stationary, the Lagrange multiplier can easily be identified as $\lambda = s - x$; consequently

$$u_{n+1}(x, t) = \frac{2ak^2e^{kx}}{(1 + ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3}t + \int_0^t (s - x) \left(\frac{\partial^2 u_n}{\partial t^2} - \left((u_n)_{xx} + 3(u_n^2)_{xx} + (u_n)_{xxxx} \right) \right) ds. \tag{6.51}$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$\begin{aligned}
 & u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \\
 &= \frac{2ak^2e^{kx}}{(1+ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t \\
 &+ p \int_0^t (s-x) \left(\left(\frac{\partial^2 u_0}{\partial t^2} + p \frac{\partial^2 u_1}{\partial t^2} + p^2 \frac{\partial^2 u_2}{\partial t^2} + \dots \right) \right. \\
 &\quad \left. - \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + \dots \right) \right) ds \\
 &- p \int_0^t (s-x) \left(\left(\frac{\partial^4 u_0}{\partial x^4} + p \frac{\partial^4 u_1}{\partial x^4} + p^2 \frac{\partial^4 u_2}{\partial x^4} + \dots \right) \right. \\
 &\quad \left. + 3 \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + \dots \right)^2 \right) ds .
 \end{aligned} \tag{6.52}$$

Comparing the coefficients of like powers of p ,

$$\begin{aligned}
 p^{(0)} : u_0(x, t) &= \frac{2e^x}{(1+e^x)^2}, \\
 p^{(1)} : u_1(x, t) &= \frac{2e^x}{(1+e^x)^2} + \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \frac{2e^x(1-4e^x+e^{2x})}{(1+e^x)^4}t^2, \\
 p^{(2)} : u_2(x, t) &= \frac{2e^x}{(1+e^x)^2} + \frac{2ak^3\sqrt{1+k^2}(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \frac{2e^x(1-4e^x+e^{2x})}{(1+e^x)^4}t^2 \\
 &\quad - \frac{2\sqrt{2}e^x(-1+e^x)(1-10e^x+e^{2x})}{3(1+e^x)^5}t^3 \\
 &\quad + \frac{e^x(1-4e^x+e^{2x})(1-44e^x+78e^{2x}-44e^{3x}+e^{4x})}{3(1+e^x)^8}t^4, \\
 &\vdots
 \end{aligned} \tag{6.53}$$

Table 6: Error estimates.

x_i	t_j 0.01	0.02	0.04	0.1	0.2	0.5
-1	2.80886 E - 14	1.79667 E - 12	1.15235 E - 10	2.83355 E - 8	1.83899 E - 6	4.74681 E - 4
-0.8	6.27276 E - 14	4.01362 E - 12	2.57471 E - 10	6.33178 E - 8	4.10454 E - 6	1.04489 E - 3
-0.6	6.08402 E - 14	3.90188 E - 12	2.25663 E - 10	6.18024 E - 8	4.02299 E - 6	1.03093 E - 3
-0.4	1.16573 E - 14	7.41129 E - 13	4.82756 E - 11	1.23843 E - 8	8.53800 E - 6	2.46302 E - 4
-0.2	5.53446 E - 14	3.53395 E - 12	2.25663 E - 10	5.47485 E - 8	3.47264 E - 6	8.35783 E - 4
0	8.63198 E - 14	5.53357 E - 12	2.54174 E - 10	8.65197 E - 8	5.54893 E - 6	1.37353 E - 3
0.2	5.56222 E - 14	3.55044 E - 12	2.27779 E - 10	5.60362 E - 8	3.63600 E - 6	9.29612 E - 4
0.4	1.14353 E - 14	7.14928 E - 13	4.49107 E - 11	1.03370 E - 8	5.93842 E - 7	9.61260 E - 5
0.6	6.06182 E - 14	3.87551 E - 12	2.47218 E - 10	5.97562 E - 8	3.76275 E - 6	8.79002 E - 4
0.8	6.23945 E - 14	3.99519 E - 12	2.55127 E - 10	6.18881 E - 8	3.92220 E - 6	9.36404 E - 4
1	2.79776 E - 14	1.78946 E - 12	1.14307 E - 10	2.77684 E - 8	1.76607 E - 6	4.28986 E - 4

The series solution is given as

$$\begin{aligned}
 u(x, t) = & \frac{2e^x}{(1 + e^x)^2} + \frac{2ak^3\sqrt{1 + k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3}t + \frac{2e^x(1 - 4e^x + e^{2x})}{(1 + e^x)^4}t^2 \\
 & - \frac{2\sqrt{2}e^x(-1 + e^x)(1 - 10e^x + e^{2x})}{3(1 + e^x)^5}t^3 \\
 & + \frac{e^x(1 - 4e^x + e^{2x})(1 - 44e^x + 78e^{2x} - 44e^{3x} + e^{4x})}{3(1 + e^x)^8}t^4 \\
 & + \frac{8e^{2x}(1 - 10e^x + 20e^{2x} - 10e^{3x} + e^{4x})}{(1 + e^x)^8}t^4 \\
 & - \frac{\sqrt{2}e^x(-1 + e^x)(1 - 56e^x + 246e^{2x} - 56e^{3x} + e^{4x})}{15(1 + e^x)^7}t^5 \\
 & + \frac{e^x(1 - 452e^x + 19149e^{2x} - 207936e^{3x} + 807378e^{4x} - 1256568e^{5x})}{45(1 + e^x)^{12}}t^6 \\
 & + \frac{e^x(807378e^{6x} - 207936e^{7x} + 19149e^{8x} - 452e^{9x} + e^{10x})}{45(1 + e^x)^{12}}t^6 + \dots
 \end{aligned} \tag{6.54}$$

Table 6 exhibits the absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution.

Figure 4 depicts the series solution $u(x, t)$.

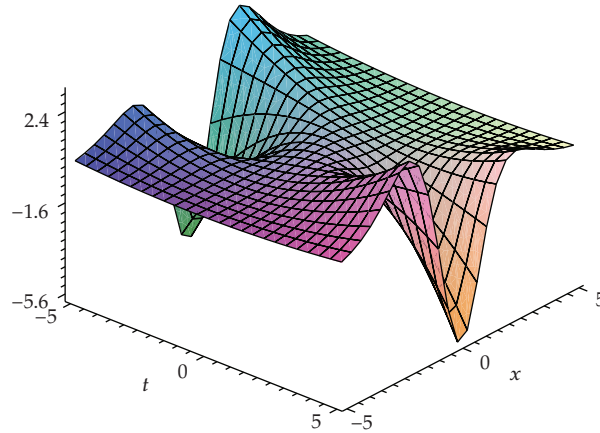


Figure 4

Example 8. Consider Fisher's equation of the following form:

$$u_t(x, t) - u_{xx}(x, t) - u(x, t)(1 - u(x, t)) = 0, \quad (6.55)$$

with initial conditions

$$u(x, 0) = \beta. \quad (6.56)$$

The correction functional is given by

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left(\frac{\partial u_n(x, s)}{\partial s} - \frac{\partial^2 \tilde{u}_n(x, s)}{\partial x^2} - \tilde{u}_n(x, \tau)(1 - \tilde{u}_n(x, s)) \right) ds. \quad (6.57)$$

Making the above correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$; consequently, we obtain the following iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left(\frac{\partial u_n(x, s)}{\partial s} - \frac{\partial^2 u_n(x, s)}{\partial x^2} - u_n(x, s)(1 - u_n(x, s)) \right) ds. \quad (6.58)$$

Applying the variational iteration method using He's polynomials (5.1) (VIMHPS), we get

$$\begin{aligned}
 u_0 + pu_1 + p^2u_2 + \dots = u_0(x, t) - p \int_0^t & \left(\left(\frac{\partial u_0(x, s)}{\partial s} + p \frac{\partial u_1(x, s)}{\partial s} + p^2 \frac{\partial u_2(x, s)}{\partial s} + \dots \right) \right) ds \\
 & + p \int_0^t \left(\left(\frac{\partial^2 u_0(x, s)}{\partial x^2} + p \frac{\partial^2 u_1(x, s)}{\partial x^2} + \dots \right) \right. \\
 & \left. + ((u_0 + pu_1 + \dots)(1 - (u_0 + pu_1 + \dots))) \right) ds.
 \end{aligned} \tag{6.59}$$

Comparing the coefficient of like powers of p , following approximants are obtained:

$$\begin{aligned}
 p^{(0)} : u_0(x, t) &= \beta, \\
 p^{(1)} : u_1(x, t) &= \beta + \beta(1 - \beta)t, \\
 p^{(2)} : u_2(x, t) &= \beta + \beta(1 - \beta)t + \frac{t^2}{2!}\beta(1 - \beta)(1 - 3\beta + 2\beta^2) - \frac{t^3}{3}\beta^2(-1 + \beta)^2, \\
 p^{(3)} : u_3(x, t) &= \beta + \beta(1 - \beta)t + \frac{t^3}{3!}\beta(1 - \beta)(1 - 6\beta + 6\beta^2) + \frac{t^4}{3}(-1 + \beta)^2\beta^2(-1 + 2\beta) \\
 &\quad - \frac{t^5}{60}(-1 + \beta)^2\beta^2(3 - 20\beta + 20\beta^2) + \frac{t^6}{18}(-1 + \beta)^3\beta^3(-1 + 2\beta) - \frac{t^7}{63}(-1 + \beta)^4\beta^4, \\
 &\vdots
 \end{aligned} \tag{6.60}$$

The series solution is given by

$$\begin{aligned}
 u(x, t) = & \beta + \beta(1 - \beta)t + \frac{t^3}{3!}\beta(1 - \beta)(1 - 6\beta + 6\beta^2) + \frac{t^4}{3}(-1 + \beta)^2\beta^2(-1 + 2\beta) \\
 & - \frac{t^5}{60}(-1 + \beta)^2\beta^2(3 - 20\beta + 20\beta^2) + \frac{t^6}{18}(-1 + \beta)^3\beta^3(-1 + 2\beta) - \frac{t^7}{63}(-1 + \beta)^4\beta^4 + \dots,
 \end{aligned} \tag{6.61}$$

and in a closed-form by

$$u(x, t) = \frac{\beta \exp t}{1 - \beta + \beta \exp t}. \tag{6.62}$$

Example 9. Consider the following seventh-order generalized KdV (SOG-KdV) equation:

$$u_t + uu_x + u_{xxx} - u_{xxxxx} + \sigma u_{xxxxxx} = 0, \tag{6.63}$$

Table 7: Numerical results for Fisher's equation.

t	$\beta = 0.2$		$\beta = 0.8$	
	*E (ADM)	*E(VIMHPS)	*E (ADM)	*E(VIMHPS)
0	0	0	0	0
0.2	6.19201 E - 06	1.96407 E - 06	5.57339 E - 06	2.63705 E - 06
0.4	1.03635 E - 04	2.60211 E - 05	1.13137 E - 04	4.7283 E - 05
0.6	5.45505 E - 04	1.05333 E - 04	4.00147 E - 04	2.63379 E - 04
0.8	1.78050 E - 03	2.54998 E - 04	1.18584 E - 03	9.01304 E - 04
1	4.45699 E - 03	4.515414 E - 04	1.99502 E - 03	1.60455 E - 05

* Error= exact solution- series solution.

where $\sigma = \beta\delta/\gamma^2$, with initial conditions

$$u(x, 0) = a_0 + a_6 \sec h^6(kx), \quad (6.64)$$

where $K = 5/\sqrt{1538}$, $a_0 = c - 18000/769^2$, $a_6 = 519750/769^2$, and c is an arbitrary parameter. The correction functional is given as

$$u_{n+1}(x, t) = u_n + \int_0^t \lambda(s) \left(\frac{\partial u_n}{\partial s} + \tilde{u}_n \frac{\partial \tilde{u}_n}{\partial x} + \frac{\partial^3 \tilde{u}_n}{\partial x^3} - \frac{\partial^5 \tilde{u}_n}{\partial x^5} + \sigma \frac{\partial^7 \tilde{u}_n}{\partial x^7} \right) ds. \quad (6.65)$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$; consequently

$$u_{n+1}(x, t) = u_n - \int_0^t \left(\frac{\partial u_n}{\partial s} + u_n \frac{\partial u_n}{\partial x} + \frac{\partial^3 u_n}{\partial x^3} - \frac{\partial^5 u_n}{\partial x^5} + \sigma \frac{\partial^7 u_n}{\partial x^7} \right) ds. \quad (6.66)$$

Applying the variational iteration method using He's polynomials (VIMHPS), we get

$$\begin{aligned} & u_0 + pu_1 + p^2u_2 + \dots \\ &= u_0 - p \int_0^t \left(\left(\frac{\partial u_0}{\partial s} + p \frac{\partial u_1}{\partial s} + p^2 \frac{\partial u_2}{\partial s} + \dots \right) + \left(u_0 \frac{\partial u_0}{\partial x} + pu_1 \frac{\partial u_1}{\partial x} + p^2 u_2 \frac{\partial u_2}{\partial x} + \dots \right) \right) ds \\ & \quad - p \int_0^t \left(\left(\frac{\partial^3 u_0}{\partial x^3} + p \frac{\partial^3 u_1}{\partial x^3} + \dots \right) - \left(\frac{\partial^5 u_0}{\partial x^5} + p \frac{\partial^5 u_1}{\partial x^5} + \dots \right) \right. \\ & \quad \left. + \sigma \left(\frac{\partial^7 u_0}{\partial x^7} + p \frac{\partial^7 u_1}{\partial x^7} + \dots \right) \right) ds. \end{aligned} \quad (6.67)$$

Comparing the coefficient of like powers of p ,

$$\begin{aligned}
 p^{(0)} : u_0(x, t) &= a_0 + a_6 \sec h^6(kx), \\
 p^{(1)} : u_1(x, t) &= a_0 + a_6 \sec h^6(kx) + \frac{3a_6 kt}{16} \\
 &\quad \times \left\{ 10a_0 + 32a_6 - 312k^2 - 26400k^4 - 9866112\delta k^6 \right. \\
 &\quad \quad + \left(15a_0 - 256k^2 - 10480k^4 + 9932224\delta k^6 \right) \cosh(2kx) \\
 &\quad \quad + \left(6a_0 - 8k^2 + 14624k^4 - 1443968\delta k^6 \right) \cosh(4kx) \\
 &\quad \quad \left. + \left(a_0 + 36k^2 - 1296k^4 + 46656\delta k^6 \right) \cosh(6kx) \right\} \sec h^{12}(kx) \tanh(kx), \\
 &\quad \vdots
 \end{aligned} \tag{6.68}$$

The series solution is given by

$$\begin{aligned}
 u(x, t) &= a_0 + a_6 \sec h^6(kx) + \frac{3a_6 kt}{16} \\
 &\quad \times \left\{ 10a_0 + 32a_6 - 312k^2 - 26400k^4 - 9866112\delta k^6 \right. \\
 &\quad \quad + \left(15a_0 - 256k^2 - 10480k^4 + 9932224\delta k^6 \right) \cosh(2kx) \\
 &\quad \quad + \left(6a_0 - 8k^2 + 14624k^4 - 1443968\delta k^6 \right) \cosh(4kx) \\
 &\quad \quad \left. + \left(a_0 + 36k^2 - 1296k^4 + 46656\delta k^6 \right) \cosh(6kx) \right\} \sec h^{12}(kx) \tanh(kx) - \frac{3a_6 kt^2}{4096} \\
 &\quad \times \left[1320a_0^2 + 6400a_0a_6 + 40960a_6^2 - 93120a_0k^2 - 642560a_6k^2 + 3592320k^4 \right. \\
 &\quad \quad - 718464a_0k^4 - 223897600a_6k^4 + 1066859520k^6 - 1066859520a_0\delta k^6 \\
 &\quad \quad - 371744399360a_6\delta k^6 + 151760209920k^8 \\
 &\quad \quad + 303520419840\delta k^{10} + 186385174609920\delta^2 k^{12} \\
 &\quad \quad + \left(2079a_0^2 + 6912a_0a_6 - 36864a_6^2 - 124488a_0k^2 - 291840a_6k^2 \right. \\
 &\quad \quad + 3662064k^4 - 7324128a_0k^4 + 83202048a_6k^4 + 643886208k^6 - 643886208a_0\delta k^6 \\
 &\quad \quad + 520393113600a_6\delta k^6 + 10077306624k^8 + 20154613248\delta k^8 \\
 &\quad \quad \left. - 115427212572672\delta k^{10} - 318167001264623616\delta^2 k^{12} \right) \cosh(2kx)
 \end{aligned}$$

$$\begin{aligned}
& + \left(924a_0^2 - 1536a_0a_6 - 14112a_0k^2 + 592896a_6k^2 \right. \\
& \quad - 1544256k^4 + 3088512a_0k^4 + 208736256a_6k^4 - 1003580928k^6 \\
& \quad + 1003580928a_0\delta k^6 - 173675003904a_6\delta k^6 - 180653147136k^8 \\
& \quad - 361306294272\delta k^8 - 159045011693568\delta k^{10} \\
& \quad \left. + 129148767835766784\delta^2 k^{12} \right) \cosh(4kx) \\
& + \left(77a_0^2 - 2816a_0a_6 + 36904a_0k^2 - 2283568k^4 \right. \\
& \quad + 4567136a_0\delta k^6 + 4399738112k^8 + 8799476224\delta k^8 + 114650957797376\delta k^{10} \\
& \quad \left. - 27618434663723008\delta^2 k^{12} \right) \cosh(6kx) \\
& - \left(168a_0^2 + 768a_0a_6 - 25536a_0k^2 + 69120a_6k^2 \right. \\
& \quad + 594048k^4 - 1188096a_0k^4 - 8460288a_6k^4 - 118291504k^6 \\
& \quad + 118293504a_0\delta k^6 + 1164533760a_6\delta k^6 - 35208886272k^8 \\
& \quad - 70417772544\delta k^8 + 25860049453056\delta k^{10} \\
& \quad \left. - 2901989324193792\delta^2 k^{12} \right) \cosh(8kx) \\
& - \left(105a_0^2 - 5880a_0k^2 - 132720k^4 - 67643520k^6 \right. \\
& \quad + 67643520a_0\delta k^6 + 7719962880k^8 + 15439925760\delta k^8 - 2194541905920\delta k^{10} \\
& \quad \left. + 132260300820480\delta^2 k^{12} \right) \cosh(10kx) \\
& - \left(28a_0^2 + 224a_0k^2 - 53312k^4 + 106626a_0k^4 \right. \\
& \quad + 11178496k^6 - 11178496a_0\delta k^6 - 436093852k^8 - 872187904\delta k^8 \\
& \quad \left. + 61463535616\delta k^{10} - 2068416315392\delta^2 k^{12} \right) \cosh(12kx) \\
& - \left(3a_0^2 + 216a_0k^2 + 3888k^4 - 7776a_0k^4 - 279936k^6 \right. \\
& \quad + 279936a\delta k^6 + 5038848k^8 + 10077696\delta k^8 - 362797056\delta k^{10} \\
& \quad \left. + 6530347008\delta^2 k^{12} \right) \cosh(14kx) \Big] \sec h^{20}(kx) + \dots .
\end{aligned} \tag{6.69}$$

The closed-form solution is given as

$$u(x, t) = a_0 + a_6 \sec h^6(k(x - ct)), \tag{6.70}$$

where $k = 5/\sqrt{1538}$, $a_0 = c - 18000/769^2$, $a_6 = 519750/769^2$, and c is an arbitrary parameter.

Table 8: Error estimates ($\delta = .01$).

x	Error t_j			
	$j = .001$	$j = .01$	$j = .05$	$j = .1$
-1.0	3.1×10^{-5}	4.6×10^{-4}	5.6×10^{-3}	1.9×10^{-2}
-.8	2.6×10^{-5}	4.0×10^{-4}	5.2×10^{-3}	1.8×10^{-2}
-.6	2.0×10^{-5}	3.4×10^{-4}	4.7×10^{-3}	1.7×10^{-2}
-.4	1.4×10^{-5}	2.7×10^{-4}	4.3×10^{-3}	1.5×10^{-2}
-.2	7.8×10^{-6}	2.0×10^{-4}	3.9×10^{-3}	1.4×10^{-2}
0.0	1.4×10^{-6}	1.4×10^{-4}	3.5×10^{-3}	1.4×10^{-2}
.2	5.0×10^{-6}	7.8×10^{-5}	3.2×10^{-3}	1.3×10^{-2}
.4	1.1×10^{-5}	1.8×10^{-5}	3.0×10^{-3}	1.3×10^{-2}
.6	1.7×10^{-5}	3.6×10^{-5}	2.8×10^{-3}	1.3×10^{-2}
.8	2.3×10^{-5}	8.7×10^{-5}	2.7×10^{-3}	1.3×10^{-2}
1.0	2.8×10^{-5}	1.3×10^{-4}	2.6×10^{-3}	1.3×10^{-2}

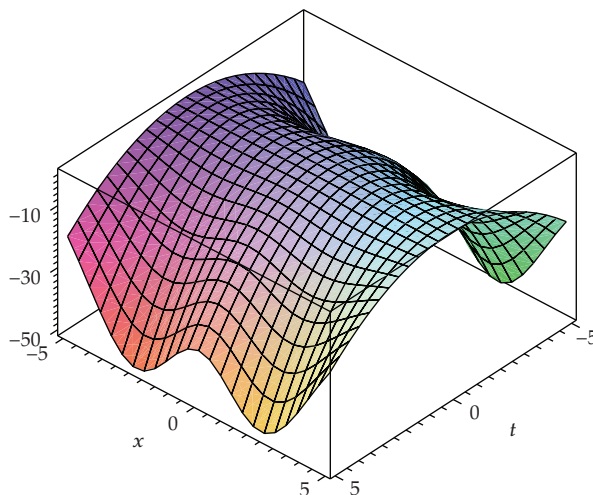
**Figure 5**

Figure 5 depicts series solution at $\delta = .01$.

7. Conclusion

In this paper, we applied variational iteration method using He's polynomials (VIMHPS) for solving various initial and boundary value problems. The proposed modified method (VIMHPS) is employed without using linearization, discretization, transformation, or restrictive assumptions, absorb the positive features of the coupled techniques, and hence is very much compatible with the diversified and versatile nature of the physical problems. Moreover, the modification based upon He's polynomials (VIMHPS) is easier to implement and is more user friendly as compared to the decomposition method where Adomian's polynomials along with their complexities are used.

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