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Research Article

Regularity of Weakly Well-Posed Characteristic Boundary Value Problems

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We study the boundary value problem for a linear first-order partial differential system with characteristic boundary of constant multiplicity. We assume the problem to be "weakly" well posed, in the sense that a unique L^2-solution exists, for sufficiently smooth data, and obeys an a priori energy estimate with a finite loss of tangential/conormal regularity. This is the case of problems that do not satisfy the uniform Kreiss-Lopatinskiĭ condition in the hyperbolic region of the frequency domain. Provided that the data are sufficiently smooth, we obtain the regularity of solutions, in the natural framework of weighted conormal Sobolev spaces.

1. Introduction and Main Results

For $n \geq 2$, let $\mathbb{R}^n_+$ denote the n-dimensional positive half-space

$$\mathbb{R}^n_+ := \left\{ x = (x_1, x'),\ x_1 > 0,\ x' := (x_2, \dots, x_n) \in \mathbb{R}^{n-1} \right\}. \tag{1.1}$$

The boundary of $\mathbb{R}^n_+$ will be systematically identified with $\mathbb{R}^{n-1}_{x'}$.

We are interested in the following stationary boundary value problem (BVP):

$$(\gamma + L + \mathcal{B})u = F, \quad \text{in } \mathbb{R}^n_+, \tag{1.2}$$

$$Mu = G, \quad \text{on } \mathbb{R}^{n-1}, \tag{1.3}$$

where L is the first-order linear partial differential operator

$$L = \sum_{j=1}^{n} A_j \partial_j; \tag{1.4}$$

for each $j = 1, \dots, n$, the short notation $\partial_j := \partial/\partial x_j$ is used.

The coefficients A_j $(j = 1, \dots, n)$ of L are $N \times N$ matrix-valued functions in $C^\infty_{(0)}(\mathbb{R}^n_+)$, the space of restrictions to $\mathbb{R}^n_+$ of functions of $C^\infty_0(\mathbb{R}^n)$. In (1.2), $\mathcal{B}$ stands for a lower-order term whose form and nature will be specified later; compare to Theorem 1.1 and Section 3.2.

The source term F, as well as the unknown u, is a $\mathbb{R}^N$-valued function of x; we may assume that they are both supported in the unitary positive half-ball $\mathbb{B}^+ := \{x = (x_1, x') : x_1 \geq 0, \ |x| < 1\}$.

The BVP has *characteristic boundary of constant multiplicity* $1 \leq r < N$ in the following sense; the coefficient A_1 of the normal derivative in L displays the blockwise structure

$$A_1(x) = \begin{pmatrix} A_1^{I,I} & A_1^{I,II} \\ A_1^{II,I} & A_1^{II,II} \end{pmatrix}, \tag{1.5}$$

where $A_1^{I,I}, A_1^{I,II}, A_1^{II,I}, A_1^{II,II}$ are, respectively, $r \times r$, $r \times (N-r)$, $(N-r) \times r$, $(N-r) \times (N-r)$ submatrices, such that

$$A^{I,II}_{1|x_1=0} = 0, \qquad A^{II,I}_{1|x_1=0} = 0, \qquad A^{II,II}_{1|x_1=0} = 0, \tag{1.6}$$

and $A_1^{I,I}$ is invertible over $\mathbb{B}^+$. According to the representation above, we split the unknown u as $u = (u^I, u^{II})$; $u^I \in \mathbb{R}^r$ and $u^{II} \in \mathbb{R}^{N-r}$ are said to be, respectively, the *noncharacteristic* and the *characteristic* components of u.

Concerning the boundary condition (1.3), M is assumed to be the matrix $(I_d \ 0)$, where I_d denotes the identity matrix of order d, 0 is the zero matrix of size $d \times (N-d)$, and d is a given positive integer $\leq r$. The datum G is a given $\mathbb{R}^d$-valued function of $x' = (x_2, \dots, x_n)$ and is supported in the unitary $(n-1)$-dimensional ball $B(0,1) := \{|x'| < 1\}$.

Section 4 will be devoted to prove the following regularity result.

Theorem 1.1. *Let k, r, s be fixed nonnegative integer numbers such that $s \geq r \geq 0$, $s > 0$, and suppose that the coefficients A_j $(j = 1, \dots, n)$ of the operator L in* (1.4) *are given in $C^\infty_{(0)}(\mathbb{R}^n_+)$ and A_1 fulfils conditions* (1.5), (1.6). *One assumes that for any $h > 0$ there exist some constants $C_0 = C_0(h) > 0$, $\gamma_0 = \gamma_0(h) \geq 1$ such that for every $\gamma \geq \gamma_0$, for every operator $\mathcal{B} = \mathrm{Op}^\gamma_\sharp(b)$, whose symbol b belongs to Γ^0 and satisfies $|b|_{0,k} \leq h$, and for all functions $F \in \mathcal{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+)$, and $G \in H^s_\gamma(\mathbb{R}^{n-1})$, the corresponding BVP* (1.2)-(1.3) *admits a unique solution $u \in L^2(\mathbb{R}^n_+)$, with $u^I_{|x_1=0} \in L^2(\mathbb{R}^{n-1})$, and the following a priori energy estimate is satisfied:*

$$\gamma \|u\|^2_{L^2(\mathbb{R}^n_+)} + \left\| u^I_{|x_1=0} \right\|^2_{L^2(\mathbb{R}^{n-1})} \leq C_0 \left(\frac{1}{\gamma^{2s+1}} \|F\|^2_{\mathcal{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+)} + \frac{1}{\gamma^{2s}} \|G\|^2_{H^s_\gamma(\mathbb{R}^{n-1})} \right). \tag{1.7}$$

Then, for every $m \in \mathbb{N}$ and any matrix-valued function $B \in C^\infty_{(0)}(\mathbb{R}^n_+)$, there exist some constants $C_m > 0$, γ_m (with $\gamma_m \geq \gamma_{m-1} \geq 1$) such that if $\gamma \geq \gamma_m$ and we are given arbitrary functions

$F \in \mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)$ and $G \in H^{s+m}_{\gamma}(\mathbb{R}^{n-1})$, the unique L^2-solution u of (1.2)-(1.3) *(with data F, G, and lower-order term $\mathcal{B} \equiv$ multiplication by B) belongs to $H^m_{\tan,\gamma}(\mathbb{R}^n_+)$, $u^I_{|x_1=0} \in H^m_{\gamma}(\mathbb{R}^{n-1})$ and the a priori estimate of order m*

$$\gamma\|u\|^2_{H^m_{\tan,\gamma}(\mathbb{R}^n_+)} + \left\|u^I_{|x_1=0}\right\|^2_{H^m_{\gamma}(\mathbb{R}^{n-1})} \le C_m\left(\frac{1}{\gamma^{2s+1}}\|F\|^2_{\mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)} + \frac{1}{\gamma^{2s}}\|G\|^2_{H^{s+m}_{\gamma}(\mathbb{R}^{n-1})}\right) \tag{1.8}$$

is satisfied.

The function spaces involved in the statement of Theorem 1.1, as well as the norms appearing in (1.7), (1.8), will be described in Section 2. The kind of lower-order operator $\mathcal{B}$ involved in (1.2), that is allowed in Theorem 1.1, will be introduced in Section 3.2.

The BVP (1.2)-(1.3), with the aforedescribed structure, naturally arises from the study of a mixed evolution problem for a symmetric (or Friedrichs'symmetrizable) hyperbolic system, with characteristic boundary. The analysis of the regularity of the stationary problem, presented in this work, plays an important role for the study of the regularity of time-dependent hyperbolic problems, constituting the final goal of our investigation and developed in [1]. In view of the well-posedness property that problems (1.2)-(1.3) enjoy in the statement of Theorem 1.1, here we do not need to assume the hyperbolicity of the linear operator L in (1.4); the only condition required on the structure of L is that expressed by conditions (1.5) and (1.6). In the hyperbolic problems, the number d of the scalar boundary conditions prescribed in (1.3) equals the number of positive eigenvalues of A_1 on $\{x_1 = 0\} \cap \mathbb{B}^+$ (the so-called *incoming characteristics* of problem (1.2)-(1.3)), compare to [1]; this value d remains constant along the boundary, as a combined effect of the hyperbolicity and the fact that $A_{1|\{x_1=0\}\cap\mathbb{B}^+}$ has constant rank.

In [2], the regularity of weak solutions to the characteristic BVP (1.2)-(1.3) was studied, under the assumption that the problem is *strongly L^2-well posed*, namely, that a unique L^2-solution exists for arbitrarily given L^2-data and the solution obeys an a priori energy inequality *without loss of regularity with respect to the data*; this means that the L^2-norms of the interior and boundary values of the solution are measured by the L^2-norms of the corresponding data F, G.

The statement of Theorem 1.1 extends the result of [2, Theorem 15], to the case where only a *weak well posedness* property is assumed on the BVP (1.2)-(1.3). Here, the L^2-solvability of (1.2)-(1.3) requires an additional regularity of the corresponding data F, G; the integer s represents the minimal amount of regularity, needed for data, in order to estimate the L^2-norm of the solution u in the interior of the domain, and its trace on the boundary, by the energy inequality (1.7).

Several problems, appearing in a variety of different physical contexts, such as fluid dynamics and magneto-hydrodynamics, exhibit a finite loss of derivatives with respect to the data, as considered by estimate (1.7) in the statement of Theorem 1.1. This is the case of some problems that do not satisfy the so-called *uniform Kreiss-Lopatinskiĭ* condition; see, for example, [3, 4]. For instance, when the *Lopatinskiĭ determinant* associated to the problem has a simple root in the *hyperbolic* region, estimating the L^2-norm of the solution makes the loss of one tangential derivative with respect to the data; see, for example, [5, 6]. In [7], Coulombel and Guès show that, in this case, the loss of regularity of order one is optimal; they also prove that the weak well posedness, with the loss of one derivative, is independent of Lipschitzean lower-order terms, but *not* independent of bounded lower-order terms. This is a major difference with the strongly well-posed case, where there is no loss of derivatives and one can treat lower-order terms as source terms in the energy estimates. Also, this yields that the techniques we used in [2], for studying the regularity of strongly L^2-well-posed BVPs, cannot be successfully performed in the case of weakly well-posed problems (see Section 4 for a better explanation).

The paper is organized as follows. In Section 2 we introduce the function spaces to be used in the following and the main related notations. In Section 3 we collect some technical tools, and the basic concerned results, that will be useful for the proof of the regularity of BVP (1.2)-(1.3), given in Section 4.

A final Appendix contains the proof of the most of the technical results used in Section 4.

2. Function Spaces

The purpose of this section is to introduce the main function spaces to be used in the following and collect their basic properties.

For $j = 1, 2, \dots, n$, we set

$$Z_1 := x_1\partial_1, \qquad Z_j := \partial_j, \quad \text{for } j \geq 2. \tag{2.1}$$

Then, for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the *conormal* derivative Z^α is defined by

$$Z^\alpha := Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}; \tag{2.2}$$

we also write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for the usual partial derivative corresponding to α.

For $\gamma \geq 1$ and $s \in \mathbb{R}$, we set

$$\lambda^{s,\gamma}(\xi) := \left(\gamma^2 + |\xi|^2\right)^{s/2} \tag{2.3}$$

and, in particular, $\lambda^{s,1} := \lambda^s$.

The Sobolev space of order $s \in \mathbb{R}$ in $\mathbb{R}^n$ is defined to be the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\lambda^s \widehat{u} \in L^2(\mathbb{R}^n)$, being $\widehat{u}$ the Fourier transform of u; in particular, for $s \in \mathbb{N}$, the Sobolev space of order s reduces to the set of all functions $u \in L^2(\mathbb{R}^n)$, for which $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq s$.

Throughout the paper, for real $\gamma \geq 1$, $H^s_\gamma(\mathbb{R}^n)$ will denote the Sobolev space of order s, equipped with the γ-depending norm $\|\cdot\|_{s,\gamma}$ defined by

$$\|u\|^2_{s,\gamma} := (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{2s,\gamma}(\xi) |\widehat{u}(\xi)|^2 d\xi, \tag{2.4}$$

where ($\xi = (\xi_1, \dots, \xi_n)$ are the dual Fourier variables of $x = (x_1, \dots, x_n)$). The norms defined by (2.4), with different values of the parameter γ, are equivalent to each other. For $\gamma = 1$ we set for brevity $\|\cdot\|_s := \|\cdot\|_{s,1}$ (and, accordingly, $H^s(\mathbb{R}^n) := H^s_1(\mathbb{R}^n)$).

It is clear that, for $s \in \mathbb{N}$, the norm in (2.4) turns out to be equivalent, *uniformly with respect to* γ, to the norm $\|\cdot\|_{H^s_\gamma(\mathbb{R}^n)}$ defined by

$$\|u\|^2_{H^s_\gamma(\mathbb{R}^n)} := \sum_{|\alpha|\leq s} \gamma^{2(s-|\alpha|)} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

Another useful remark about the parameter depending norms defined in (2.4) is provided by the following counterpart of the usual Sobolev imbedding inequality:

$$\|u\|_{s,\gamma} \leq \gamma^{s-r} \|u\|_{r,\gamma}, \tag{2.6}$$

for arbitrary $s \leq r$ and $\gamma \geq 1$.

In Section 4, the ordinary Sobolev spaces, endowed with the weighted norms above, will be considered in $\mathbb{R}^{n-1}$ (interpreted as the boundary of the half-space $\mathbb{R}^n_+$); regardless to the different dimension, the same notations and conventions as before will be used there.

Let us introduce now some classes of function spaces of Sobolev type, defined over the half-space $\mathbb{R}^n_+$; these spaces will be used to measure the regularity of solutions to characteristic BVPs with sufficiently smooth data (cf. Theorem 1.1 and Section 4).

Given an integer $m \geq 1$, the *conormal Sobolev space* of order m is defined as the set of functions $u \in L^2(\mathbb{R}^n_+)$ such that $Z^\alpha u \in L^2(\mathbb{R}^n_+)$, for all multi-indices α with $|\alpha| \leq m$. Agreeing with the notations set for the usual Sobolev spaces, for $\gamma \geq 1$, $H^m_{\tan,\gamma}(\mathbb{R}^n_+)$ will denote the conormal space of order m equipped with the γ-depending norm

$$\|u\|^2_{H^m_{\tan,\gamma}(\mathbb{R}^n_+)} := \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)} \|Z^\alpha u\|^2_{L^2(\mathbb{R}^n_+)}, \tag{2.7}$$

and we again write $H^m_{\tan}(\mathbb{R}^n_+) := H^m_{\tan,1}(\mathbb{R}^n_+)$.

For later use, we need to consider also a class of *mixed tangential/conormal* spaces, where different orders of tangential and conormal smoothness are allowed. Namely, for every $m, r \in \mathbb{N}$, with $m \geq r$, we let $\mathscr{H}^{m,r}_{\tan}(\mathbb{R}^n_+)$ denote the space of all functions $u \in L^2(\mathbb{R}^n_+)$ such that $Z^\alpha u \in L^2(\mathbb{R}^n_+)$, whenever $|\alpha| \leq m$ and $0 \leq \alpha_1 \leq r$: here derivatives Z^α are required belonging to L^2 up to the order m, but *conormal derivatives* (namely, derivatives *involving the operator* Z_1) are allowed only up to the lower order r, the remaining $m - r$ derivatives being *purely tangential* (i.e, involving only differentiation with respect to tangential variables x'). This space is provided with the expected γ-depending norm

$$\|u\|^2_{\mathscr{H}^{m,r}_{\tan,\gamma}(\mathbb{R}^n_+)} := \sum_{|\alpha|\leq m,\, 0\leq\alpha_1\leq r} \gamma^{2(m-|\alpha|)} \|Z^\alpha u\|^2_{L^2(\mathbb{R}^n_+)}. \tag{2.8}$$

The notation $\mathscr{H}^{m,r}_{\tan,\gamma}(\mathbb{R}^n_+)$ is used, here and below, with the same meaning as for usual and conormal Sobolev spaces (accordingly, one has $\mathscr{H}^{m,r}_{\tan}(\mathbb{R}^n_+) = \mathscr{H}^{m,r}_{\tan,1}(\mathbb{R}^n_+)$).

For a given Banach space Y (with norm $\|\cdot\|_Y$) and $1 \leq p \leq +\infty$, $L^p(0,+\infty; Y)$ will denote the space of the Y-valued measurable functions on $(0,+\infty)$ such that $\int_0^{+\infty} \|u(t)\|^p_Y dt < +\infty$.

It is easy to see that the following identities hold true for $\mathscr{H}^{m,r}_{\tan}(\mathbb{R}^n_+)$, in the border cases $r = 0$ and $r = m$:

$$\mathscr{H}^{m,0}_{\tan}(\mathbb{R}^n_+) = L^2\Big(0, +\infty; H^m\Big(\mathbb{R}^{n-1}\Big)\Big), \qquad \mathscr{H}^{m,m}_{\tan}(\mathbb{R}^n_+) = H^m_{\tan}(\mathbb{R}^n_+). \tag{2.9}$$

Actually all of the previously collected observations and properties of γ-weighted norms on usual Sobolev spaces can be readily extended to the weighted norms defined on conormal and mixed spaces.

Remark 2.1. The above-considered tangential-conormal spaces $\mathscr{H}^{m,r}_{\tan}(\mathbb{R}^n_+)$ can be viewed as a conormal counterpart, by the action of the $\sharp$ mapping introduced below, of corresponding mixed spaces of Sobolev type in $\mathbb{R}^n$, studied in Hörmander's [8].

3. Preliminaries and Technical Tools

In this section, we collect several technical tools that will be used in the subsequent analysis (cf. Section 4).

We start by recalling the definition of two operators $\sharp$ and $\natural$, introduced by Nishitani and Takayama in [9], with the main property of mapping isometrically square integrable (resp., essentially bounded) functions over the half-space $\mathbb{R}^n_+$ onto square integrable (resp., essentially bounded) functions over the full space $\mathbb{R}^n$.

The mappings $\sharp : L^2(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n)$ and $\natural : L^\infty(\mathbb{R}^n_+) \to L^\infty(\mathbb{R}^n)$ are, respectively, defined by

$$w^\sharp(x) := w(e^{x_1}, x')e^{x_1/2}, \qquad a^\natural(x) = a(e^{x_1}, x'), \quad \forall x = (x_1, x') \in \mathbb{R}^n. \tag{3.1}$$

They are both norm preserving bijections.

It is also useful to notice that the above operators can be extended to the set $\mathfrak{D}'(\mathbb{R}^n_+)$ of Schwartz distributions in $\mathbb{R}^n_+$. It is easily seen that both $\sharp$ and $\natural$ are topological isomorphisms of the space $C^\infty_0(\mathbb{R}^n_+)$ of test functions in $\mathbb{R}^n_+$ (resp., $C^\infty(\mathbb{R}^n_+)$) onto the space $C^\infty_0(\mathbb{R}^n)$ of test functions in $\mathbb{R}^n$ (resp., $C^\infty(\mathbb{R}^n)$). Therefore, a standard duality argument leads to define $\sharp$ and $\natural$ on $\mathfrak{D}'(\mathbb{R}^n_+)$, by setting for every $\varphi \in C^\infty_0(\mathbb{R}^n)$

$$\begin{aligned} \langle u^\sharp, \varphi\rangle &:= \langle u, \varphi^{\sharp^{-1}}\rangle, \\ \Big\langle u^\natural, \varphi\Big\rangle &:= \Big\langle u, \varphi^\flat\Big\rangle \end{aligned} \tag{3.2}$$

($\langle\cdot,\cdot\rangle$ is used to denote the duality pairing between distributions and test functions either in the half-space $\mathbb{R}^n_+$ or the full space $\mathbb{R}^n$). In the right-hand sides of (3.2), $\sharp^{-1}$ is just the inverse operator of $\sharp$, while the operator $\flat$ is defined by

$$\varphi^\flat(x) = \frac{1}{x_1}\varphi(\log x_1, x'), \quad \forall x_1 > 0, \ x' \in \mathbb{R}^{n-1}, \tag{3.3}$$

for functions $\varphi \in C^\infty_0(\mathbb{R}^n)$. The operators $\sharp^{-1}$ and $\flat$ arise by explicitly calculating the formal adjoints of $\sharp$ and $\natural$, respectively.

Of course, one has that $u^\sharp, u^\natural \in \mathfrak{D}'(\mathbb{R}^n)$; moreover the following relations can be easily verified (cf. [9]):

$$(\psi u)^\sharp = \psi^\natural u^\sharp, \tag{3.4}$$

$$\partial_j\left(u^\natural\right) = \left(Z_j u\right)^\natural, \quad j = 1, \ldots, n, \tag{3.5}$$

$$\partial_1\left(u^\sharp\right) = (Z_1 u)^\sharp + \frac{1}{2}u^\sharp, \tag{3.6}$$

$$\partial_j\left(u^\sharp\right) = \left(Z_j u\right)^\sharp, \quad j = 2, \ldots, n, \tag{3.7}$$

whenever $u \in \mathfrak{D}'(\mathbb{R}^n_+)$ and $\psi \in C^\infty(\mathbb{R}^n_+)$ (in (3.4) $u \in L^2(\mathbb{R}^n_+)$ and $\psi \in L^\infty(\mathbb{R}^n_+)$ are even allowed).

From formulas (3.6), (3.7) and the L^2-boundedness of $\sharp$, it also follows that $\sharp : H^m_{\tan,\gamma}(\mathbb{R}^n_+) \to H^m_\gamma(\mathbb{R}^n)$ is a topological isomorphism, for each integer $m \geq 1$ and real $\gamma \geq 1$.

Following [9] (see also [2]), in the next subsection the lastly mentioned property of $\sharp$ will be exploited to shift some remarkable properties of the ordinary Sobolev spaces in $\mathbb{R}^n$ to the functional framework of conormal Sobolev spaces over the half-space $\mathbb{R}^n_+$.

In the end, we observe that the operator $\sharp$ continuously maps the space $C^\infty_{(0)}(\mathbb{R}^n_+)$ into the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions in $\mathbb{R}^n$ (note also that the same is no longer true for the image of $C^\infty_{(0)}(\mathbb{R}^n_+)$ under the operator $\natural$, which is only included into the space $C^\infty_b(\mathbb{R}^n)$ of infinitely smooth functions in $\mathbb{R}^n$, with bounded derivatives of all orders).

3.1. Parameter-Depending Norms on Sobolev Spaces

We recall a classical characterization of ordinary Sobolev spaces in $\mathbb{R}^n$, according to Hörmander's [8], based upon the uniform boundedness of a suitable family of parameter-depending norms.

For given $s \in \mathbb{R}$, $\gamma \geq 1$ and for each $\delta \in]0,1]$ a norm in $H^{s-1}(\mathbb{R}^n)$ is defined by setting

$$\|u\|^2_{s-1,\gamma,\delta} := (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{2s,\gamma}(\xi)\lambda^{-2,\gamma}(\delta\xi)|\widehat{u}(\xi)|^2 d\xi. \tag{3.8}$$

According to Section 2, for $\gamma = 1$ and any $0 < \delta \leq 1$, we set $\|\cdot\|_{s-1,\delta} := \|\cdot\|_{s-1,1,\delta}$; the family of δ-weighted norms $\{\|\cdot\|_{s-1,\delta}\}_{0<\delta\leq 1}$ was deeply studied in [8]; easy arguments (relying essentially on a γ-rescaling of functions) lead to get the same properties for the norms $\{\|\cdot\|_{s-1,\gamma,\delta}\}_{0<\delta\leq 1}$ defined in (3.8) with an arbitrary $\gamma \geq 1$.

Of course, one has $\|\cdot\|_{s-1,\gamma,1} = \|\cdot\|_{s-1,\gamma}$ (cf. (2.4), with $s-1$ instead of s). It is also clear that, for each fixed $\delta \in]0,1[$, the norm $\|\cdot\|_{s-1,\gamma,\delta}$ is equivalent to $\|\cdot\|_{s-1,\gamma}$ in $H^{s-1}_\gamma(\mathbb{R}^n)$, *uniformly with respect to* γ; notice, however, that the constants appearing in the equivalence inequalities will generally depend on δ (see (3.18)).

The next characterization of Sobolev spaces readily follows by taking account of the parameter γ into the arguments used in [8, Theorem 2.4.1].

Proposition 3.1. *For every $s \in \mathbb{R}$ and $\gamma \geq 1$, $u \in H^s_\gamma(\mathbb{R}^n)$ if and only if $u \in H^{s-1}_\gamma(\mathbb{R}^n)$, and the set $\{\|u\|_{s-1,\gamma,\delta}\}_{0<\delta\leq 1}$ is bounded. In this case, one has*

$$\|u\|_{s-1,\gamma,\delta} \uparrow \|u\|_{s,\gamma}, \quad \text{as } \delta \downarrow 0. \tag{3.9}$$

In order to show the regularity result stated in Theorem 1.1, it is useful to provide the conormal Sobolev space $H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$, $m \in \mathbb{N}$, $\gamma \geq 1$, with a family of parameter-depending norms satisfying analogous properties to those of norms defined in (3.8). Nishitani and Takayama [9] introduced such norms in the "unweighted" case $\gamma = 1$, just applying the ordinary Sobolev norms $\|\cdot\|_{m-1,\delta}$ in (3.8) to the pull-back of functions on $\mathbb{R}^n_+$, by the $\sharp$ operator; then these norms were used in [2] to characterize the conormal regularity of functions.

Following [9], for $\gamma \geq 1$, $\delta \in]0,1]$, and all $u \in H^{m-1}_{\tan}(\mathbb{R}^n_+)$, we set

$$\|u\|^2_{\mathbb{R}^n_+, m-1,\tan,\gamma,\delta} := \left\|u^\sharp\right\|^2_{m-1,\gamma,\delta} = (2\pi)^{-n}\int_{\mathbb{R}^n} \lambda^{2m,\gamma}(\xi)\lambda^{-2,\gamma}(\delta\xi)\left|\widehat{u^\sharp}(\xi)\right|^2 d\xi. \tag{3.10}$$

Because $\sharp$ is an isomorphism of $H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ onto $H^{m-1}_\gamma(\mathbb{R}^n)$, the family of norms $\{\|\cdot\|_{\mathbb{R}^n_+,m-1,\tan,\gamma,\delta}\}_{0<\delta\leq 1}$ keeps all the properties enjoyed by the family of norms defined in (3.8).

In particular, the same characterization of ordinary Sobolev spaces on $\mathbb{R}^n$, given by Proposition 3.1, applies also to conormal Sobolev spaces in $\mathbb{R}^n_+$ (cf. [2, 9]).

Proposition 3.2. *For every positive integer m and $\gamma \geq 1$, $u \in H^m_{\tan,\gamma}(\mathbb{R}^n_+)$ if and only if $u \in H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$, and the set $\{\|u\|_{\mathbb{R}^n_+,m-1,\tan,\gamma,\delta}\}_{0<\delta\leq 1}$ is bounded. In this case, one has*

$$\|u\|_{\mathbb{R}^n_+,m-1,\tan,\gamma,\delta} \uparrow \|u\|_{\mathbb{R}^n_+,m,\tan,\gamma}, \quad \text{as } \delta \downarrow 0. \tag{3.11}$$

As regards to the mixed space $\mathscr{H}^{m,r}_{\tan}(\mathbb{R}^n_+)$, it is worthwhile noticing that it can be endowed with the γ-weighted norm defined, by the Fourier transformation, as

$$\|u\|^2_{\mathbb{R}^n_+,m,r,\tan,\gamma} := (2\pi)^{-n}\int_{\mathbb{R}^n} \lambda^{2r,\gamma}(\xi)\lambda(\xi')^{2(m-r),\gamma}\left|\widehat{u^\sharp}(\xi)\right|^2 d\xi; \tag{3.12}$$

here and below $\xi' := (\xi_2,\ldots,\xi_n)$ denotes the Fourier dual variables of the tangential space variables $x' = (x_2,\ldots,x_n)$, and, with a slight abuse of notation, we write $\lambda^{r,\gamma}(\xi')$ to mean in fact $\lambda^{r,\gamma}(0,\xi')$.

Of course, the norm in (3.12) is equivalent, uniformly with respect to γ, to the norm (2.8).

3.2. A Class of Conormal Operators

The $\sharp$ operator, defined at the beginning of Section 3, can be used to allow pseudodifferential operators in $\mathbb{R}^n$ *acting conormally* on functions only defined over the positive half-space $\mathbb{R}^n_+$. Then the standard machinery of pseudodifferential calculus (in the parameter depending version introduced in [10, 11]) can be rearranged into a functional calculus properly behaved on conormal Sobolev spaces described in Section 2. In Section 4, this calculus will be usefully applied to study the conormal regularity of the stationary BVP (1.2)-(1.3).

Let us introduce the pseudodifferential symbols, with a parameter, to be used later; here we closely follow the terminology and notations of [12].

Definition 3.3. A parameter-depending pseudodifferential symbol of order $m \in \mathbb{R}$ is a real-(or complex-) valued measurable function $a(x,\xi,\gamma)$ on $\mathbb{R}^n \times \mathbb{R}^n \times [1,+\infty[$, such that a is C^∞ with respect to x and ξ, and for all multi-indices $\alpha,\beta \in \mathbb{N}^n$ there exists a positive constant $C_{\alpha,\beta}$ satisfying

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi,\gamma)\right| \le C_{\alpha,\beta}\lambda^{m-|\alpha|,\gamma}(\xi), \tag{3.13}$$

for all $x,\xi \in \mathbb{R}^n$ and $\gamma \ge 1$.

The same definition as above extends to functions $a(x,\xi,\gamma)$ taking values in the space $\mathbb{R}^{N\times N}$ (resp., $\mathbb{C}^{N\times N}$) of $N\times N$ real (resp., complex) valued matrices, for all integers $N>1$ (where the module $|\cdot|$ is replaced in (3.13) by any equivalent norm in $\mathbb{R}^{N\times N}$ (resp., $\mathbb{C}^{N\times N}$)). We denote by Γ^m the set of γ-depending symbols of order $m\in\mathbb{R}$ (the same notation being used for both scalar-or matrix-valued symbols). Γ^m is equipped with the obvious norms

$$|a|_{m,k} := \max_{|\alpha|+|\beta|\le k} \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,\, \gamma\ge 1} \lambda^{-m+|\alpha|,\gamma}(\xi)\left|\partial_\xi^\alpha\partial_x^\beta a(x,\xi,\gamma)\right|, \quad \forall k\in\mathbb{N}, \tag{3.14}$$

which turn it into a Fréchet space. For all $m,m' \in \mathbb{R}$, with $m \le m'$, the continuous imbedding $\Gamma^m \subset \Gamma^{m'}$ can be easily proven.

For all $m\in\mathbb{R}$, the function $\lambda^{m,\gamma}$ is of course a (scalar-valued) symbol in Γ^m.

To perform the analysis of Section 4, it is important to consider the behavior of the weight function $\lambda^{m,\gamma}(\cdot)\lambda^{-1,\gamma}(\delta\cdot)$, involved in the definition of the parameter-depending norms in (3.8), (3.10), as a γ-depending symbol according to Definition 3.3.

In order to simplify the forthcoming statements, henceforth the following short notations will be used:

$$\lambda_\delta^{m-1,\gamma}(\xi) := \lambda^{m,\gamma}(\xi)\lambda^{-1,\gamma}(\delta\xi), \qquad \widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi) := \left(\lambda_\delta^{m-1,\gamma}(\xi)\right)^{-1}\left(=\lambda^{-m,\gamma}(\xi)\lambda^{1,\gamma}(\delta\xi)\right), \tag{3.15}$$

for all real numbers $m\in\mathbb{R}$, $\gamma\ge 1$, and $\delta\in]0,1]$. One has the obvious identities $\lambda_1^{m-1,\gamma}(\xi) \equiv \lambda^{m-1,\gamma}(\xi)$, $\widetilde{\lambda}_1^{-m+1,\gamma}(\xi) \equiv \lambda_1^{-m+1,\gamma}(\xi) \equiv \lambda^{-m+1,\gamma}(\xi)$. However, to avoid confusion in the following, it is worthwhile to remark that functions $\lambda_\delta^{-m+1,\gamma}(\xi)$ and $\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi)$ are no longer the same as soon as δ becomes strictly smaller than 1; indeed (3.15) gives $\lambda_\delta^{-m+1,\gamma}(\xi) = \lambda^{-m+2,\gamma}(\xi)\lambda^{-1,\gamma}(\delta\xi)$.

A straightforward application of Leibniz's rule leads to the following result.

Lemma 3.4. *For every $m \in \mathbb{R}$ and all $\alpha \in \mathbb{N}^n$, there exists a positive constant $C_{m,\alpha}$ such that*

$$\left|\partial_\xi^\alpha \lambda_\delta^{m-1,\gamma}(\xi)\right| \leq C_{m,\alpha}\lambda_\delta^{m-1-|\alpha|,\gamma}(\xi), \quad \forall \xi \in \mathbb{R}^n,\ \forall \gamma \geq 1,\ \forall \delta \in\ ]0,1]. \tag{3.16}$$

Because of estimates (3.16), $\lambda_\delta^{m-1,\gamma}(\xi)$ can be regarded as a γ-depending symbol, in two different ways. On one hand, combining estimates (3.16) with the trivial inequality

$$\lambda^{-1,\gamma}(\delta\xi) \leq 1 \tag{3.17}$$

immediately gives that $\{\lambda_\delta^{m-1,\gamma}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^m.

On the other hand, the left inequality in

$$\delta\lambda^{1,\gamma}(\xi) \leq \lambda^{1,\gamma}(\delta\xi) \leq \lambda^{1,\gamma}(\xi), \quad \forall \xi \in \mathbb{R}^n,\ \forall \delta \in\ ]0,1], \tag{3.18}$$

together with (3.16), also gives

$$\left|\partial_\xi^\alpha \lambda_\delta^{m-1,\gamma}(\xi)\right| \leq C_{m,\alpha}\delta^{-1}\lambda^{m-1-|\alpha|,\gamma}(\xi), \quad \forall \xi \in \mathbb{R}^n,\ \forall \gamma \geq 1. \tag{3.19}$$

According to Definition 3.3, (3.19) means that $\lambda_\delta^{m-1,\gamma}$ actually belongs, for each fixed δ, to Γ^{m-1}; nevertheless, the family $\{\lambda_\delta^{m-1,\gamma}\}_{0<\delta\leq 1}$ is unbounded as a subset of Γ^{m-1}.

For later use, we also need to study the behavior of functions $\widetilde{\lambda}_\delta^{-m+1,\gamma}$ as γ-depending symbols.

Analogously to Lemma 3.4, one can prove the following result.

Lemma 3.5. *For all $m \in \mathbb{R}$ and $\alpha \in \mathbb{N}^n$, there exists $\widetilde{C}_{m,\alpha} > 0$ such that*

$$\left|\partial_\xi^\alpha \widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi)\right| \leq \widetilde{C}_{m,\alpha}\widetilde{\lambda}_\delta^{-m+1-|\alpha|,\gamma}(\xi), \quad \forall \xi \in \mathbb{R}^n,\ \forall \gamma \geq 1,\ \forall \delta \in\ ]0,1]. \tag{3.20}$$

In particular, Lemma 3.5 implies that the family $\{\widetilde{\lambda}_\delta^{-m+1,\gamma}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^{-m+1} (it suffices to combine (3.20) with the right inequality in (3.18)).

Any symbol $a = a(x,\xi,\gamma) \in \Gamma^m$ defines a *pseudodifferential operator* $\mathrm{Op}^\gamma(a) = a(x,D,\gamma)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, by the standard formula

$$\forall u \in \mathcal{S}(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n, \quad \mathrm{Op}^\gamma(a)u(x) = a(x,D,\gamma)u(x) =: (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a(x,\xi,\gamma)\widehat{u}(\xi)d\xi, \tag{3.21}$$

where, of course, we denote $x\cdot\xi := \sum_{j=1}^n x_j\xi_j$. a is called the symbol of the operator (3.21), and m is its order. It comes from the classical theory that $\mathrm{Op}^\gamma(a)$ defines a linear-bounded operator

$$\mathrm{Op}^\gamma(a) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n); \tag{3.22}$$

moreover, the latter extends to a linear-bounded operator on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions in $\mathbb{R}^n$.

An exhaustive account of the symbolic calculus for pseudodifferential operators with symbols in Γ^m can be found in [11] (see also [12]). Here, we just recall the following result, concerning the product and the commutator of two pseudodifferential operators.

Proposition 3.6. *Let $a \in \Gamma^m$ and $b \in \Gamma^l$, for $l, m \in \mathbb{R}$. Then $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)$ is a pseudodifferential operator with symbol in Γ^{m+l}; moreover, if one lets $a\#b$ denote the symbol of the product, one has for every integer $N \geq 1$*

$$a\#b - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b \in \Gamma^{m+l-N}. \tag{3.23}$$

Under the same assumptions, the commutator $[\mathrm{Op}^\gamma(a), \mathrm{Op}^\gamma(b)] := \mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b) - \mathrm{Op}^\gamma(b)\mathrm{Op}^\gamma(a)$ is again a pseudodifferential operator with symbol $c \in \Gamma^{m+l}$. If one further assumes that one of the two symbols a or b is scalar-valued (so that a and b commute in the pointwise product), then the symbol c of $[\mathrm{Op}^\gamma(a), \mathrm{Op}^\gamma(b)]$ has order $m + l - 1$.

We point out that when the symbol $b \in \Gamma^l$ of the preceding statement does not depend on the x variables (i.e., $b = b(\xi, \gamma)$), then the symbol $a\#b$ of the product $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)$ reduces to the pointwise product of symbols a and b; in this case, the *asymptotic* formula (3.23) is replaced by the *exact* formula

$$(a\#b)(x, \xi, \gamma) = a(x, \xi, \gamma) b(\xi, \gamma). \tag{3.24}$$

According to (3.15), (3.21), we write

$$\lambda_\delta^{m-1,\gamma}(D) := \mathrm{Op}^\gamma\left(\lambda_\delta^{m-1,\gamma}\right), \qquad \widetilde{\lambda}_\delta^{-m+1,\gamma}(D) := \mathrm{Op}^\gamma\left(\widetilde{\lambda}_\delta^{-m+1,\gamma}\right). \tag{3.25}$$

In view of (3.15) and (3.24), the operator $\lambda_\delta^{m-1,\gamma}(D)$ is invertible, and its two-sided inverse is given by $\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$.

Starting from the symbolic classes Γ^m, $m \in \mathbb{R}$, we introduce now the class of *conormal operators* in $\mathbb{R}^n_+$, to be used in the sequel.

Let $a(x, \xi, \gamma)$ be a γ-depending symbol in Γ^m, $m \in \mathbb{R}$. The *conormal operator with symbol* a, denoted by $\mathrm{Op}^\gamma_\sharp(a)$ (or equivalently $a(x, Z, \gamma)$), is defined by setting

$$\forall u \in C^\infty_{(0)}(\mathbb{R}^n_+), \quad \left(\mathrm{Op}^\gamma_\sharp(a) u\right)^\sharp = \left(\mathrm{Op}^\gamma(a)\right)\left(u^\sharp\right). \tag{3.26}$$

In other words, the operator $\mathrm{Op}^\gamma_\sharp(a)$ is the composition of mappings

$$\mathrm{Op}^\gamma_\sharp(a) = \sharp^{-1} \circ \mathrm{Op}^\gamma(a) \circ \sharp. \tag{3.27}$$

As we already noted, $u^\sharp \in \mathcal{S}(\mathbb{R}^n)$ whenever $u \in C^\infty_{(0)}(\mathbb{R}^n_+)$; hence formula (3.26) makes sense and gives that $\mathrm{Op}^\gamma_\sharp(a)u$ is a C^∞-function in $\mathbb{R}^n_+$. Also $\mathrm{Op}^\gamma_\sharp(a) : C^\infty_{(0)}(\mathbb{R}^n_+) \to C^\infty(\mathbb{R}^n_+)$ is a linear-bounded operator that extends to a linear-bounded operator from the space of distributions $u \in \mathfrak{D}'(\mathbb{R}^n_+)$ satisfying $u^\sharp \in \mathcal{S}'(\mathbb{R}^n)$ into $\mathfrak{D}'(\mathbb{R}^n_+)$ itself. (In principle, $\mathrm{Op}^\gamma_\sharp(a)$ could be defined by (3.26) over all functions $u \in C^\infty(\mathbb{R}^n_+)$, such that $u^\sharp \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathrm{Op}^\gamma_\sharp(a)$ defines a linear-bounded operator on the latter function space, provided that it is equipped with the topology induced, via $\sharp$, from the Fréchet topology of $\mathcal{S}(\mathbb{R}^n)$.) Throughout the paper, we continue to denote this extension by $\mathrm{Op}^\gamma_\sharp(a)$ (or $a(x, Z, \gamma)$ equivalently).

As an immediate consequence of (3.27), we have that for all symbols $a \in \Gamma^m$, $b \in \Gamma^l$, with $m, l \in \mathbb{R}$, there holds

$$\forall u \in C^\infty_{(0)}(\mathbb{R}^n_+), \quad \mathrm{Op}^\gamma_\sharp(a)\mathrm{Op}^\gamma_\sharp(b)u = \left(\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)\left(u^\sharp\right)\right)^{\sharp^{-1}}. \tag{3.28}$$

Then, it is clear that a functional calculus of conormal operators can be straightforwardly borrowed from the corresponding pseudodifferential calculus in $\mathbb{R}^n$; in particular we find that products and commutators of conormal operators are still operators of the same type, and their symbols are computed according to the rules collected in Proposition 3.6.

Below, let us consider the main examples of conormal operators that will be met in Section 4.

As a first example, we quote the multiplication by a matrix-valued function $B \in C^\infty_{(0)}(\mathbb{R}^n_+)$. It is clear that this makes an operator of order zero according to (3.26); indeed (3.4) gives for any vector-valued $u \in C^\infty_{(0)}(\mathbb{R}^n_+)$

$$(Bu)^\sharp(x) = B^\natural(x)u^\sharp(x), \tag{3.29}$$

and $B^\natural$ is a C^∞-function in $\mathbb{R}^n$, with bounded derivatives of any order, hence a symbol in Γ^0.

We remark that, when computed for $B^\natural$, the norm of order $k \in \mathbb{N}$, defined on symbols by (3.14), just reduces to

$$\left|B^\natural\right|_{0,k} = \max_{|\alpha|\le k}\left\|\partial^\alpha B^\natural\right\|_{L^\infty(\mathbb{R}^n)} = \max_{|\alpha|\le k}\|Z^\alpha B\|_{L^\infty(\mathbb{R}^n_+)}, \tag{3.30}$$

where the second identity above exploits formulas (3.5) and that $\natural$ maps isometrically $L^\infty(\mathbb{R}^n_+)$ onto $L^\infty(\mathbb{R}^n)$.

Now, let $\mathcal{L} := \gamma I_N + \sum_{j=1}^n A_j(x)Z_j$ be a first-order linear partial differential operator, with matrix-valued coefficients $A_j \in C^\infty_{(0)}(\mathbb{R}^n_+)$ for $j = 1, \ldots, n$ and $\gamma \ge 1$. Since the leading part of $\mathcal{L}$ only involves conormal derivatives, applying (3.4), (3.6), and (3.7) then gives

$$\left(\gamma u + \sum_{j=1}^n A_j Z_j u\right)^\sharp = \left(\gamma I - \frac{1}{2}A_1^\natural\right)u^\sharp + \sum_{j=1}^n A_j^\natural \partial_j u^\sharp = \mathrm{Op}^\gamma(a)u^\sharp, \tag{3.31}$$

where $a = a(x, \xi, \gamma) := (\gamma I_N - (1/2)A_1^\natural(x)) + i\sum_{j=1}^n A_j^\natural(x)\xi_j$ is a symbol in Γ^1. Then $\mathcal{L}$ is a conormal operator of order 1, according to (3.26).

In Section 4, we will be mainly interested in the family of conormal operators

$$\lambda_\delta^{m-1,\gamma}(Z) := \mathrm{Op}_\sharp^\gamma\left(\lambda_\delta^{m-1,\gamma}\right), \qquad \widetilde{\lambda}_\delta^{-m+1,\gamma}(Z) := \mathrm{Op}_\sharp^\gamma\left(\widetilde{\lambda}_\delta^{-m+1,\gamma}\right). \tag{3.32}$$

The operators $\lambda_\delta^{m-1,\gamma}(Z)$ are involved in the characterization of conormal regularity provided by Proposition 3.2 (remember that, after Lemma 3.4, $\lambda_\delta^{m-1,\gamma} \in \Gamma^{m-1}$). Indeed, from Plancherel's formula and the fact that the operator $\sharp$ preserves the L^2-norm, the following identities

$$\|u\|_{\mathbb{R}^n_+,m-1,\tan,\gamma,\delta} \equiv \left\|\lambda_\delta^{m-1,\gamma}(Z)u\right\|_{L^2(\mathbb{R}^n_+)} \tag{3.33}$$

can be straightforwardly established; hence, Proposition 3.2 can be restated in terms of the boundedness, with respect to δ, of the L^2-norms of functions $\lambda_\delta^{m-1,\gamma}(Z)u$. This observation is the key point that leads to the analysis performed in Section 4.

Another main feature of the conormal operators (3.32) is that $\widetilde{\lambda}_\delta^{-m+1,\gamma}(Z)$ provides a two-sided inverse of $\lambda_\delta^{m-1,\gamma}(Z)$; this comes at once from the analogous property of the operators in (3.25) and formulas (3.26), (3.28).

3.3. Sobolev Continuity of Conormal Operators

Proposition 3.7. *If $s, m \in \mathbb{R}$ then for all $a \in \Gamma^m$ the pseudodifferential operator $\mathrm{Op}^\gamma(a)$ extends as a linear-bounded operator from $H_\gamma^{s+m}(\mathbb{R}^n)$ into $H_\gamma^s(\mathbb{R}^n)$, and the operator norm of such an extension is uniformly bounded with respect to γ.*

We refer the reader to [11] for a detailed proof of Proposition 3.7. A thorough analysis shows that the norm of $\mathrm{Op}^\gamma(a)$, as a linear-bounded operator from $H_\gamma^{s+m}(\mathbb{R}^n)$ to $H_\gamma^s(\mathbb{R}^n)$, actually depends only on a norm of type (3.14) of the symbol a, besides the Sobolev order s and the symbolic order m (cf. [11] for detailed calculations). This observation entails, in particular, that the operator norm is uniformly bounded with respect to γ and other additional parameters from which the symbol of the operator might possibly depend, as a bounded mapping.

From the Sobolev continuity of pseudodifferential operators quoted above, and using that the operator $\sharp$ maps isomorphically conormal Sobolev spaces in $\mathbb{R}^n_+$ onto ordinary Sobolev spaces in $\mathbb{R}^n$, we easily derive the following result.

Proposition 3.8. *If $m \in \mathbb{Z}$ and $a \in \Gamma^m$, then the conormal operator $\mathrm{Op}_\sharp^\gamma(a)$ extends to a linear-bounded operator from $H_{\tan,\gamma}^{s+m}(\mathbb{R}^n_+)$ to $H_{\tan,\gamma}^s(\mathbb{R}^n_+)$, for every integer $s \geq 0$, such that $s + m \geq 0$; moreover the operator norm of such an extension is uniformly bounded with respect to γ.*

Remark 3.9. We point out that the statement above only deals with integer orders of symbols and conormal Sobolev spaces. The reason is that, in Section 2, conormal Sobolev spaces were only defined for positive integer orders. In principle, this lack could be removed by extending the definition of conormal spaces $H_{\tan}^s(\mathbb{R}^n_+)$ to any real order s: this could be trivially done, just defining $H_{\tan}^s(\mathbb{R}^n_+)$ to be the pull-back, by the operator $\sharp$, of functions in $H^s(\mathbb{R}^n)$. However, this extension to fractional exponents seems to be useless for the subsequent developments.

As regards to the action of conormal operators on the mixed spaces $\mathcal{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+)$, similar arguments to those used in the proof of Proposition 3.8 lead to the following.

Proposition 3.10. *Let $a = a(x,\xi,\gamma)$ be a symbol in Γ^m, for $m \in \mathbb{Z}$. Then for all integers $r, s \in \mathbb{N}$, such that $s \geq r$, $s > 0$, and $r + m \geq 0$, $\mathrm{Op}^{\gamma}_{\sharp}(a) = a(x, Z, \gamma)$ extends by continuity to a linear-bounded operator*

$$\mathrm{Op}^{\gamma}_{\sharp}(a) : \mathcal{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+) \longrightarrow \mathcal{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+). \tag{3.34}$$

Moreover, the operator norm of such an extension is uniformly bounded with respect to γ.

4. Proof of Theorem 1.1

This section is entirely devoted to the proof of Theorem 1.1.

4.1. The Strategy of the Proof: Comparison with the Strongly Well-Posed Case

As it was already pointed out in the Introduction, in order to solve the BVP (1.2)-(1.3) in L^2, Theorem 1.1 requires an additional tangential/conormal regularity of the corresponding data. The precise increase of regularity needed for the data is prescribed by the energy inequality (1.7): to estimate the L^2-norm of the solution, in the interior and on the boundary of the domain, we lose r conormal derivatives and $s - r$ tangential derivatives with respect to the interior source term F, and s (tangential) derivatives with respect to the boundary datum G.

In [2], the conormal regularity of weak solutions to the BVP (1.2)-(1.3) was studied, under the assumption that no loss of derivatives occurs in the estimate of the solution by the data. To prove the result of [2, Theorem 15], the solution u to (1.2)-(1.3) was regularized by a family of tangential mollifiers J_ε, $0 < \varepsilon < 1$, defined by Nishitani and Takayama in [9] as a suitable combination of the operator $\sharp$ and the standard Friedrichs'mollifiers. The essential point of the analysis performed in [2] was to notice that the mollified solution $J_\varepsilon u$ solves the same problem (1.2)-(1.3), as the original solution u. The data of the problem for $J_\varepsilon u$ come from the regularization, by J_ε, of the data involved in the original problem for u; furthermore, an additional term $[J_\varepsilon, L]u$, where $[J_\varepsilon, L]$ is the commutator between the differential operator L and the tangential mollifier J_ε, appears into the equation satisfied by $J_\varepsilon u$. Because the energy estimate associated to a strong L^2-well-posed problem does not lose derivatives, actually this term can be incorporated into the source term of the equation satisfied by $J_\varepsilon u$.

In the case of Theorem 1.1, where the a priori estimate (1.7) exhibits a finite loss of regularity with respect to the data, this technique seems to be unsuccessful, since $[J_\varepsilon, L]u$ cannot be absorbed into the right-hand side without losing derivatives on the solution u; on the other hand, it seems that the same term cannot be merely reduced to a lower-order term involving the smoothed solution $J_\varepsilon u$, as well (this should require a sharp control of the error term $u - J_\varepsilon u$).

These observations lead to develop another technique, where the tangential mollifier J_ε is replaced by the family of operators (3.32), involved in the characterization of regularity given by Proposition 3.2. Instead of studying the problem satisfied by the smoothed solution $J_\varepsilon u$, here we consider the problem satisfied by $\lambda_\delta^{m-1,\gamma}(Z)u$. As before, a new term $[\lambda_\delta^{m-1,\gamma}(Z), L]u$ appears which takes account of the commutator between the differential operator L and the conormal operator $\lambda_\delta^{m-1,\gamma}(Z)$. Since we assume the weak well-posedness of the BVP (1.2)-(1.3) to be preserved under lower order terms, the approach consists of treating the commutator $[\lambda_\delta^{m-1,\gamma}(Z), L]u$ as a lower-order term within the interior equation for $\lambda_\delta^{m-1,\gamma}(Z)u$ (see (4.10)) (differently from the strongly L^2-well-posed case studied in [2], the stability of problem (1.2)-(1.3) under lower-order perturbations is no longer a trivial consequence of the well-posedness itself. In Theorem 1.1, this stability is required as an additional hypothesis about the BVP); this is made possible by taking advantage from the invertibility of the operator $\lambda_\delta^{m-1,\gamma}(Z)$.

We argue by induction on the integer order $m \geq 1$. Let us take arbitrary data $F \in \mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)$, $G \in H^{s+m}_\gamma(\mathbb{R}^{n-1})$, and fix an arbitrary matrix-valued function $B \in C^\infty_{(0)}(\mathbb{R}^n_+)$ (as the lower order term in the interior equation (1.2)).

Because of the inductive hypothesis, we already know that the unique L^2-solution u to (1.2)-(1.3) actually belongs to $H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ and $u^I_{|x_1=0}$ belongs to $H^{m-1}_\gamma(\mathbb{R}^{n-1})$, provided that γ is taken to be larger than a certain constant $\gamma_{m-1} \geq 1$; moreover the solution u obeys the estimate (1.8) of order $m-1$

$$\gamma\|u\|^2_{H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)} + \left\|u^I_{|x_1=0}\right\|^2_{H^{m-1}_\gamma(\mathbb{R}^{n-1})} \leq C_{m-1}\left(\frac{1}{\gamma^{2s+1}}\|F\|^2_{\mathscr{H}^{s+m-1,r+m-1}_{\tan,\gamma}(\mathbb{R}^n_+)} + \frac{1}{\gamma^{2s}}\|G\|^2_{H^{s+m-1}_\gamma(\mathbb{R}^{n-1})}\right), \tag{4.1}$$

where the positive constant C_{m-1}, as well as γ_{m-1}, only depends on the smoothness order m and the L^∞-norm of a finite number (depending on m itself) of conormal derivatives of B (cf. (3.30)), besides the coefficients A_j $(1 \leq j \leq n)$ of L and the integer numbers r and s.

In order to increase the conormal regularity of the solution u by order one, we are going to act on u by the conormal operator $\lambda_\delta^{m-1,\gamma}(Z)$; then we consider the analogue of the original problem (1.2)-(1.3) satisfied by $\lambda_\delta^{m-1,\gamma}(Z)u$.

4.2. A Modified Version of the Conormal Operator $\lambda_\delta^{m-1,\gamma}(Z)$

Due to some technical reasons that will be clarified in Section 4.3, we need to slightly modify the conormal operator $\lambda_\delta^{m-1,\gamma}(Z)$ to be applied to the solution u of the original BVP (1.2)-(1.3), as was described in the preceding section.

The first step is to decompose the weight function $\lambda_\delta^{m-1,\gamma}$ as the sum of two contributions. To do so, we proceed as follows. Firstly, we take an arbitrary positive, even function $\chi \in C^\infty(\mathbb{R}^n)$ with the following properties:

$$0 \leq \chi(x) \leq 1, \quad \forall x \in \mathbb{R}^n, \qquad \chi(x) \equiv 1, \quad \text{for } |x| \leq \frac{1}{2}, \qquad \chi(x) \equiv 0, \quad \text{for } |x| > 1. \tag{4.2}$$

Then, we set

$$\lambda_{\chi,\delta}^{m-1,\gamma}(\xi) := \chi(D)\left(\lambda_{\delta}^{m-1,\gamma}\right)(\xi) = \left(\mathcal{F}^{-1}\chi * \lambda_{\delta}^{m-1,\gamma}\right)(\xi),$$
$$r_{m,\delta}(\xi,\gamma) := \lambda_{\delta}^{m-1,\gamma}(\xi) - \lambda_{\chi,\delta}^{m-1,\gamma}(\xi) = (I-\chi(D))\left(\lambda_{\delta}^{m-1,\gamma}\right)(\xi). \tag{4.3}$$

The following result (the proof of which is postponed to Appendix A) shows that the function $\lambda_{\chi,\delta}^{m-1,\gamma}$ essentially behaves like $\lambda_{\delta}^{m-1,\gamma}$.

Lemma 4.1. *Let the function $\chi \in C^\infty(\mathbb{R}^n)$ satisfy the assumptions in* (4.2). *Then $\lambda_{\chi,\delta}^{m-1,\gamma}$ is a symbol in Γ^{m-1}; moreover for every multi-index $\alpha \in \mathbb{N}^n$, there exists a positive constant $C_{m,\alpha}$, independent of γ and δ, such that*

$$\left|\partial_\xi^\alpha \lambda_{\chi,\delta}^{m-1,\gamma}(\xi)\right| \le C_{m,\alpha}\lambda_{\delta}^{m-1-|\alpha|,\gamma}(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{4.4}$$

An immediate consequence of Lemma 4.1 and (4.3) is that $r_{m,\delta}$ is also a γ-depending symbol in Γ^{m-1}.

Let us define with the obvious meaning of the notations

$$\lambda_{\chi,\delta}^{m-1,\gamma}(D) := \mathrm{Op}^\gamma\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right), \qquad r_{m,\delta}(D,\gamma) := \mathrm{Op}^\gamma(r_{m,\delta}),$$
$$\lambda_{\chi,\delta}^{m-1,\gamma}(Z) := \mathrm{Op}_\sharp^\gamma\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right), \qquad r_{m,\delta}(Z,\gamma) := \mathrm{Op}_\sharp^\gamma(r_{m,\delta}). \tag{4.5}$$

The second important result is concerned with the conormal operator $r_{m,\delta}(Z,\gamma) = \mathrm{Op}_\sharp^\gamma(r_{m,\delta})$ and tells that it essentially behaves as a *regularizing* operator on conormal Sobolev spaces.

Lemma 4.2. *For every $k \in \mathbb{N}$, the conormal operator $r_{m,\delta}(Z,\gamma)$ extends as a linear-bounded operator, still denoted by $r_{m,\delta}(Z,\gamma)$, from $L^2(\mathbb{R}^n_+)$ to $H^k_{\tan,\gamma}(\mathbb{R}^n_+)$. Moreover there exists a positive constant $C_{m,k}$, depending only on k and m, such that for all $\gamma \ge 1$ and $\delta \in]0,1]$*

$$\|r_{m,\delta}(Z,\gamma)u\|_{H^k_{\tan,\gamma}(\mathbb{R}^n_+)} \le C_{m,k}\gamma^k\|u\|_{L^2(\mathbb{R}^n_+)}, \quad \forall u \in L^2(\mathbb{R}^n_+). \tag{4.6}$$

Remark 4.3. In the framework of the general theory of pseudodifferential operators, the procedure adopted to define the symbol $\lambda_{\chi,\delta}^{m-1,\gamma}$ is standard and is used to modify an arbitrary symbol in such a way to make *properly supported* the corresponding pseudodifferential operator (see [13] for the definition of a properly supported operator and an extensive description of the method). As a general issue, one can prove that the resulting properly supported operator differs from the original one by a regularizing remainder. Essentially, an easy adaptation of the same arguments to the framework of conormal spaces in $\mathbb{R}^n_+$ can be employed to prove the regularizing action of the conormal operator $r_{m,\delta}(Z,\gamma)$ stated by Lemma 4.2.

According to (4.3), we decompose

$$\lambda_{\delta}^{m-1,\gamma}(Z) = \lambda_{\chi,\delta}^{m-1,\gamma}(Z) + r_{m,\delta}(Z,\gamma). \tag{4.7}$$

As a consequence of Lemmas 4.1 and 4.2, the role of the family of conormal operators $\{\lambda_{\delta}^{m-1,\gamma}(Z)\}_{0<\delta\leq 1}$ in the characterization of the conormal regularity provided by Proposition 3.2 (cf. also (3.33)) can also be played by the family of "modified" operators $\{\lambda_{\chi,\delta}^{m-1,\gamma}(Z)\}_{0<\delta\leq 1}$, namely, we have the following.

Corollary 4.4. *For every positive integer m and $\gamma \geq 1$, $u \in H^m_{\tan,\gamma}(\mathbb{R}^n_+)$ if and only if $u \in H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ and the set $\{\|\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u\|_{L^2(\mathbb{R}^n_+)}\}_{0<\delta\leq 1}$ is bounded.*

In order to suitably handle the commutator between the differential operator L and the conormal operator $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)$, that comes from writing down the problem satisfied by $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u$ (see Sections 4.3.1 and 4.3.2), it is useful to analyze the behavior of the pseudodifferential operators $\lambda_{\chi,\delta}^{m-1,\gamma}(D)$, when interacting with another pseudodifferential operator by composition and commutation. The following lemma analyzes these situations; for later use, it is convenient to replace in our reasoning the function $\lambda_{\chi,\delta}^{m-1,\gamma}$ by a general γ-depending symbol a_δ preserving the same kind of decay properties as in (4.4).

Lemma 4.5. *Let $\{a_\delta\}_{0<\delta\leq 1}$ be a family of symbols $a_\delta = a_\delta(x,\xi,\gamma) \in \Gamma^{r-1}$, $r \in \mathbb{R}$, such that for all multi-indices $\alpha,\beta \in \mathbb{N}^n$ there exists a positive constant $C_{r,\alpha,\beta}$, independent of γ and δ, for which*

$$\left|\partial_\xi^\alpha \partial_x^\beta a_\delta(x,\xi,\gamma)\right| \leq C_{r,\alpha,\beta}\lambda_\delta^{r-1-|\alpha|,\gamma}(\xi), \quad \forall x,\xi \in \mathbb{R}^n. \tag{4.8}$$

Let $b = b(x,\xi,\gamma)$ be another symbol in Γ^l, for $l \in \mathbb{R}$.

Then, for every $\delta \in]0,1]$, the product $\mathrm{Op}^\gamma(a_\delta)\mathrm{Op}^\gamma(b)$ is a pseudodifferential operator with symbol $a_\delta\#b$ in Γ^{l+r-1}. Moreover, for all multi-indices $\alpha,\beta \in \mathbb{N}^n$ there exists a constant $C_{r,l,\alpha,\beta}$, independent of γ and δ, such that

$$\left|\partial_\xi^\alpha \partial_x^\beta (a_\delta\#b)(x,\xi,\gamma)\right| \leq C_{r,l,\alpha,\beta}\lambda_\delta^{l+r-1-|\alpha|,\gamma}(\xi), \quad \forall x,\xi \in \mathbb{R}^n. \tag{4.9}$$

Under the same hypotheses, $\mathrm{Op}^\gamma(a_\delta)\mathrm{Op}^\gamma(b)\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$ is a pseudodifferential operator with symbol $(a_\delta\#b)\widetilde{\lambda}_\delta^{-m+1,\gamma}$ in Γ^{l+r-m}; moreover, $\{(a_\delta\#b)\widetilde{\lambda}_\delta^{-m+1,\gamma}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^{l+r-m}. Eventually, if the symbol a_δ is scalar-valued, $[\mathrm{Op}^\gamma(a_\delta),\mathrm{Op}^\gamma(b)]\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$ is a pseudodifferential operator with symbol $c_\delta \in \Gamma^{l+r-m-1}$, and $\{c_\delta\}_{0<\delta\leq 1}$ is a bounded subset of $\Gamma^{l+r-m-1}$.

The proof of Lemma 4.5 is postponed to Appendix A.

Remark 4.6. That $\mathrm{Op}^\gamma(a_\delta)\mathrm{Op}^\gamma(b)$, $\mathrm{Op}^\gamma(a_\delta)\mathrm{Op}^\gamma(b)\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$ and $[\mathrm{Op}^\gamma(a_\delta),\mathrm{Op}^\gamma(b)]\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$ have symbols belonging, respectively, to Γ^{l+r-1}, Γ^{l+r-m} and $\Gamma^{l+r-m-1}$ (for scalar-valued a_δ) follows at once from the standard rules of symbolic calculus summarized in Proposition 3.6. The nontrivial part of the statement above (although deduced from the asymptotic formula

(3.23) with a minor effort) is the one asserting that the symbol of $\mathrm{Op}^{\gamma}(a_{\delta})\mathrm{Op}^{\gamma}(b)$ enjoys estimates (4.9); indeed, these estimates give the precise dependence on δ of the decay at infinity of this symbol. Then the remaining assertions in Lemma 4.5 easily follow from (4.9) itself.

Remark 4.7. In view of Proposition 3.8, the results on symbols collected in Lemma 4.5 can be used to study the conormal Sobolev continuity of the related conormal operators.

To be definite, for every nonnegative integer number s, such that $s + l + r - m$ is also nonnegative, Proposition 3.8 and Lemma 4.5 imply that the conormal operator $\mathrm{Op}^{\gamma}_{\sharp}(a_{\delta})\mathrm{Op}^{\gamma}_{\sharp}(b)\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z)$ extends as a linear-bounded mapping from $H^{s+l+r-m}_{\tan,\gamma}(\mathbb{R}^n_+)$ into $H^{s}_{\tan,\gamma}(\mathbb{R}^n_+)$; moreover, its operator norm is uniformly bounded with respect to γ and δ.

If in addition $s+l+r-m \geq 1$ and a_{δ} are scalar-valued, then $[\mathrm{Op}^{\gamma}_{\sharp}(a_{\delta}), \mathrm{Op}^{\gamma}_{\sharp}(b)]\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z)$ extends as a linear-bounded operator from $H^{s+l+r-m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ into $H^{s}_{\tan,\gamma}(\mathbb{R}^n_+)$, and again its operator norm is uniformly bounded with respect to γ and δ.

These mapping properties will be usefully applied in Sections 4.3 and 4.5.

4.3. The Interior Equation

We follow the strategy already explained in Section 4.1, where now the role of the operator $\lambda^{m-1,\gamma}_{\delta}(Z)$ is replaced by $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)$. Since $\lambda^{m-1,\gamma}_{\chi,\delta} \in \Gamma^{m-1}$ (because of Lemma 4.1) and, for $\gamma \geq \gamma_{m-1}$, $u \in H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ (from the inductive hypothesis), after Proposition 3.8 we know that $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u \in L^2(\mathbb{R}^n_+)$.

Applying $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)$ to (1.2) (where $\mathcal{B}$ is just the multiplication by B), we find that $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u$ must solve

$$(\gamma + L + B)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right) + \left[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), L + B\right]u = \lambda^{m-1,\gamma}_{\chi,\delta}(Z)F, \quad \text{in } \mathbb{R}^n_+. \tag{4.10}$$

We are going now to show that the commutator term $[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), L+B]u$ in the above equation can be actually considered as a lower-order term with respect to $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u$.

To this end, we may decompose this term as the sum of two contributions corresponding, respectively, to the *tangential* and *normal* components of L.

First, in view of (1.5), (1.6), we may write the coefficient A_1 of the normal derivative ∂_1 in the expression (1.4) of L as

$$A_1 = A_1^1 + A_1^2, \qquad A_1^1 := \begin{pmatrix} A_1^{I,I} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^2_{1|x_1=0} = 0, \tag{4.11}$$

hence

$$A_1^2\partial_1 = H_1 Z_1, \tag{4.12}$$

where $H_1(x) = x_1^{-1} A_1^2(x) \in C^\infty_{(0)}(\mathbb{R}^n_+)$. Accordingly, we split L as

$$L = A_1^1 \partial_1 + L_{\tan}, \qquad L_{\tan} := H_1 Z_1 + \sum_{j=2}^{n} A_j Z_j. \tag{4.13}$$

Consequently, we have

$$\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L + B\right] u = \left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^1 \partial_1\right] u + \left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B\right] u. \tag{4.14}$$

Note that $L_{\tan} + B$ is just a conormal operator of order 1, according to the terminology introduced in Section 3.2.

4.3.1. The Tangential Commutator

Firstly, we study the *tangential commutator* $[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B]u$. Using the identity $\widetilde{\lambda}_\delta^{-m+1,\gamma}(Z)\lambda_\delta^{m-1,\gamma}(Z) = I$ and (4.7), the latter can be written in terms of $\lambda_{\chi,\delta}^{\gamma,m-1}(Z)u$, modulo some smoothing reminder. Indeed we compute

$$\begin{aligned}
\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B\right] u &= \left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B\right] \widetilde{\lambda}_\delta^{-m+1,\gamma}(Z) \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z) u + r_{m,\delta}(Z,\gamma) u\right) \\
&= \left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B\right] \widetilde{\lambda}_\delta^{-m+1,\gamma}(Z) \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z) u\right) + s_{m,\delta}(x, Z, \gamma) u,
\end{aligned} \tag{4.15}$$

where we have set for short

$$s_{m,\delta}(x, Z, \gamma) := \left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B\right] \widetilde{\lambda}_\delta^{-m+1,\gamma}(Z) r_{m,\delta}(Z, \gamma). \tag{4.16}$$

4.3.2. The Normal Commutator

First of all, we notice that, due to the structure of the matrix A_1^1, the commutator $[\lambda_{\chi,\delta}^{\gamma,m-1}(Z), A_1^1 \partial_1]$ acts nontrivially only on the noncharacteristic component of the vector function u; namely, we have:

$$\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^1 \partial_1\right] u = \begin{pmatrix} \left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I} \partial_1\right] u^I \\ 0 \end{pmatrix}. \tag{4.17}$$

Therefore, we focus on the study of the first nontrivial component of the commutator term. Note that the commutator $[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I} \partial_1]$ cannot be merely treated by the tools of the conormal calculus developed in Section 3.2, because of the presence of the effective normal derivative ∂_1 (recall that $A_1^{I,I}$ is invertible). This section is devoted to the study of the normal commutator $[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I} \partial_1] u^I$. The following result is of fundamental importance.

Proposition 4.8. *For all $\delta \in]0,1]$, $\gamma \geq 1$ and $m \in \mathbb{N}$, there exists a symbol $q_{m,\delta}(x,\xi,\gamma) \in \Gamma^{m-2}$ such that*

$$\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1\right]w = q_{m,\delta}(x,Z,\gamma)(\partial_1 w), \quad \forall w \in C_{(0)}^{\infty}(\mathbb{R}_+^n). \tag{4.18}$$

Moreover, the symbol $q_{m,\delta}(x,\xi,\gamma)$ obeys the following estimates. For all $\alpha,\beta \in \mathbb{N}^n$, there exists a positive constant $C_{m,\alpha,\beta}$, independent of γ and δ, such that

$$\left|\partial_\xi^\alpha \partial_x^\beta q_{m,\delta}(x,\xi,\gamma)\right| \leq C_{m,\alpha,\beta}\lambda_\delta^{m-2-|\alpha|,\gamma}(\xi), \quad \forall x,\xi \in \mathbb{R}^n. \tag{4.19}$$

Proof. That $q_{m,\delta}(x,\xi,\gamma)$, satisfying estimates (4.19), is a symbol in Γ^{m-2} actually follows arguing from (4.19) and inequalities (3.18) as was already done for $\lambda_\delta^{m-1,\gamma}(\xi)$ and $\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi)$ (see Section 3.2).

For given $w \in C_{(0)}^{\infty}(\mathbb{R}_+^n)$, let us explicitly compute $([\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1]w)^\sharp$; using the identity $(\partial_1 w)^\sharp = e^{-x_1}(Z_1 w)^\sharp$ and that $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)$ and Z_1 commute, we find for every $x \in \mathbb{R}^n$

$$\begin{aligned}
&\left(\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1\right]w\right)^\sharp(x) \\
&\quad = \lambda_{\chi,\delta}^{m-1,\gamma}(D)\left(A_1^{I,I,\natural}e^{-(\cdot)_1}(Z_1 w)^\sharp\right)(x) - A_1^{I,I,\natural}(x)e^{-x_1}\left(Z_1\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w\right)^\sharp(x) \\
&\quad = \lambda_{\chi,\delta}^{m-1,\gamma}(D)\left(A_1^{I,I,\natural}e^{-(\cdot)_1}(Z_1 w)^\sharp\right)(x) - A_1^{I,I,\natural}(x)e^{-x_1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)Z_1 w\right)^\sharp(x) \\
&\quad = \lambda_{\chi,\delta}^{m-1,\gamma}(D)\left(A_1^{I,I,\natural}e^{-(\cdot)_1}(Z_1 w)^\sharp\right)(x) - A_1^{I,I,\natural}(x)e^{-x_1}\lambda_{\chi,\delta}^{m-1,\gamma}(D)(Z_1 w)^\sharp(x).
\end{aligned} \tag{4.20}$$

Observing that $\lambda_{\chi,\delta}^{m-1,\gamma}(D)$ acts on the space $\mathcal{S}(\mathbb{R}^n)$ as the convolution by the inverse Fourier transform of $\lambda_{\chi,\delta}^{m-1,\gamma}$, the preceding expression can be equivalently restated as follows:

$$\begin{aligned}
\left(\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1\right]w\right)^\sharp(x) &= \mathcal{F}^{-1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right) * A_1^{I,I,\natural}e^{-(\cdot)_1}(Z_1 w)^\sharp(x) \\
&\quad - A_1^{I,I,\natural}(x)e^{-x_1}\mathcal{F}^{-1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right) * (Z_1 w)^\sharp \\
&= \left\langle \mathcal{F}^{-1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right), A_1^{I,I,\natural}(x-\cdot)e^{-(x_1-(\cdot)_1)}(Z_1 w)^\sharp(x-\cdot)\right\rangle \\
&\quad - A_1^{I,I,\natural}(x)e^{-x_1}\langle \mathcal{F}^{-1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right), (Z_1 w)^\sharp(x-\cdot)\rangle \\
&= \left\langle \eta_\delta^{m-1,\gamma}, \chi(\cdot)A_1^{I,I,\natural}(x-\cdot)e^{-(x_1-(\cdot)_1)}(Z_1 w)^\sharp(x-\cdot)\right\rangle \\
&\quad - \left\langle \eta_\delta^{m-1,\gamma}, \chi(\cdot)A_1^{I,I,\natural}(x)e^{-x_1}(Z_1 w)^\sharp(x-\cdot)\right\rangle \\
&= \left\langle \eta_\delta^{m-1,\gamma}, \chi(\cdot)A_1^{I,I,\natural}(x-\cdot)(\partial_1 w)^\sharp(x-\cdot)\right\rangle \\
&\quad - \left\langle \eta_\delta^{m-1,\gamma}, \chi(\cdot)A_1^{I,I,\natural}(x)e^{-(\cdot)_1}(\partial_1 w)^\sharp(x-\cdot)\right\rangle \\
&= \left\langle \eta_\delta^{m-1,\gamma}, \chi(\cdot)\left[A_1^{I,I,\natural}(x-\cdot) - A_1^{I,I,\natural}(x)e^{-(\cdot)_1}\right](\partial_1 w)^\sharp(x-\cdot)\right\rangle,
\end{aligned} \tag{4.21}$$

where $\eta_\delta^{m-1,\gamma} := \mathcal{F}^{-1}(\lambda_\delta^{m-1,\gamma})$ and the identity $\mathcal{F}^{-1}(\lambda_{\chi,\delta}^{m-1,\gamma}) = \chi\eta_\delta^{m-1,\gamma}$ (following at once from (4.3)) has been used. Just for brevity, let us further set

$$\mathcal{K}(x,y) := \left[A_1^{I,I,\natural}(x-y) - A_1^{I,I,\natural}(x)e^{-y_1}\right]\chi(y). \tag{4.22}$$

Thus the identity above reads as

$$\left(\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1\right]w\right)^{\sharp}(x) = \left\langle \eta_\delta^{m-1,\gamma},\ \mathcal{K}(x,\cdot)(\partial_1 w)^{\sharp}(x-\cdot)\right\rangle, \tag{4.23}$$

where the "kernel" $\mathcal{K}(x,y)$ is a bounded function in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, with bounded derivatives of all orders. This regularity of $\mathcal{K}$ is due to the presence of the function χ in formula (4.22); actually the vanishing of χ at infinity prevents the blow-up of the exponential factor e^{-y_1}, as $y_1 \to -\infty$. We point out that this is just the step of our analysis of the normal commutator, where this function χ is needed.

After (4.22), we also have that $\mathcal{K}(x,0)=0$; then, by a Taylor expansion with respect to y, we can represent the kernel $\mathcal{K}(x,y)$ as follows:

$$\mathcal{K}(x,y) = \sum_{k=1}^{n} b_k(x,y)y_k, \tag{4.24}$$

where $b_k(x,y)$ are given bounded functions in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, with bounded derivatives; it comes from (4.22) and (4.2) that b_k can be defined in such a way that for some $\varepsilon>1$ and all $x\in\mathbb{R}^n$ there holds

$$\operatorname{supp} b_k(x,\cdot) \subseteq \{|y|\le\varepsilon\}. \tag{4.25}$$

Inserting (4.24) in (4.23) and using standard properties of the Fourier transform we get

$$\begin{aligned}
\left(\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1\right]w\right)^{\sharp}(x) &= \left\langle \eta_\delta^{m-1,\gamma},\ \sum_{k=1}^{n} b_k(x,\cdot)(\cdot)_k(\partial_1 w)^{\sharp}(x-\cdot)\right\rangle\\
&= \sum_{k=1}^{n}\left\langle (\cdot)_k\mathcal{F}^{-1}\left(\lambda_\delta^{m-1,\gamma}\right),\ b_k(x,\cdot)(\partial_1 w)^{\sharp}(x-\cdot)\right\rangle\\
&= -\sum_{k=1}^{n}\left\langle \mathcal{F}^{-1}\left(D_k\lambda_\delta^{m-1,\gamma}\right),\ b_k(x,\cdot)(\partial_1 w)^{\sharp}(x-\cdot)\right\rangle\\
&= -\sum_{k=1}^{n}\left\langle D_k\lambda_\delta^{m-1,\gamma},\ \mathcal{F}^{-1}\left(b_k(x,\cdot)(\partial_1 w)^{\sharp}(x-\cdot)\right)\right\rangle\\
&= -\sum_{k=1}^{n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\mathcal{F}^{-1}\left(b_k(x,\cdot)(\partial_1 w)^{\sharp}(x-\cdot)\right)(\xi)d\xi\\
&= -\sum_{k=1}^{n}(2\pi)^{-n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\left(\int_{\mathbb{R}^n} e^{i\xi\cdot y}b_k(x,y)(\partial_1 w)^{\sharp}(x-y)dy\right)d\xi,
\end{aligned} \tag{4.26}$$

where we have set $D_k := -i\partial_{\xi_k}$ (for each $k = 1, \dots, n$); note that for $w \in C^\infty_{(0)}(\mathbb{R}^n_+)$ and any $x \in \mathbb{R}^n$, the function $b_k(x,\cdot)(\partial_1 w)^\sharp(x-\cdot)$ belongs to $\mathcal{S}(\mathbb{R}^n)$; hence the last expression in (4.26) makes sense. Henceforth, we replace $(\partial_1 w)^\sharp$ by any function $v \in \mathcal{S}(\mathbb{R}^n)$. Our next goal is writing the integral operator

$$(2\pi)^{-n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\left(\int_{\mathbb{R}^n} e^{i\xi\cdot y} b_k(x,y)v(x-y)dy\right)d\xi \tag{4.27}$$

as a pseudodifferential operator.

Firstly, we make use of the inversion formula for the Fourier transformation and Fubini's theorem to recast (4.27) as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{i\xi\cdot y} b_k(x,y)v(x-y)dy \\
&\quad = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot y} b_k(x,y)\left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\eta}\widehat{v}(\eta)d\eta\right)dy \\
&\quad = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\eta}\left(\int_{\mathbb{R}^n} e^{-iy\cdot(\eta-\xi)} b_k(x,y)dy\right)\widehat{v}(\eta)d\eta \\
&\quad = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\eta}\widehat{b}_k(x,\eta-\xi)\widehat{v}(\eta)d\eta;
\end{aligned} \tag{4.28}$$

for every index k, $\widehat{b}_k(x,\zeta)$ denotes the partial Fourier transform of $b_k(x,y)$ with respect to y. Then, inserting (4.28) into (4.27), we obtain

$$\begin{aligned}
&(2\pi)^{-n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\left(\int_{\mathbb{R}^n} e^{i\xi\cdot y} b_k(x,y)v(x-y)dy\right)d\xi \\
&\quad = (2\pi)^{-2n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\left(\int_{\mathbb{R}^n} e^{ix\cdot\eta}\widehat{b}_k(x,\eta-\xi)\widehat{v}(\eta)d\eta\right)d\xi.
\end{aligned} \tag{4.29}$$

Recall that, for each $x \in \mathbb{R}^n$, the function $y \mapsto b_k(x,y)$ belongs to $C_0^\infty(\mathbb{R}^n)$ (and its compact support does not depend on x, see (4.25)); thus, for each $x \in \mathbb{R}^n$, $\widehat{b}_k(x,\zeta)$ is rapidly decreasing in ζ. Because of the estimates for derivatives of $\lambda_\delta^{m-1,\gamma}$ and since $\widehat{v}(\eta)$ is also rapidly decreasing, Fubini's theorem can be used to change the order of the integrations within (4.29). So we get

$$\begin{aligned}
&(2\pi)^{-2n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\left(\int_{\mathbb{R}^n} e^{ix\cdot\eta}\widehat{b}_k(x,\eta-\xi)\widehat{v}(\eta)d\eta\right)d\xi \\
&\quad = (2\pi)^{-2n}\int_{\mathbb{R}^n} e^{ix\cdot\eta}\left(\int_{\mathbb{R}^n} \widehat{b}_k(x,\eta-\xi)D_k\lambda_\delta^{m-1,\gamma}(\xi)d\xi\right)\widehat{v}(\eta)d\eta \\
&\quad = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\eta}q_{k,m,\delta}(x,\eta,\gamma)\widehat{v}(\eta)d\eta,
\end{aligned} \tag{4.30}$$

where we have set

$$q_{k,m,\delta}(x,\xi,\gamma) := (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{b}_k(x,\eta) D_k \lambda_\delta^{m-1,\gamma}(\xi-\eta)\,d\eta. \tag{4.31}$$

Notice that formula (4.31) defines $q_{k,m,\delta}$ as the convolution of the functions $\widehat{b}_k(x,\cdot)$ and $D_k\lambda_\delta^{m-1,\gamma}$; hence $q_{k,m,\delta}$ is a well-defined C^∞-function in $\mathbb{R}^n \times \mathbb{R}^n$.

The proof of Proposition 4.8 will be accomplished, once the following lemma is proved.

Lemma 4.9. *For every $m \in \mathbb{N}$, $k = 1,\ldots,n$ and all $\alpha,\beta \in N^n$, there exists a positive constant $C_{k,m,\alpha,\beta}$, independent of γ and δ, such that*

$$\left|\partial_\xi^\alpha \partial_x^\beta q_{k,m,\delta}(x,\xi,\gamma)\right| \le C_{k,m,\alpha,\beta}\lambda_\delta^{m-2-|\alpha|,\gamma}(\xi), \quad \forall x,\xi \in \mathbb{R}^n. \tag{4.32}$$

It comes from Lemma 4.9 and the left inequality in (3.18) that, for each index k, the function $q_{k,m,\delta}$ is a symbol in Γ^{m-2}; notice however that the set $\{q_{k,m,\delta}\}_{0<\delta\le 1}$ is bounded in Γ^{m-1} but not in Γ^{m-2}. The proof of Lemma 4.9 is postponed to Appendix A.

Now, we continue the proof of Proposition 4.8.

End of the Proof of Proposition 4.8

The last row of (4.30) provides the desired representation of (4.27) as a pseudodifferential operator; actually it gives the identity

$$(2\pi)^{-n}\int_{\mathbb{R}^n} D_k\lambda_\delta^{m-1,\gamma}(\xi)\left(\int_{\mathbb{R}^n} e^{i\xi\cdot y} b_k(x,y)v(x-y)\,dy\right)d\xi = \mathrm{Op}^\gamma(q_{k,m,\delta})v(x), \tag{4.33}$$

for every $v \in \mathcal{S}(\mathbb{R}^n)$.

Inserting the above formula (with $v = (\partial_1 w)^\sharp$) into (4.26) finally gives

$$\left(\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), A_1^{I,I}\partial_1\right]w\right)^\sharp(x) = \mathrm{Op}^\gamma(q_{m,\delta})(\partial_1 w)^\sharp(x), \tag{4.34}$$

where $q_{m,\delta} = q_{m,\delta}(x,\xi,\gamma)$ is the symbol in Γ^{m-2} defined by

$$q_{m,\delta}(x,\xi,\gamma) := -\sum_{k=1}^n q_{k,m,\delta}(x,\xi,\gamma). \tag{4.35}$$

Of course, formula (4.18) is equivalent to (4.34), in view of (3.26). Estimates (4.19) are satisfied by $q_{m,\delta}$ by summation over k of the similar estimates satisfied by $q_{m,\delta,k}$ (cf. Lemma 4.9).

This ends the proof of Proposition 4.8. □

Now, we are going to show how the representation in (4.18) can be exploited to treat the normal commutator as a lower-order term in (4.10) satified by $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u$.

Firstly, we notice that formula (4.18) has been deduced for smooth functions w, while u is just an L^2-function (actually it belongs to $H^{m-1}_{\tan}(\mathbb{R}^n_+)$, by the inductive hypothesis). In order to use (4.18) for u, we need to approximate the latter by smooth functions. This can be done by the help of [14, Proposition 1, Theorem 1]; from there, we know that there exists a sequence $\{u_\nu\}_\nu$ in $C^\infty_{(0)}(\mathbb{R}^n_+)$ such that

$$\begin{gathered} u_\nu \longrightarrow u, \quad \text{in } L^2(\mathbb{R}^n_+), \\ (L+B)u_\nu \longrightarrow (L+B)u, \quad \text{in } L^2(\mathbb{R}^n_+), \\ u^I_{\nu|x_1=0} \longrightarrow u^I_{|x_1=0}, \quad \text{in } H^{-1/2}\left(\mathbb{R}^{n-1}\right), \text{ as } \nu \longrightarrow +\infty. \end{gathered} \tag{4.36}$$

For each index ν, the regular function $u_\nu \in C^\infty_{(0)}(\mathbb{R}^n_+)$ solves the same BVP as the function u, with new data F_ν, G_ν defined by

$$F_\nu := (\gamma + L + B)u_\nu, \qquad G_\nu := Mu_{\nu|x_1=0}. \tag{4.37}$$

It immediately follows from (4.36) that the regular data F_ν, G_ν approximate the original data F, G by

$$F_\nu \longrightarrow F, \quad \text{in } L^2(\mathbb{R}^n_+), \qquad G_\nu \longrightarrow G, \quad \text{in } H^{-1/2}\left(\mathbb{R}^{n-1}\right), \text{ as } \nu \longrightarrow +\infty. \tag{4.38}$$

The same analysis performed to the BVP (1.2)-(1.3) can be applied to the BVP solved by u_ν, for each ν; in particular, we find that $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu$ satisfies the analogue to (4.10), where F is replaced by F_ν. Because of the regularity of u_ν, formula (4.18) can be used to represent the normal commutator term $[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), A^{I,I}_1\partial_1]u^I_\nu$. Directly from system $(\gamma + L + B)u_\nu = F_\nu$, $\partial_1 u^I_\nu$ can be represented in terms of tangential derivatives of u_ν only, as follows:

$$\partial_1 u^I_\nu = \left(A^{I,I}_1\right)^{-1} F^I_\nu + \mathcal{L}^I u_\nu, \tag{4.39}$$

where $\mathcal{L}^I = \mathcal{L}^I(x, Z, \gamma)$ denotes the tangential partial differential operator

$$\mathcal{L}^I u_\nu := -\left(A^{I,I}_1\right)^{-1}\left[\gamma u^I_\nu + H_1 Z_1 u^{II}_\nu + \left(\sum_{j=2}^n A_j Z_j u_\nu + Bu_\nu\right)^I\right], \tag{4.40}$$

and we have set $H_1 := x_1^{-1}A^{I,II}_1$ (recall that $H_1 \in C^\infty_{(0)}(\mathbb{R}^n_+)$ since $A^{I,II}_{1|x_1=0} = 0$). Inserting (4.39) into (4.18), written for $w = u^I_\nu$, leads to

$$\left[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), A^{I,I}_1\partial_1\right]u^I_\nu = q_{m,\delta}(x, Z, \gamma)\left(\left(A^{I,I}_1\right)^{-1} F^I_\nu + \mathcal{L}^I u_\nu\right). \tag{4.41}$$

On the other hand, plugging $\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)\lambda_{\chi,\delta}^{m-1,\gamma}(Z)+\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)r_{m,\delta}(Z,\gamma)=I$ into (4.41) gives

$$\begin{aligned}\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z),A_1^{I,I}\partial_1\right]u_\nu^I &= q_{m,\delta}(x,Z,\gamma)\left(\left(A_1^{I,I}\right)^{-1}F_\nu^I\right)\\ &\quad+q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u_\nu\right)\\ &\quad+q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)r_{m,\delta}(Z,\gamma)u_\nu.\end{aligned}\tag{4.42}$$

Now, we let $\nu\to+\infty$.

From (1.2) (written for u and u_ν), and using that $u_\nu^\sharp\to u^\sharp$ in $L^2(\mathbb{R}^n)$, one finds

$$(A_1\partial_1u_\nu)^\sharp\longrightarrow(A_1\partial_1u)^\sharp,\quad\text{in }\mathcal{S}'(\mathbb{R}^n),\tag{4.43}$$

hence (because $\sharp^{-1}$ is a linear continuous operator from $\mathcal{S}'(\mathbb{R}^n)\subset\mathfrak{D}'(\mathbb{R}^n)$ to $\mathfrak{D}'(\mathbb{R}^n_+)$)

$$\lambda_{\chi,\delta}^{m-1,\gamma}(Z)\left(A_1^{I,I}\partial_1u_\nu^I\right)\longrightarrow\lambda_{\chi,\delta}^{m-1,\gamma}(Z)\left(A_1^{I,I}\partial_1u^I\right),\quad\text{in }\mathfrak{D}'(\mathbb{R}^n_+).\tag{4.44}$$

On the other hand, $u_\nu^\sharp\to u^\sharp$ in $L^2(\mathbb{R}^n)$ implies that

$$\lambda_{\chi,\delta}^{m-1,\gamma}(Z)(u_\nu)\longrightarrow\lambda_{\chi,\delta}^{m-1,\gamma}(Z)(u),\quad\text{in }\mathfrak{D}'(\mathbb{R}^n_+),\tag{4.45}$$

hence

$$A_1^{I,I}\partial_1\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)\left(u_\nu^I\right)\right)\longrightarrow A_1^{I,I}\partial_1\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)\left(u^I\right)\right),\quad\text{in }\mathfrak{D}'(\mathbb{R}^n_+).\tag{4.46}$$

Adding (4.44), (4.46) then gives

$$\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z),A_1^{I,I}\partial_1\right]u_\nu^I\longrightarrow\left[\lambda_{\chi,\delta}^{m-1,\gamma}(Z),A_1^{I,I}\partial_1\right]u^I,\quad\text{in }\mathfrak{D}'(\mathbb{R}^n_+).\tag{4.47}$$

As to the right-hand side of (4.42), all the involved operators, acting on F_ν and u_ν, are conormal. Hence the L^2-convergences $u_\nu\to u$ and $F_\nu\to F$ and the fact that conormal operators continuously extend to the space of distributions $u\in\mathfrak{D}'(\mathbb{R}^n_+)$, for which $u^\sharp\in\mathcal{S}'(\mathbb{R}^n)$, give the convergences

$$\begin{gathered}q_{m,\delta}(x,Z,\gamma)\left(\left(A_1^{I,I}\right)^{-1}F_\nu^I\right)\longrightarrow q_{m,\delta}(x,Z,\gamma)\left(\left(A_1^{I,I}\right)^{-1}F^I\right),\\ q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u_\nu\right)\longrightarrow q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u\right),\\ q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)r_{m,\delta}(Z,\gamma)u_\nu\longrightarrow q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}_{\delta}^{-m+1,\gamma}(Z)r_{m,\delta}(Z,\gamma)u,\quad\text{in }\mathfrak{D}'(\mathbb{R}^n_+).\end{gathered}\tag{4.48}$$

Therefore, the uniqueness of the limit in $\mathfrak{D}'(\mathbb{R}^n_+)$ together with (4.47), (4.48) implies that (4.42) holds true for the L^2-solution u of (1.2)-(1.3), that is,

$$\begin{aligned}\left[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), A^{I,I}_1\partial_1\right]u^I &= q_{m,\delta}(x,Z,\gamma)\left(\left(A^{I,I}_1\right)^{-1}F^I\right)\\ &\quad + q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right)\\ &\quad + q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z)r_{m,\delta}(Z,\gamma)u.\end{aligned} \tag{4.49}$$

Let us come back to the commutator term $[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), L+B]u$ appearing in the interior equation (4.10).

Substituting (4.15), (4.49) into (4.14) gives for this term the following representation:

$$\left[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), L+B\right]u = \rho_{m,\delta}(x,Z,\gamma)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right) + \tau_{m,\delta}(x,Z,\gamma)u + \eta_{m,\delta}(x,Z,\gamma)F, \tag{4.50}$$

where we have set for short

$$\begin{aligned}\rho_{m,\delta}(x,Z,\gamma) &:= \left[\lambda^{m-1,\gamma}_{\chi,\delta}(Z), L_{\tan}+B\right]\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z) + \begin{pmatrix} q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z)\\ 0\end{pmatrix},\\ \tau_{m,\delta}(x,Z,\gamma) &:= s_{m,\delta}(x,Z,\gamma) + \begin{pmatrix} q_{m,\delta}(x,Z,\gamma)\mathcal{L}^I\widetilde{\lambda}^{-m+1,\gamma}_{\delta}(Z)r_{m,\delta}(Z,\gamma)\\ 0\end{pmatrix},\\ \eta_{m,\delta}(x,Z,\gamma) &:= \begin{pmatrix} q_{m,\delta}(x,Z,\gamma)\left(A^{I,I}_1\right)^{-1} & 0\\ 0 & 0\end{pmatrix}.\end{aligned} \tag{4.51}$$

Consequently, the interior equation (4.10) can be restated as

$$(\gamma+L+B+\rho_{m,\delta}(x,Z,\gamma))\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right)+\tau_{m,\delta}(x,Z,\gamma)u+\eta_{m,\delta}(x,Z,\gamma)F=\lambda^{m-1,\gamma}_{\chi,\delta}(Z)F, \quad \text{in } \mathbb{R}^n_+. \tag{4.52}$$

Since $L_{\tan}+B$ and $\mathcal{L}^I$ are conormal operators with symbol in Γ^1, Lemma 4.5 and Proposition 4.8 imply that $\rho_{m,\delta}(x,Z,\gamma)$ is a conormal operator with symbol in Γ^0, for each $0<\delta\leq 1$; moreover, it amounts that the family of symbols $\{\rho_{m,\delta}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^0. Therefore, $\rho_{m,\delta}(x,Z,\gamma)$ can be regarded, jointly with B, as a lower-order term in (4.52), as considered in the statement of Theorem 1.1.

Concerning the terms $\tau_{m,\delta}(x,Z,\gamma)u$, $\eta_{m,\delta}(x,Z,\gamma)F$, they can be both moved into the right-hand side of (4.52), to be treated as a part of the interior source term, as will be detailed in Section 4.5.

4.4. The Boundary Condition

Now we are going to seek for an appropriate boundary condition to be coupled with the interior equation (4.10), in order to state a BVP solved by $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u$.

To this end, it is worthwhile to make an additional hypothesis about the smooth function χ involved in the definition of $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)$. We assume that χ has the form

$$\forall x = (x_1, x') \in \mathbb{R}^n, \quad \chi(x) = \chi_1(x_1)\widetilde{\chi}(x'), \tag{4.53}$$

where $\chi_1 \in C^\infty(\mathbb{R})$ and $\widetilde{\chi} \in C^\infty(\mathbb{R}^{n-1})$ are given positive even functions, to be chosen in such a way that conditions (4.2) are made satisfied.

As it was done for the analysis of the normal commutator (cf. Proposition 4.8), we start our reasoning by dealing with smooth functions. In this case, following closely the arguments employed to prove Proposition 4.8 and Lemma 4.9, we are able to get the following.

Proposition 4.10. *Assume that χ obeys the assumptions* (4.2), (4.53). *Then, for all $\delta \in]0,1]$, $\gamma \geq 1$, and $m \in \mathbb{N}$, the function $b'_{m,\delta}(\xi',\gamma)$ defined by*

$$b'_{m,\delta}(\xi',\gamma) := (2\pi)^{-n}\int_{\mathbb{R}^n} \lambda_\delta^{m-1,\gamma}(\eta_1, \eta' + \xi')\left(e^{(\cdot)_1/2}\chi_1\right)^{\wedge_1}(\eta_1)\widehat{\widetilde{\chi}}(\eta')d\eta, \quad \forall \xi' \in \mathbb{R}^{n-1}, \tag{4.54}$$

(where $\xi' := (\xi_2,\ldots,\xi_n)$ are the Fourier dual variables of the tangential variables $x' = (x_2,\ldots,x_n)$) is a γ-depending symbol in $\mathbb{R}^{n-1}$ belonging to Γ^{m-1}. Moreover, for all functions $w \in C^\infty_{(0)}(\mathbb{R}^n_+)$ there holds

$$\forall x' \in \mathbb{R}^{n-1}, \quad \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w\right)_{|x_1=0}(x') = b'_{m,\delta}(D',\gamma)\left(w_{|x_1=0}\right)(x'), \tag{4.55}$$

where one has set $D' = (D_2,\ldots,D_n)$, $D_j = -i\partial_j$ (for $j = 2,\ldots,n$) and $b'_{m,\delta}(D',\gamma)$ stands for the ordinary pseudodifferential operator in $\mathbb{R}^{n-1}$ with symbol $b'_{m,\delta}$.

Eventually, the following estimates are satisfied by the symbol $b'_{m,\delta}(\xi',\gamma)$: for all $\alpha' = (\alpha_2,\ldots,\alpha_n) \in \mathbb{N}^{n-1}$ there exists a positive constant $C_{m,\alpha'}$, independent of γ and δ, such that

$$\left|\partial_{\xi'}^{\alpha'} b'_{m,\delta}(\xi',\gamma)\right| \leq C_{m,\alpha'}\lambda_\delta^{m-1-|\alpha'|,\gamma}(\xi'), \quad \forall \xi' \in \mathbb{R}^{n-1}. \tag{4.56}$$

Proof. That $b'_{m,\delta}$ belongs to Γ^{m-1} immediately follows from estimates (4.56), using the $(n-1)$-dimensional counterpart of (3.18).

Let $w \in C^\infty_{(0)}(\mathbb{R}^n_+)$; to find a symbol $b'_{m,\delta}$ satisfying (4.55), from (4.3) we firstly compute

$$\begin{aligned}\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w\right)^\sharp(x) &= \lambda_{\chi,\delta}^{m-1,\gamma}(D)\left(w^\sharp\right)(x) = \left(\mathcal{F}^{-1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right) * w^\sharp\right)(x)\\ &= \langle \mathcal{F}^{-1}\left(\lambda_{\chi,\delta}^{m-1,\gamma}\right), w^\sharp(x-\cdot)\rangle\\ &= \left\langle \mathcal{F}^{-1}\left(\lambda_\delta^{m-1,\gamma}\right), \chi(\cdot)e^{(x_1-(\cdot)_1)/2}w\left(e^{x_1-(\cdot)_1}, x'-(\cdot)'\right)\right\rangle, \quad \forall (x_1,x') \in \mathbb{R}^n,\end{aligned} \tag{4.57}$$

hence

$$\begin{aligned}
&\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w(x) \\
&\quad = \left\langle \mathcal{F}^{-1}\left(\lambda_{\delta}^{m-1,\gamma}\right), \chi(\cdot)e^{-(\cdot)_1/2}w\left(x_1e^{-(\cdot)_1}, x' - (\cdot)'\right)\right\rangle \\
&\quad = \left\langle \lambda_{\delta}^{m-1,\gamma}, \mathcal{F}^{-1}\left(\chi(\cdot)e^{-(\cdot)_1/2}w\left(x_1e^{-(\cdot)_1}, x' - (\cdot)'\right)\right)\right\rangle \\
&\quad = (2\pi)^{-n}\int \lambda_{\delta}^{m-1,\gamma}(\xi)\left(\int e^{i\xi\cdot y}\chi(y)e^{-y_1/2}w\left(x_1e^{-y_1}, x'-y'\right)dy\right)d\xi, \quad \forall x_1 > 0,\ \forall x' \in \mathbb{R}^{n-1}.
\end{aligned} \tag{4.58}$$

The regularity of w legitimates all the above calculations. Setting $x_1 = 0$ in the last expression above, we deduce the corresponding expression for the trace on the boundary of $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w$

$$\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w\right)_{|x_1=0}(x') = (2\pi)^{-n}\int \lambda_{\delta}^{m-1,\gamma}(\xi)\left(\int e^{i\xi\cdot y}\chi(y)\mathrm{e}^{-y_1/2}\left(w_{|x_1=0}\right)(x'-y')dy\right)d\xi. \tag{4.59}$$

Now we substitute (4.53) into the y-integral appearing in the last expression above; then Fubini's theorem gives

$$\begin{aligned}
&\int e^{i\xi\cdot y}\chi_1(y_1)\widetilde{\chi}(y')e^{-y_1/2}\left(w_{|x_1=0}\right)(x'-y')dy \\
&\quad = \int e^{i\xi'\cdot y'}\left(\int e^{i\xi_1y_1}e^{-y_1/2}\chi_1(y_1)dy_1\right)\widetilde{\chi}(y')\left(w_{|x_1=0}\right)(x'-y')dy' \\
&\quad = \left(e^{(\cdot)_1/2}\chi_1\right)^{\wedge_1}(\xi_1)\int e^{i\xi'\cdot y'}\widetilde{\chi}(y')\left(w_{|x_1=0}\right)(x'-y')dy',
\end{aligned} \tag{4.60}$$

where $\wedge_1$ is used to denote the one-dimensional Fourier transformation with respect to y_1. Writing, by the inversion formula, $(w_{|x_1=0})(x'-y') = (2\pi)^{-n+1}\int e^{i(x'-y')\cdot\eta'}\widehat{w_{|x_1=0}}(\eta')d\eta'$ and using once more Fubini's theorem, we further obtain

$$\begin{aligned}
&\int e^{i\xi'\cdot y'}\widetilde{\chi}(y')\left(w_{|x_1=0}\right)(x'-y')dy' \\
&\quad = (2\pi)^{-n+1}\int e^{i\xi'\cdot y'}\widetilde{\chi}(y')\left(\int e^{i(x'-y')\cdot\eta'}\widehat{w_{|x_1=0}}(\eta')d\eta'\right)dy' \\
&\quad = \int e^{ix'\cdot\eta'}\left((2\pi)^{-n+1}\int e^{i(\xi'-\eta')\cdot y'}\widetilde{\chi}(y')dy'\right)\widehat{w_{|x_1=0}}(\eta')d\eta' \\
&\quad = (2\pi)^{-n+1}\int e^{ix'\cdot\eta'}\widehat{\widetilde{\chi}}(\xi'-\eta')\widehat{w_{|x_1=0}}(\eta')d\eta';
\end{aligned} \tag{4.61}$$

$\wedge$ is used here to denote the $(n-1)$-dimensional Fourier transformation with respect to x'. Inserting (4.60), (4.61) into (4.59) then leads to

$$\begin{aligned}&\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w\right)_{|x_1=0}(x')\\&\quad=(2\pi)^{-n}\int\lambda_{\delta}^{m-1,\gamma}(\xi)\left(e^{(\cdot)_1/2}\chi_1\right)^{\wedge_1}(\xi_1)\left((2\pi)^{-n+1}\int e^{ix'\cdot\eta'}\widehat{\widehat{\chi}}(\xi'-\eta')\widehat{w_{|x_1=0}}(\eta')d\eta'\right)d\xi.\end{aligned}\tag{4.62}$$

Because $(e^{(\cdot)_1/2}\chi_1)^{\wedge_1}\in\mathcal{S}(\mathbb{R})$, $\widehat{\widehat{\chi}}\in\mathcal{S}(\mathbb{R}^{n-1})$, and $\widehat{w_{|x_1=0}}\in\mathcal{S}(\mathbb{R}^{n-1})$, the double integral

$$\iint e^{ix'\cdot\eta'}\lambda_{\delta}^{m-1,\gamma}(\xi)\left(e^{(\cdot)_1/2}\chi_1\right)^{\wedge_1}(\xi_1)\widehat{\widehat{\chi}}(\xi'-\eta')\widehat{w_{|x_1=0}}(\eta')d\eta'd\xi \tag{4.63}$$

converges absolutely; hence Fubini's theorem allows to exchange the order of the integrations in (4.62) and find

$$\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)w\right)_{|x_1=0}(x')=(2\pi)^{-n+1}\int e^{ix'\cdot\eta'}b'_{m,\delta}(\eta',\gamma)\widehat{w_{|x_1=0}}(\eta')d\eta', \tag{4.64}$$

where $b'_{m,\delta}(\eta',\gamma)$ is defined by (4.54). This shows the identity (4.55).

The proof of estimates (4.56) is similar to that of estimates (4.32) in Lemma 4.9 (see Appendix A); so we will omit it. □

Let us now illustrate how formula (4.55) can be used to derive the desired boundary condition satisfied by $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u$.

Again, let u be the L^2-solution to the original BVP (1.2)-(1.3) and $\{u_\nu\}_{\nu=1}^{+\infty}$ the corresponding sequence in $C^\infty_{(0)}(\mathbb{R}^n_+)$, approximating u in the sense of (4.36).

The last convergence in (4.36) and the Sobolev continuity of standard pseudodifferential operators gives in particular that

$$b'_{m,\delta}(D',\gamma)\left(u^I_{\nu|x_1=0}\right)\longrightarrow b'_{m,\delta}(D',\gamma)\left(u^I_{|x_1=0}\right),\quad\text{in } H^{-m+1/2}\left(\mathbb{R}^{n-1}\right). \tag{4.65}$$

On the other hand, (4.36) and (4.52) (written for u and u_ν) can be used to prove that $(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I)_{|x_1=0}$ and $(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I_\nu)_{|x_1=0}$, for each ν, are traces well defined in $H^{-1/2}(\mathbb{R}^{n-1})$ and the convergence

$$\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I_\nu\right)_{|x_1=0}\longrightarrow\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I\right)_{|x_1=0} \tag{4.66}$$

holds true, at least in $\mathfrak{D}'(\mathbb{R}^{n-1})$. The proof of this assertion is postponed to Appendix A (see Lemma A.2).

Since u_ν are smooth functions, from Proposition 4.10, it comes that for each ν:

$$\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I_\nu\right)_{|x_1=0}=b'_{m,\delta}(D',\gamma)\left(u^I_{\nu|x_1=0}\right). \tag{4.67}$$

Then, letting $\nu \to +\infty$, (4.65) and (4.66) yield

$$\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I\right)_{|x_1=0} = b'_{m,\delta}(D',\gamma)\left(u^I_{|x_1=0}\right). \tag{4.68}$$

Recalling that $M = (I_d, 0)$, from the boundary condition (1.3) and (4.68), we immediately find

$$\begin{aligned}\left(M\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u\right)\right)_{|x_1=0} &= M\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I\right)_{|x_1=0} = Mb'_{m,\delta}(D',\gamma)\left(u^I_{|x_1=0}\right)\\ &= b'_{m,\delta}(D',\gamma)\left(Mu_{|x_1=0}\right) = b'_{m,\delta}(D',\gamma)G.\end{aligned} \tag{4.69}$$

4.5. Derivation of the Conormal Regularity at the Order m

We are now in the position to get the desired conormal regularity of the solution u to (1.2)-(1.3), under the assumptions that $F \in \mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)$, $G \in H^{s+m}_\gamma(\mathbb{R}^{n-1})$. From now on, assume that $\gamma \geq \gamma_{m-1}$; so far, from the inductive hypothesis we know that, for such a γ, u belongs to $H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$, $u^I_{|x_1=0} \in H^{m-1}_\gamma(\mathbb{R}^{n-1})$ and the estimate (4.1) is satisfied (see the end of Section 4.1). Because of the calculations made in Sections 4.3 and 4.4, it follows that $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u$ is an L^2 solution of the problem

$$\begin{aligned}&(\gamma + L + B + \rho_{m,\delta}(x,Z,\gamma))\left(\lambda_{\chi,\delta}^{\gamma,m-1}(Z)u\right)\\ &\quad = \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z) - \eta_{m,\delta}(x,Z,\gamma)\right)F - \tau_{m,\delta}(x,Z,\gamma)u, \quad \text{in } \mathbb{R}^n_+,\end{aligned} \tag{4.70}$$

$$M\left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u\right) = b'_{m,\delta}(D',\gamma)G, \quad \text{on } \mathbb{R}^{n-1}. \tag{4.71}$$

The previous one reads as the original BVP (1.2)-(1.3) solved by u, where the role of the lower-order term $\mathcal{B}$ is played here by the conormal operator $B + \rho_{m,\delta}(x,Z,\gamma)$. As already discussed in the end of Section 4.3, in view of Proposition 3.8 and Lemma 4.5, the symbol of $B + \rho_{m,\delta}(x,Z,\gamma)$ belongs to Γ^0, and $\{\rho_{m,\delta}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^0.

As regards to the terms $\tau_{m,\delta}(x,Z,\gamma)u$ and $\eta_{m,\delta}(x,Z,\gamma)F$ appearing into the right-hand side of (4.70), they can be regarded as a part of the source term in the interior equation (4.70) (this is the reason why they have been moved in the right-hand side of (4.70)).

Let us firstly focus on $\tau_{m,\delta}(x,Z,\gamma)u$. After Lemma 4.5 and Proposition 3.8 (see also Proposition 4.8, Remark 4.7, and formulas (4.16), (4.51)), we know that for any $k \in \mathbb{N}$ the operators $[\lambda_{\chi,\delta}^{m-1,\gamma}(Z), L_{\tan} + B]\widetilde{\lambda}_\delta^{-m+1,\gamma}(Z)$ and $q_{m,\delta}(x,Z,\gamma)\mathscr{L}^I\widetilde{\lambda}_\delta^{-m+1,\gamma}(Z)$ extend as linear-bounded mappings from $H^k_{\tan,\gamma}(\mathbb{R}^n_+)$ into itself, and their operator norms are uniformly bounded with respect to γ and δ. Combining with the result of Lemma 4.2, it follows that a positive constant $C_s > 0$, independent of γ and δ, can be found in such a way that

$$\|\tau_{m,\delta}(x,Z,\gamma)u\|_{\mathscr{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+)} \leq \|\tau_{m,\delta}(x,Z,\gamma)u\|_{H^s_{\tan,\gamma}(\mathbb{R}^n_+)} \leq C_s\gamma^s\|u\|_{L^2(\mathbb{R}^n_+)}. \tag{4.72}$$

Concerning now the term $(\lambda_{\chi,\delta}^{m-1,\gamma}(Z) - \eta_{m,\delta}(x,Z,\gamma))F$, after Lemma 4.5 and Proposition 4.8 we already know that the symbol $\eta_{m,\delta}(x,\xi,\gamma)$ has order $m-2$ and obeys the same decay estimates as in (4.19). From (3.17) it then follows that $\{\eta_{m,\delta}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^{m-1}; because $\{\lambda_{\chi,\delta}^{m-1,\gamma}\}_{0<\delta\leq 1}$ is also a bounded subset of Γ^{m} (as a consequence of (4.4) and (3.17) again), after Proposition 3.10 we conclude that there exists a positive constant $C_{m,s,r}$ such that

$$\left\| \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z) - \eta_{m,\delta}(x, Z, \gamma) \right) F \right\|_{\mathscr{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+)} \leq C_{m,s,r} \|F\|_{\mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)}. \tag{4.73}$$

As regards to the boundary datum $b'_{m,\delta}(D', \gamma)G$ in (4.71), the family of symbols $\{b'_{m,\delta}\}_{0<\delta\leq 1}$ in $\mathbb{R}^{n-1}$ defines a bounded subset of Γ^m; this follows from estimates (4.56) and the inequality (3.17) (in dimension $n-1$). Therefore, the Sobolev continuity of ordinary pseudodifferential operators in $\mathbb{R}^{n-1}$ implies the existence of a positive constant $C_{m,s}$, independent of γ and δ, such that:

$$\left\| b'_{m,\delta}(D', \gamma)G \right\|_{H^s_\gamma(\mathbb{R}^{n-1})} \leq C_{m,s} \|G\|_{H^{s+m}_\gamma(\mathbb{R}^{n-1})}. \tag{4.74}$$

From the assumptions made about the BVP (1.2)-(1.3) in Theorem 1.1, we find some constants $\widetilde{\gamma}_m \geq 1$, $\widetilde{C}_m > 0$ such that for all $\gamma \geq \gamma_m := \max\{\gamma_{m-1}, \widetilde{\gamma}_m\}$ and every $\delta \in]0,1]$, $\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u$ is the only L^2-solution of (4.70)-(4.71) and obeys the estimate

$$\begin{aligned} &\gamma \left\| \lambda_{\chi,\delta}^{m-1,\gamma}(Z)u \right\|^2_{L^2(\mathbb{R}^n_+)} + \left\| \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I \right)_{|x_1=0} \right\|^2_{L^2(\mathbb{R}^{n-1})} \\ &\quad \leq \widetilde{C}_m \left(\frac{1}{\gamma^{2s+1}} \left\| \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z) - \eta_{m,\delta}(x, Z, \gamma) \right) F - \tau_{m,\delta}(x, Z, \gamma)u \right\|^2_{\mathscr{H}^{s,r}_{\tan,\gamma}(\mathbb{R}^n_+)} \right. \\ &\qquad \left. + \frac{1}{\gamma^{2s}} \left\| b'_{m,\delta}(D', \gamma)G \right\|^2_{H^s_\gamma(\mathbb{R}^{n-1})} \right). \end{aligned} \tag{4.75}$$

We remark that, according to the statement of Theorem 1.1, the constants $\widetilde{\gamma}_m$, $\widetilde{C}_m$ are only dependent on a δ, γ-uniform upper bound of the kth-order norm (3.14), computed on the symbol of $B + \rho_{m,\delta}(x, Z, \gamma)$, besides the coefficients A_j, $1 \leq j \leq n$, and the integer numbers r, s.

Then, using (4.72)–(4.74) and (1.7) to estimate the right-hand side of (4.75), we get

$$\begin{aligned} &\gamma \left\| \lambda_{\chi,\delta}^{m-1,\gamma}(Z)u \right\|^2_{L^2(\mathbb{R}^n_+)} + \left\| \left(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I \right)_{|x_1=0} \right\|^2_{L^2(\mathbb{R}^{n-1})} \\ &\quad \leq \widetilde{C}_{m,s,r} \left(\frac{1}{\gamma^{2s+1}} \|F\|^2_{\mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)} + \frac{1}{\gamma^{2s}} \|G\|^2_{H^{s+m}_\gamma(\mathbb{R}^{n-1})} \right), \end{aligned} \tag{4.76}$$

where $\widetilde{C}_{m,s,r}$ is a suitable positive constant independent of $\gamma \geq \gamma_m$ and δ. Since the L^2-norms $\|\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u\|_{L^2(\mathbb{R}^n_+)}$ are bounded by (4.76), uniformly with respect to $\delta \in]0,1]$, Corollary 4.4 gives $u \in H^m_{\tan,\gamma}(\mathbb{R}^n_+)$.

As to the Sobolev regularity of the trace on the boundary of the noncharacteristic component u^I of the solution, the estimate (4.76) gives a bound of $\|(\lambda_{\chi,\delta}^{m-1,\gamma}(Z)u^I)_{|x_1=0}\|_{L^2(\mathbb{R}^{n-1})} = \|b'_{m,\delta}(D',\gamma)(u^I_{|x_1=0})\|_{L^2(\mathbb{R}^{n-1})}$ uniform with respect to $\delta \in]0,1]$ (cf. (4.68)). Then $u^I_{|x_1=0} \in H^m_\gamma(\mathbb{R}^{n-1})$ can be derived from the next result, the proof of which will be given in Appendix A.

Lemma 4.11. *For $m \in \mathbb{N}$ and $\delta \in]0,1]$, let $b'_{m,\delta}(\xi',\gamma)$ be defined by (4.54). Then there exists a symbol $\beta_{m,\delta}(\xi',\gamma) \in \Gamma^{m-2}$ such that*

$$b'_{m,\delta}(\xi',\gamma) = \lambda_\delta^{m-1,\gamma}(\xi') + \beta_{m,\delta}(\xi',\gamma), \quad \forall \xi' \in \mathbb{R}^{n-1}. \tag{4.77}$$

In addition, the symbol $\beta_{m,\delta}$ satisfies the following estimates: for every $\alpha' \in \mathbb{N}^{n-1}$, there exists a positive constant $C_{m,\alpha'}$, independent of γ and δ, such that

$$\left|\partial_{\xi'}^{\alpha'}\beta_{m,\delta}(\xi',\gamma)\right| \le C_{m,\alpha'}\lambda_\delta^{m-2-|\alpha'|,\gamma}(\xi'), \quad \forall \xi' \in \mathbb{R}^{n-1}. \tag{4.78}$$

Arguing as was done to derive Corollary 4.4 from Lemma 4.2, from Lemma 4.11 we deduce the following.

Corollary 4.12. *For every positive integer m and $\gamma \ge 1$, $v \in H^m_\gamma(\mathbb{R}^{n-1})$ if and only if $v \in H^{m-1}_\gamma(\mathbb{R}^{n-1})$ and the set $\{\|b'_{m,\delta}(D',\gamma)v\|_{L^2(\mathbb{R}^{n-1})}\}_{0<\delta\le 1}$ is bounded.*

After the result of Corollary 4.12, we conclude that $u^I_{|x_1=0} \in H^m_\gamma(\mathbb{R}^{n-1})$.

It remains to prove that the solution of (1.2)-(1.3) satisfies the a priori estimate (1.8) of order m. From estimate (4.76) and the use of the identities (4.7), (4.68), (4.77), we also deduce

$$\begin{aligned}\gamma\left\|\lambda_\delta^{m-1,\gamma}(Z)u\right\|^2_{L^2(\mathbb{R}^n_+)} &+ \left\|\lambda_\delta^{m-1,\gamma}(D')\left(u^I_{|x_1=0}\right)\right\|^2_{L^2\left(\mathbb{R}^{n-1}\right)} \\ &\le C'_{m,s,r}\left(\frac{1}{\gamma^{2s+1}}\|F\|^2_{\mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)} + \frac{1}{\gamma^{2s}}\|G\|^2_{H^{s+m}_\gamma\left(\mathbb{R}^{n-1}\right)}\right) \\ &\quad + 2\gamma\|r_{m,\delta}(Z,\gamma)u\|^2_{L^2(\mathbb{R}^n_+)} + 2\left\|\beta_{m,\delta}(D',\gamma)\left(u^I_{|x_1=0}\right)\right\|^2_{L^2\left(\mathbb{R}^{n-1}\right)},\end{aligned} \tag{4.79}$$

where the positive constant $C'_{m,s,r}$ is again independent of $\gamma \ge \gamma_m$ and δ. On the other hand, using Lemma 4.2 and that $\{\beta_{m,\delta}\}_{0<\delta\le 1}$ is a bounded subset of Γ^{m-1} (that follows at once from Lemma 4.11 and inequality (3.17)), one can estimate

$$\gamma\|r_{m,\delta}(Z,\gamma)u\|^2_{L^2(\mathbb{R}^n_+)} + \left\|\beta_{m,\delta}(D',\gamma)\left(u^I_{|x_1=0}\right)\right\|^2_{L^2\left(\mathbb{R}^{n-1}\right)} \le C_m\left(\gamma\|u\|^2_{L^2(\mathbb{R}^n_+)} + \left\|u^I_{|x_1=0}\right\|^2_{H^{m-1}_\gamma\left(\mathbb{R}^{n-1}\right)}\right) \tag{4.80}$$

with positive constant C_m independent, once again, of γ and δ.

In the end, combining (4.79), (4.80) and using the a priori estimate (4.1) of order $m-1$ on u, which holds true by the inductive assumption, we conclude that there exists a constant $C''_{m,s,r} > 0$ such that

$$\gamma\left\|\lambda_{\delta}^{m-1,\gamma}(Z)u\right\|^2_{L^2(\mathbb{R}^n_+)} + \left\|\lambda_{\delta}^{m-1,\gamma}(D')\left(u^I_{|x_1=0}\right)\right\|^2_{L^2\left(\mathbb{R}^{n-1}\right)} \leq C''_{m,s,r}\left(\frac{1}{\gamma^{2s+1}}\|F\|^2_{\mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)} + \frac{1}{\gamma^{2s}}\|G\|^2_{H^{s+m}_{\gamma}\left(\mathbb{R}^{n-1}\right)}\right) \tag{4.81}$$

for all $\gamma \geq \gamma_m$ and $\delta \in]0,1]$.

The energy estimate (1.8) of order m hence follows by letting $\delta \to 0$ into the left-hand side of (4.81) (for an arbitrarily fixed $\gamma \geq \gamma_m$) and exploiting the results of Propositions 3.1 and 3.2.

Appendices

A. Proof of Some Technical Lemmata

A.1. Proof of Lemma 4.1

The proof that $\lambda_{\chi,\delta}^{m-1,\gamma}$ obeys estimates (4.4) relies on the following γ-weighted version of Peetre's inequality.

For all $s \in \mathbb{R}$, $\gamma \geq 1$, and $\xi, \eta \in \mathbb{R}^n$

$$\lambda^{s,\gamma}(\xi) \leq 2^{|s|}\lambda^{s,\gamma}(\xi-\eta)\lambda^{|s|}(\eta). \tag{A.1}$$

The proof of (A.1) follows by an easy account of the parameter γ into the arguments used to prove the classical Peetre's inequality (cf., e.g., [11], [15, Lemma 1.18]). As an easy consequence of (A.1) and (3.15), it can be also proved that the following holds:

$$\lambda_{\delta}^{s-1,\gamma}(\xi) \leq 2^{|s|+1}\lambda_{\delta}^{s-1,\gamma}(\xi-\eta)\lambda^{|s|}(\eta)\lambda^{1}(\delta\eta), \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{A.2}$$

for an arbitrary $\delta \in]0,1]$.

For an arbitrary $\alpha \in \mathbb{N}^n$, we use (A.2) with $s = m - |\alpha|$ and (3.16) to find

$$\left|\partial^{\alpha}\lambda_{\chi,\delta}^{m-1,\gamma}(\xi)\right| = \left|\left(\mathcal{F}^{-1}\chi * \partial^{\alpha}\lambda_{\delta}^{m-1,\gamma}\right)(\xi)\right| \leq \int\left|\mathcal{F}^{-1}\chi(\eta)\right|\left|\partial^{\alpha}\lambda_{\delta}^{m-1,\gamma}(\xi-\eta)\right|d\eta \leq C_{m,\alpha}2^{m+1+|\alpha|}\lambda_{\delta}^{m-1-|\alpha|,\gamma}(\xi)\int\left|\mathcal{F}^{-1}\chi(\eta)\right|\lambda^{m+1+|\alpha|}(\eta)d\eta. \tag{A.3}$$

Since $\mathcal{F}^{-1}\chi\lambda^{m+1+|\alpha|} \in \mathcal{S}(\mathbb{R}^n)$, the integral above is finite; moreover it does not depend on γ and δ. Therefore (A.3) is precisely an estimate of type (4.4). That $\lambda_{\chi,\delta}^{m-1,\gamma} \in \Gamma^{m-1}$ follows at once from the same arguments applied to $\lambda_{\delta}^{m-1,\gamma}$ (see (3.19)).

A.2. Proof of Lemma 4.5

The symbol of $\mathrm{Op}^\gamma(a_\delta)\mathrm{Op}^\gamma(b)$ is $a_\delta \# b$. Because of (3.17), we already know that $\{a_\delta\}_{0<\delta\le 1}$ is a bounded subset of Γ^r. Then the rules of symbolic calculus give that $a_\delta \# b$ is a symbol in Γ^{r+l}, for every δ; moreover, from (3.23) one has

$$a_\delta \# b(x,\xi,\gamma) = \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a_\delta(x,\xi,\gamma)\partial_x^\alpha b(x,\xi,\gamma) + \mathcal{R}_{N,\delta}(x,\xi,\gamma), \tag{A.4}$$

for every integer $N \ge 1$, and $\{\mathcal{R}_{N,\delta}\}_{0<\delta\le 1}$ is a bounded subset of Γ^{r+l-N}.

In particular, setting $N = 1$ in (A.4) gives

$$a_\delta \# b(x,\xi,\gamma) = a_\delta(x,\xi,\gamma)b(x,\xi,\gamma) + \mathcal{R}_{1,\delta}(x,\xi,\gamma), \tag{A.5}$$

where $\mathcal{R}_{1,\delta} \in \Gamma^{r+l-1}$ and for all $\alpha,\beta \in \mathbb{N}^n$ there exists $C_{r,l,\alpha,\beta} > 0$, *independent of* γ *and* δ, such that

$$\left|\partial_\xi^\alpha \partial_x^\beta \mathcal{R}_{1,\delta}(x,\xi,\gamma)\right| \le C_{r,l,\alpha,\beta}\lambda^{r+l-1-|\alpha|,\gamma}(\xi), \quad \forall x,\xi \in \mathbb{R}^n. \tag{A.6}$$

Then, combining (A.6) with the right inequality in (3.18), we easily derive that $\mathcal{R}_{1,\delta}$ satisfies the estimates (4.9). By Leibniz's rule and the use of (4.8), one can trivially check that estimates (4.9) are satisfied by the product of symbols $a_\delta(x,\xi,\gamma)b(x,\xi,\gamma)$ as well. That $a_\delta \# b \in \Gamma^{r+l-1}$ follows again from estimates (4.9) themselves and the left inequality in (3.18).

As regards to the remaining assertions about the symbols of the operators $\mathrm{Op}^\gamma(a_\delta)\mathrm{Op}^\gamma(b)\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$ and $[\mathrm{Op}^\gamma(a_\delta), \mathrm{Op}^\gamma(b)]\widetilde{\lambda}_\delta^{-m+1,\gamma}(D)$ (in the case of scalar-valued a_δ), they follow at once from Leibniz's rule and Proposition 3.6, combined with the estimates (4.9) and (3.20).

A.3. Proof of Lemma 4.9

Recall that we have defined for each $k = 1,\dots,n$

$$q_{k,m,\delta}(x,\xi,\gamma) := (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{b}_k(x,\eta)D_k\lambda_\delta^{m-1,\gamma}(\xi-\eta)d\eta, \tag{A.7}$$

where the functions $b_k = b_k(x,y)$ (cf. (4.24)) are given in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, have bounded derivatives in $\mathbb{R}^n \times \mathbb{R}^n$, and satisfy for all $x \in \mathbb{R}^n$

$$\operatorname{supp} b_k(x,\cdot) \subseteq \{|y| \le \varepsilon\}, \tag{A.8}$$

with some constant $\varepsilon > 1$. Recall also that $\widehat{b}_k(x,\zeta)$ denotes the partial Fourier transform of $b_k(x,y)$ with respect to y.

In the sequel, we remove the subscript k for simplicity.

The following lemma is concerned with the behavior at infinity of $\widehat{b}(x,\zeta)$.

Lemma A.1. *Let the function $b = b(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ obey all of the preceding assumptions. Then, for every positive integer N and all multi-indices $\alpha \in \mathbb{N}^n$, there exists a positive constant $C_{N,\alpha}$ such that*

$$\left(1 + |\zeta|^2\right)^N \left|\partial_x^\alpha \widehat{b}(x, \zeta)\right| \le C_{N,\alpha}, \quad \forall x, \zeta \in \mathbb{R}^n. \tag{A.9}$$

Proof. Since for each $x \in \mathbb{R}^n$, the function $b(x, \cdot)$ has compact support (independent of x), integrating by parts we get for an arbitrary integer $N > 0$

$$\begin{aligned}
\left(1 + |\zeta|^2\right)^N \widehat{b}(x, \zeta) &= \sum_{|\alpha| \le N} \frac{N!}{\alpha!(N - |\alpha|)!} \int_{\{|y| \le \varepsilon\}} \zeta^{2\alpha} e^{-i\zeta \cdot y} b(x, y)\, dy \\
&= \sum_{|\alpha| \le N} \frac{N!}{\alpha!(N - |\alpha|)!} (-1)^{|\alpha|} \int_{\{|y| \le \varepsilon\}} e^{-i\zeta \cdot y} \partial_y^{2\alpha} b(x, y)\, dy,
\end{aligned} \tag{A.10}$$

from which (A.9) trivially follows, using that y-derivatives of $b(x, y)$ are bounded in $\mathbb{R}^n \times \mathbb{R}^n$ by a positive constant independent of x. □

We are going now to analyze the behavior at infinity of the derivatives of the symbol $q_{m,\delta}(x, \xi, \gamma)$ defined as in (A.7), where b_k is replaced by b. For all multi-indices $\alpha, \beta \in \mathbb{N}^n$, differentiation under the integral sign in (A.7) gives

$$\partial_\xi^\alpha \partial_x^\beta q_{m,\delta}(x, \xi, \gamma) = -i(2\pi)^{-n} \int \partial_x^\beta \widehat{b}(x, \eta) \partial^{\alpha + e^k} \lambda_\delta^{m-1,\gamma}(\xi - \eta)\, d\eta, \tag{A.11}$$

where $e^k := (0, \ldots, \underbrace{1}_{k}, \ldots, 0)$. Then using (3.16) and (A.9) and combining with (A.2), for $s = m - 1 - |\alpha|$, we obtain

$$\begin{aligned}
\left|\partial_\xi^\alpha \partial_x^\beta q_{m,\delta}(x, \xi, \gamma)\right| &\le C_{N,\beta} C_{m,\alpha} \int \lambda^{-2N}(\eta) \lambda_\delta^{m-2-|\alpha|,\gamma}(\xi - \eta)\, d\eta \\
&\le C_{N,m,\alpha,\beta} \lambda_\delta^{m-2-|\alpha|,\gamma}(\xi) \int \lambda^{m+2+|\alpha|-2N}(\eta)\, d\eta,
\end{aligned} \tag{A.12}$$

where the integral in the last line is finite, provided that the integer N is taken to be sufficiently large. This provides the estimate (4.32), with constant $C_{N,m,\alpha,\beta} \int \lambda^{m+2+|\alpha|-2N}(\eta)\, d\eta$ independent of γ and δ.

A.4. Proof of Lemma 4.11

Setting for short

$$\phi(x) := e^{x_1/2} \chi_1(x_1) \widetilde{\chi}(x'), \tag{A.13}$$

the symbol (4.54) can be rewritten as

$$b'_{m,\delta}(\xi',\gamma) = (2\pi)^{-n}\int \lambda_{\delta}^{m-1,\gamma}(\eta_1,\eta'+\xi')\widehat{\phi}(\eta)d\eta. \tag{A.14}$$

By a first-order Taylor expansion of $\eta \mapsto \lambda_{\delta}^{m-1,\gamma}(\eta_1,\eta'+\xi')$ about $\eta = 0$, we further obtain

$$\begin{aligned} &b'_{m,\delta}(\xi',\gamma)\\ &\quad= (2\pi)^{-n}\lambda_{\delta}^{m-1,\gamma}(\xi')\int \widehat{\phi}(\eta)d\eta - i(2\pi)^{-n}\sum_{j=1}^{n}\int\left(\int_0^1 \partial_j\lambda_{\delta}^{m-1,\gamma}(t\eta_1,t\eta'+\xi')dt\right)\widehat{\partial_j\phi}(\eta)d\eta. \end{aligned} \tag{A.15}$$

Then, using $(2\pi)^{-n}\int\widehat{\phi}(\eta)d\eta = \phi(0) = 1$ and the definition of ϕ (see (4.2)), (A.15) yields (4.77), where

$$\beta_{m,\delta}(\xi',\gamma) := -i(2\pi)^{-n}\sum_{j=1}^{n}\int\left(\int_0^1 \partial_j\lambda_{\delta}^{m-1,\gamma}(t\eta_1,t\eta'+\xi')dt\right)\widehat{\partial_j\phi}(\eta)d\eta. \tag{A.16}$$

To prove (4.78), differentiation under the integral sign of (A.16) gives for an arbitrary $\alpha' \in \mathbb{N}^{n-1}$

$$\partial_{\xi'}^{\alpha'}\beta_{m,\delta}(\xi',\gamma) = -i(2\pi)^{-n}\sum_{j=1}^{n}\int\left[\int_0^1\left(\partial^{e^j+(0,\alpha')}\lambda_{\delta}^{m-1,\gamma}\right)(t\eta_1,t\eta'+\xi')dt\right]\widehat{\partial_j\phi}(\eta)d\eta, \tag{A.17}$$

hence from (3.16) we get

$$\left|\partial_{\xi'}^{\alpha'}\beta_{m,\delta}(\xi',\gamma)\right| \le C_{m,\alpha'}\sum_{j=1}^{n}\int\left(\int_0^1 \lambda_{\delta}^{m-2-|\alpha'|,\gamma}(t\eta_1,t\eta'+\xi')dt\right)\left|\widehat{\partial_j\phi}(\eta)\right|d\eta. \tag{A.18}$$

Then, applying (A.2) (for $s = m-1-|\alpha'|$) to estimate the right-hand side of (A.18) and using that $\widehat{\partial_j\phi} \in \mathcal{S}(\mathbb{R}^n)$ for each $1 \le j \le n$, we get

$$\left|\partial_{\xi'}^{\alpha'}\beta_{m,\delta}(\xi',\gamma)\right| \le C'_{m,N,\alpha'}\lambda^{m-2-|\alpha'|,\gamma}(\xi')\int\lambda^{m+2+|\alpha'|-N}(\eta)d\eta, \tag{A.19}$$

for an arbitrary integer $N > 0$ and $C'_{m,N,\alpha'} > 0$ independent of γ and δ. This provides the estimate of type (4.78), once N is chosen large enough to ensure the convergence of the last integral.

A.5. A Further Technical Result

We conclude this appendix with the following result, that was involved in the arguments given in Section 4.4.

Lemma A.2. *Let* $u \in H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ *be a solution to* (1.2)-(1.3), *with data* $F \in \mathscr{H}^{s+m,r+m}_{\tan,\gamma}(\mathbb{R}^n_+)$, $G \in H^{s+m}_{\gamma}(\mathbb{R}^{n-1})$, *such that* $u^I_{|x_1=0} \in H^{m-1}_{\gamma}(\mathbb{R}^{n-1})$, *for a given integer* $m \geq 1$. *Let* $\{u_\nu\}$ *be a sequence in* $C^\infty_{(0)}(\mathbb{R}^n_+)$ *approximating the solution* u *in the sense of* (4.36). *Then the trace* $(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u^I)_{|x_1=0}$ *is well defined in* $H^{-1/2}(\mathbb{R}^{n-1})$ *and one has*

$$\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u^I_\nu\right)_{|x_1=0} \longrightarrow \left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u^I\right)_{|x_1=0}, \quad \text{in } \mathcal{D}'\left(\mathbb{R}^{n-1}\right). \tag{A.20}$$

Proof. Since $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)$ is of order $m-1$, in view of Proposition 3.8 it follows from $u \in H^{m-1}_{\tan,\gamma}(\mathbb{R}^n_+)$ that $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u \in L^2(\mathbb{R}^n_+)$. We use (4.52) to find

$$\begin{aligned}(L+B)\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u = {} & \left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z) - \eta_{m,\delta}(x,Z,\gamma)\right)F - \gamma\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u \\ & - \rho_{m,\delta}(x,Z,\gamma)\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u - \tau_{m,\delta}(x,Z,\gamma)u,\end{aligned} \tag{A.21}$$

where the operators involved in the right-hand side are defined by (4.51). Because of (4.73), (4.74) and since $\rho_{m,\delta}(x,Z,\gamma)$ is L^2-bounded (by Proposition 3.8), from (A.21) we derive that $(L+B)\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u \in L^2(\mathbb{R}^n_+)$. Then, it is known from [14] that the trace on the boundary of $(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u)^I = \lambda^{m-1,\gamma}_{\chi,\delta}(Z)u^I$ exists in $H^{-1/2}(\mathbb{R}^{n-1})$; moreover the Green formula

$$\int_{\mathbb{R}^n_+}(L+B)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right)v = \int_{\mathbb{R}^n_+}\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\left[(L+B)^*v\right] + \int_{\mathbb{R}^{n-1}}\left(A_1\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right)_{|x_1=0}v_{|x_1=0}dx' \tag{A.22}$$

holds true for all functions $v \in C^\infty_{(0)}(\mathbb{R}^n_+)$.

Notice that $u_\nu \in C^\infty_{(0)}(\mathbb{R}^n_+)$ implies that $(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu)^\sharp = \lambda^{m-1,\gamma}_{\chi,\delta}(D)(u^\sharp_\nu) \in L^2(\mathbb{R}^n)$, and then $\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu \in L^2(\mathbb{R}^n_+)$. Therefore, starting from the same equation as (4.52), where u and F are replaced by u_ν and $F_\nu := (\gamma + L + B)u_\nu$, and arguing as before, one also gets $(L+B)\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu \in L^2(\mathbb{R}^n_+)$; then $(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u^I_\nu)_{|x_1=0} \in H^{-1/2}(\mathbb{R}^{n-1})$, for each ν, and (A.22) is fulfilled, where u is replaced by u_ν.

Because the Green formulas hold for u and u_ν, (A.20) is true, granted that the convergences

$$\begin{aligned}&\int_{\mathbb{R}^n_+}(L+B)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu\right)v \longrightarrow \int_{\mathbb{R}^n_+}(L+B)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\right)v, \\ &\int_{\mathbb{R}^n_+}\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu\left[(L+B)^*v\right] \longrightarrow \int_{\mathbb{R}^n_+}\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u\left[(L+B)^*v\right]\end{aligned} \tag{A.23}$$

have been proved, whenever $v \in C^\infty_{(0)}(\mathbb{R}^n_+)$.

For each ν, we use (A.21) (and a change of variables) to get

$$\begin{aligned}\int_{\mathbb{R}^n_+}(L+B)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u_\nu\right)v &= \int_{\mathbb{R}^n}\Big[\left(\lambda^{m-1,\gamma}_{\chi,\delta}(D)-\eta_{m,\delta}(x,D,\gamma)\right)F^\sharp_\nu - \gamma\lambda^{m-1,\gamma}_{\chi,\delta}(D)u^\sharp_\nu \\ &\quad -\rho_{m,\delta}(x,D,\gamma)\lambda^{m-1,\gamma}_{\chi,\delta}(D)u^\sharp_\nu - \tau_{m,\delta}(x,D,\gamma)u^\sharp_\nu\Big]v^\sharp \\ &= \int_{\mathbb{R}^n} F^\sharp_\nu a(x,D,\gamma)^* v^\sharp + \int u^\sharp_\nu b(x,D,\gamma)^* v^\sharp,\end{aligned} \tag{A.24}$$

where we have set

$$\begin{aligned} a(x,D,\gamma) &:= \lambda^{m-1,\gamma}_{\chi,\delta}(D) - \eta_{m,\delta}(x,D,\gamma), \\ b(x,D,\gamma) &:= -\gamma\lambda^{m-1,\gamma}_{\chi,\delta}(D) - \rho_{m,\delta}(x,D,\gamma)\lambda^{m-1,\gamma}_{\chi,\delta}(D) - \tau_{m,\delta}(x,D,\gamma).\end{aligned} \tag{A.25}$$

Repeating the same calculations on $\int(L+B)(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)u)v$ also gives

$$\int_{\mathbb{R}^n_+}(L+B)\left(\lambda^{m-1,\gamma}_{\chi,\delta}(Z)(u-u_\nu)\right)v = \int_{\mathbb{R}^n}\left(F^\sharp_\nu - F^\sharp\right)a(x,D,\gamma)^* v^\sharp + \int\left(u^\sharp_\nu - u^\sharp\right)b(x,D,\gamma)^* v^\sharp. \tag{A.26}$$

Then the first convergence in (A.23) is proven, as a consequence of the convergences $F^\sharp_\nu \to F^\sharp$, $u^\sharp_\nu \to u^\sharp$ in $L^2(\mathbb{R}^n)$ and Cauchy-Schwarz's inequality.

In a completely similar way, one can check the validity of the second convergence in (A.23). □

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